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Two-Mechanism Approach in Thermo-Viscoelasticity with Internal Variables

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Two-mechanism (more general: multi-mechanism) models have become an important tool for modeling of complex material behavior. In particular, two-mechanism models have been applied for modeling of ratcheting in metal plasticity as well as of steel behavior in case of phase transformations. The characteristic trait of two-mechanism models is the additive decomposition of the inelastic (i.e., plastic or visco-elastic, e.g.) strain into two parts (sometimes called "mechanisms") in the case of small deformations. In comparison with rheological models, there is an interaction between these mechanisms allowing to describe important observable effects. We develop a general visco-elastic two-mechanism model in the framework of the internal-variables approach. As a numerical example, we simulate the movement of a rod having a special visco-elastic behavior. An applied periodic stress with non-zero mean value may lead to a ratcheting effect stemming from the coupling of mechanisms.

1 Introduction

Two-mechanism (more general: multi-mechanism) models have become an important tool for modeling of complex material behavior. The characteristic trait of two-mechanism models (abbreviated as 2M models) is the additive decomposition of the inelastic (i.e., plastic or visco-elastic, e.g.) strain into two parts (sometimes called "mechanisms") in the case of small deformations (see Figure 1). We refer to Saï (2011), Wolff et al. (2011) for current overviews. In comparison with rheological models (cf. Palmov (1998), e.g.), generally there is an interaction between the mechanisms allowing to describe important observable effects. On the other hand, rheological models consisting of elements connected in series are (simple) multi-mechanism models.
If the inelastic strain is seen as one mechanism (as it was historically first), one refers to a "unified model" (or "Chaboche" model) (cf. the survey by Chaboche (2008) and the references cited therein). In this case plastic and viscous components are considered together in the same variable. As explained in Contesti and Cailletaud (1989) and Cailletaud and Saï (1995), there are experimentally observable effects (inverse strain-rate sensibility, e.g.) which can be qualitatively correctly described by the two-mechanism approach. To our knowledge, a first systematic formulation and investigation of two mechanism models was given by Contesti and Cailletaud (1989).
An important application of two-mechanism models is cyclic plasticity including ratcheting. Investigations of ratcheting with the aid of two-mechanism models can be found in Saï and Cailletaud (2007), Hassan et al. (2008), Taleb and Hauet (2009), Taleb and Cailletaud (2010), Saï (2011), e.g. In Hassan et al. (2008), a direct comparison between a modified Chaboche model and a 2M model has been performed (in Wolff et al. (2011), this comparison has been repeated in short).

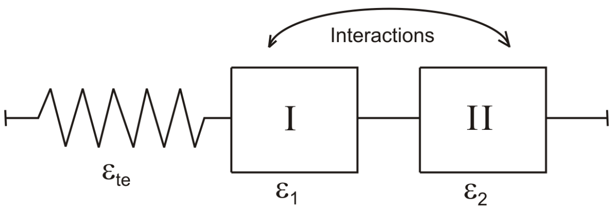

Figure 1: Scheme of a two-mechanism model. The two inelastic mechanisms 1 and 2 have their own evolution equations. Generally, they are not independent from each other. The thermoelastic strain ε_{te} is usually not regarded as a mechanism.

Another important application of two-mechanism models lies in modeling of complex material behavior of steel under phase transformations. For a direct two-mechanism approach we refer to Videau et al. (1994), Wolff et al. (2008), Aeby-Gautier and Cailletaud (2004) , e.g.
The complex material behavior of important materials (such as visco-plastic materials, shape-memory alloys, soils, granular materials, composites, biological tissues) leads to multi-mechanism models, when taking the additive decomposition of the strain tensor into account. However, in most cases, the concrete application is *not* set in the framework of multi-mechanism models. For some references see Wolff et al. (2011).
Some polymers show a material behavior similar to ratcheting in metal plasticity. This effect has been reported for an epoxy resin in Tao and Xia (2007), and in Shen et al. (2004) for epoxy polymers, e.g. In Xia et al. (2006), nonlinear viscoelastic models have been applied for description of cyclic deformation behavior of polymers. As we will show in this study, a two-mechanism model with linear viscoelastic mechanisms can describe a ratcheting effect. In Kröger et al. (2012), the mathematical model for a rod with a viscoelastic two-mechanism behavior has been analyzed and simulations have been performed.
The main aims of this paper are:

- Application of the two-mechanism approach to visco-elastic material behavior within the internal-variables framework, resulting in the development of a general thermo-visco-elastic 2M model.
- Simulations of the longitudinal movement of a rod having a special visco-elastic behavior in order to study some effects stemming from the coupling of mechanisms like ratcheting.

2 Application of the Two-Mechanism Approach in Thermo-Viscoelasticity

2.1 Preparations

Some Basics

We restrict ourselves to small deformations. Thus, the equation of momentum, the energy equation and the Clausius-Duhem inequality are given by

$$\varrho\,\ddot{\boldsymbol{u}} - \mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \tag{2.1}$$

$$\varrho\,\dot{e} + \mathrm{div}\,\boldsymbol{q} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r \tag{2.2}$$

$$-\varrho\,\dot{\psi} - \varrho\,\eta\,\dot{\theta} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{\nabla}\theta \geq 0. \tag{2.3}$$

The relations (2.1) - (2.3) have to be fulfilled in the space-time domain $\Omega\times]0,T[$. Ω is the body's reference configuration, $T > 0$ is the process time. The notation is standard: ϱ - density in the reference configuration, that means for $t = 0$, $\boldsymbol{u}$ - displacement vector, $\boldsymbol{\varepsilon}$ - linearized Green strain tensor, θ - absolute temperature, $\boldsymbol{\sigma}$ - Cauchy stress tensor, $\boldsymbol{f}$ - volume density of external forces, e - mass density of the internal energy, $\boldsymbol{q}$ - heat-flux density vector, r - volume density of heat supply, ψ - mass density of free (or Helmholtz) energy, η - mass density of entropy. The time derivative is denoted by a dot. $\boldsymbol{\alpha} : \boldsymbol{\beta}$ is the scalar product of the tensors, $\boldsymbol{q}\cdot\boldsymbol{p}$ is the scalar product of the vectors. Tensors and vectors are in bold face. We note the well-known relations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T), \qquad e = \psi + \theta\,\eta. \tag{2.4}$$

In the general case of inelastic material behavior (in case of small deformations), the full strain $\boldsymbol{\varepsilon}$ is split up into

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{in} \tag{2.5}$$

($\boldsymbol{\varepsilon}_{te}$ - thermoelastic strain, $\boldsymbol{\varepsilon}_{in}$ - inelastic strain). In many applications, in particular in metal plasticity, the inelastic strain is assumed to be traceless, i.e. $\mathrm{tr}(\boldsymbol{\varepsilon}_{in}) = 0$. To be more general, we do not assume this property. The accumulated inelastic strain is defined by

$$s_{in}(t) := \int_0^t \left(\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_{in}(\tau) : \dot{\boldsymbol{\varepsilon}}_{in}(\tau)\right)^{\frac{1}{2}}\,d\tau. \tag{2.6}$$

In many cases, the dependence on the space variable x will not be written. We propose for the free energy ψ the split

$$\psi(\varepsilon_{te}, \theta, \xi) = \psi_{te}(\varepsilon_{te}, \theta) + \psi_{in}(\theta, \xi) \tag{2.7}$$

into a thermoelastic part ψ_{te} and an inelastic one ψ_{in}. The quantities $\xi = (\xi_1, \ldots, \xi_m)$ (ξ_j - scalars or tensors) represent the internal variables. These variables will be chosen in accordance with concrete models under consideration. Moreover, they have to fulfil evolution equations which are usually ordinary differential equations (ODE) with respect to the time t

$$\dot{\xi}_j = \Xi_j(\xi, \theta, \boldsymbol{\sigma}). \tag{2.8}$$

Additionally, one poses initial conditions, i.e.

$$\xi_j(0) = \xi_{j0} \qquad \text{for } j = 1, \ldots, m. \tag{2.9}$$

In many cases, the thermoelastic part ψ_{te} of the free energy is given by

$$\psi_{te} = \frac{1}{2\varrho}\big(\boldsymbol{E}\boldsymbol{\varepsilon}_{te} : \boldsymbol{\varepsilon}_{te} - 2(\theta - \theta_0)\boldsymbol{G} : \boldsymbol{\varepsilon}_{te}\big) + C(\theta). \tag{2.10}$$

$\boldsymbol{E}$ is the fourth-order elasticity tensor, $\boldsymbol{G}$ is the second-order stress-temperature tensor. θ_0 is a fixed initial temperature, i.e. for time $t = 0$. C is the temperature-dependent calorimetric function (see Helm and Haupt (2003), e.g.). The (second-order) tensor $\boldsymbol{G}$ is symmetric, i.e.,

$$G_{ij} = G_{ji} \qquad \text{for all } i, j \in \{1, 2\}, \tag{2.11}$$

and $\boldsymbol{E}$ is positive definite, i.e., it holds

$$\boldsymbol{E}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = \sum_{k,l=1}^{3} E_{ijkl}\varepsilon_{kl}\varepsilon_{ij} > 0 \qquad \text{for all second order tensors } \boldsymbol{\varepsilon} \neq 0. \tag{2.12}$$

Note that $\boldsymbol{E}\boldsymbol{\varepsilon}$ is the application of the fourth-order tensor $\boldsymbol{E}$ to the second-order tensor $\boldsymbol{\varepsilon}$ and yields the second-order tensor $\boldsymbol{E}\boldsymbol{\varepsilon}$. Moreover, $\boldsymbol{E}$ fulfils the following symmetry relations

$$E_{ijkl} = E_{jikl} = E_{ijlk} \qquad \text{for all } i, j, k, l \in \{1, 2, 3, 4\}. \tag{2.13}$$

Therefore, in the case of totally anisotropic material behavior, only 21 components of $\boldsymbol{E}$ are independent. If there are some additional symmetries, this number is reduced up to two for isotropic behavior (cf. remark 2.1 (ii)). We note that generally $\boldsymbol{E}$ and $\boldsymbol{G}$ depend on the space variable $x \in \Omega$ (spatial inhomogeneity) and on the temperature θ. In these cases, the conditions (2.11) – (2.13) must be fulfilled for all body points x and for all admissible temperatures, respectively. For convenience, unless stated otherwise, we only consider spatially homogeneous materials. Clearly, this is not a great restriction. The temperature dependence will be accented, if it seems to be appropriate.

Using standard arguments of thermodynamics (cf. Coleman and Gurtin (1967), Lemaitre and Chaboche (1990), Maugin (1992), Besson et al. (2001), Haupt (2002), e.g.), one obtains the remaining inequality

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \varrho\sum_{j=1}^{m}\frac{\partial\psi}{\partial\xi_j} : \dot{\xi}_j - \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{\nabla}\theta \geq 0. \tag{2.14}$$

as well as the potential relations

$$\boldsymbol{\sigma} = \varrho\frac{\partial\psi_{te}}{\partial\boldsymbol{\varepsilon}_{te}}, \qquad \eta = -\frac{\partial\psi}{\partial\theta}. \tag{2.15}$$

Moreover, one defines the thermodynamic forces $\boldsymbol{X}_j$ via

$$\boldsymbol{X}_j = \varrho\frac{\partial\psi}{\partial\xi_j} \tag{2.16}$$

From (2.10) and (2.15) the relation between stress and thermoelastic strain tensor follows

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}_{te} - (\theta - \theta_0)\boldsymbol{G} \tag{2.17}$$

Usually, one assumes Fourier's law for the heat conduction

$$\boldsymbol{q} = -\kappa\,\boldsymbol{\nabla}\theta \tag{2.18}$$

with a positive heat conductivity κ (or, more generally with a positively definite heat-conductivity tensor). In case of (2.18), the heat-conduction inequality is fulfilled, i.e.

$$-\frac{1}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{\nabla}\theta \geq 0. \tag{2.19}$$

Hence, the model under consideration is thermodynamically consistent, if the Clausius-Planck inequality is fulfilled

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \sum_{j=1}^{m} \boldsymbol{X}_j : \dot{\xi}_j \geq 0. \tag{2.20}$$

By standard arguments (cf. Lemaitre and Chaboche (1990), Besson et al. (2001), Haupt (2002), e.g.), the energy equation (2.2) implies the heat-conduction equation

$$\varrho\, c_d\, \dot{\theta} - \mathrm{div}(\kappa\boldsymbol{\nabla}\theta) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \sum_{j=1}^{m} \boldsymbol{X}_j : \dot{\xi}_j + \theta\,\frac{\partial\boldsymbol{\sigma}}{\partial\theta} : \dot{\boldsymbol{\varepsilon}}_{te} + \theta\sum_{j=1}^{m}\frac{\partial\boldsymbol{X}_j}{\partial\theta} : \dot{\xi}_j + r. \tag{2.21}$$

The parameter c_d is the specific heat. In the scheme outlined above, there is only one inelastic strain $\boldsymbol{\varepsilon}_{in}$. Therefore, one can speak about a one-mechanism (1M) model.

Two-Mechanism Approach

Up to this point, there is no difference between 1M models and 2M models. From now on, we deal with 2M models. The general assertions can be extended to multi-mechanism models (in short mM models) without difficulties. In the theory of 2M models the following decomposition of the inelastic strain is crucial

$$\boldsymbol{\varepsilon}_{in} = A_1\,\boldsymbol{\varepsilon}_1 + A_2\,\boldsymbol{\varepsilon}_2, \tag{2.22}$$

with positive parameters A_1, A_2. Defining the partial stresses

$$\boldsymbol{\sigma}_j := A_j\,\boldsymbol{\sigma} \qquad\qquad j = \{1, 2\}, \tag{2.23}$$

the Clausius-Planck inequality (2.20) takes the form

$$\boldsymbol{\sigma}_1 : \dot{\boldsymbol{\varepsilon}}_1 + \boldsymbol{\sigma}_2 : \dot{\boldsymbol{\varepsilon}}_2 - \sum_{j=1}^{m} \boldsymbol{X}_j : \dot{\xi}_j \geq 0. \tag{2.24}$$

The heat-conduction equation (2.21) can be re-written in an analogous manner. In the general situation, the inelastic strains $\boldsymbol{\varepsilon}_j$ are *not* assumed to be traceless. For both $\boldsymbol{\varepsilon}_j$ we introduce *separate* accumulations

$$s_j(t) := \int_0^t \left(\frac{2}{3}\,\dot{\boldsymbol{\varepsilon}}_j(\tau) : \dot{\boldsymbol{\varepsilon}}_j(\tau)\right)^{\frac{1}{2}}\, d\tau \quad j = 1, 2. \tag{2.25}$$

Note, that s_{in} (as defined in (2.6)) is not the sum of s_1 and s_2. In order to develop concrete 2M models, one has to propose the free energy (density) ψ, evolution equations for the internal variables ξ (like in (2.8)) as well as for the inelastic strains $\dot{\boldsymbol{\varepsilon}}_j$.

Remarks 2.1.

(i) Sometimes, the thermoelastic part ψ_{te} of the free energy (cf. (2.7)) can depend on internal variables in order to take possible damage effects into account. We drop this here.

(ii) In *case of isotropy*, the thermoelastic part ψ_{te} in (2.10) is frequently given by

$$\psi_{te} := \frac{1}{\varrho}\{\mu\,\boldsymbol{\varepsilon}_{te}^* : \boldsymbol{\varepsilon}_{te}^* + \frac{K}{2}(\mathrm{tr}(\boldsymbol{\varepsilon}_{te}))^2 - 3\,K\,\alpha(\theta - \theta_0)\,\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\} + C(\theta). \tag{2.26}$$

Here are: $\mu > 0$ - shear modulus, $K > 0$ - compression modulus, α - linear heat-dilatation coefficient, $\boldsymbol{\varepsilon}_{te}^*$ - deviator of $\boldsymbol{\varepsilon}_{te}$, defined (in $3d$ case) by

$$\boldsymbol{\varepsilon}_{te}^* = \boldsymbol{\varepsilon}_{te} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\boldsymbol{I} \qquad\qquad (\boldsymbol{I}\text{ - unity tensor}). \tag{2.27}$$

Clearly, in the case of (2.26) the first potential relation in (2.15) leads to the (isotropic) Duhamel-Neumann (i.e. generalized Hooke) relation of linear thermoelasticity

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon}^*_{te} + K\,\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\boldsymbol{I} - 3K\alpha\,(\theta - \theta_0)\boldsymbol{I}, \tag{2.28}$$

which specializes the general relation (2.17).

(iii) The approach for evolution equations in (2.8) can be generalized, using functionals instead of functions. We refer to Haupt (2002) for discussion and references. Sometimes, one assumes that for given θ and $\boldsymbol{\sigma}$ the system (2.8), (2.9) is uniquely solvable. Due to objectivity arguments, a possible dependence on the stress tensor $\boldsymbol{\sigma}$ is realized via tensorial invariants. We drop this here.

(iv) The positive parameters A_1 and A_2 in the split of the inelastic strain (2.22) open opportunities for further extensions and special applications. We refer to Saï and Cailletaud (2007). A_1 and A_2 can depend on further quantities as, for instance, they can constitute phase fraction in complex materials (steel, shape memory alloys, e.g.). In this sense, here is a bridge from the macro to the meso (or micro) level of modeling. However, in many current applications, these parameters are equal to one (cf. Hassan et al. (2008), Taleb and Cailletaud (2010), Wolff and Taleb (2008), e.g.)

2.2 Two Kelvin-Voigt Bodies in Series as a Two-Mechanism Model

This subsection is a *preparation* for introducing a more general visco-elastic 2M model in the next subsection 2.3. Thus, for simplicity, in this subsection 2.2, we only consider the isothermal case. We consider two Kelvin-Voigt bodies (abbreviated as KV bodies) in series (see Figure 2). As known, a KV body consist of a Hooke element (illustrated by a spring) and a Newton element (illustrated by a damper) connected in parallel (cf. e.g. Haupt (2002), Altenbach and Altenbach (1994)).

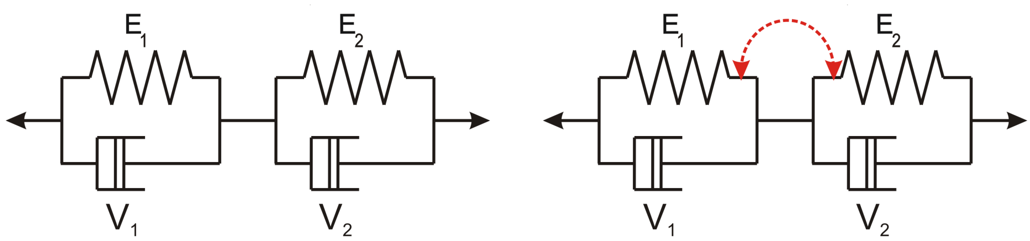

Figure 2: Two Kelvin-Voigt bodies in series - without coupling as a rheological model (left) as well as with coupling leading to a 2M model (right).

Denoting by “1” and “2” the quantities of the first and second Kelvin-Voigt body, respectively, one has the obvious relations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2, \qquad\qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2. \tag{2.29}$$

Here, we understand $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ as $3d$ strain and stress tensor, respectively, generalizing the corresponding $1d$ considerations. In Section 3, we deal with the space-dependent $1d$ case, presenting simulations for a rod. In remark 3.1, the simple case of springs and dampers is addressed which leads to ordinary differential equations.

Case of no Coupling – Rheological Model

At first, we deal with the case of no coupling between the two KV bodies. (This is for motivation of the introduction of the coupling in the next paragraph.) The stress-strain relations for the (generally non-isotropic) KV bodies are

$$\boldsymbol{\sigma} = \boldsymbol{E}_i\,\boldsymbol{\varepsilon}_i + \boldsymbol{V}_i\,\dot{\boldsymbol{\varepsilon}}_i \qquad i = 1, 2. \tag{2.30}$$

$\boldsymbol{E}_i$ and $\boldsymbol{V}_i$ are fourth-order tensors. Here, in our case, the $\boldsymbol{V}_i$ are assumed to be positive definite (cf. (2.12)), while the $\boldsymbol{E}_i$ fulfil the weaker condition of positive semi-definiteness:

$$\boldsymbol{E}_i\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq 0 \qquad \text{for all symmetric second-order tensors } \boldsymbol{\varepsilon}. \tag{2.31}$$

Moreover, $\boldsymbol{E}_i$ and $\boldsymbol{V}_i$ have the symmetry properties (2.13). In comparison with the general approach in subsection 2.1 there is no separate (thermo-)elastic part in the the split of strain (2.29). Nevertheless, the two KV bodies in series represent a special 2M model, and the above arguments leading to the Clausius-Planck inequality hold true with the exception of the first potential relation in (2.15). Re-writing (2.30) as

$$\dot{\boldsymbol{\varepsilon}}_i = \boldsymbol{V}_i^{-1}(\boldsymbol{\sigma} - \boldsymbol{E}_i\,\boldsymbol{\varepsilon}_i), \tag{2.32}$$

the quantities $E_i\,\boldsymbol{\varepsilon}_i$ can be regarded as back stresses. This can be derived, assuming the free energy as

$$\psi = \frac{1}{2\varrho}\,(\boldsymbol{E}_1\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_1 + \boldsymbol{E}_2\,\boldsymbol{\varepsilon}_2 : \boldsymbol{\varepsilon}_2), \tag{2.33}$$

and defining the back-stresses (compatible with (2.32))

$$\boldsymbol{X}_i := \varrho\,\frac{\partial\psi_{in}}{\partial\boldsymbol{\varepsilon}_i} = \boldsymbol{E}_i\,\boldsymbol{\varepsilon}_i, \qquad i = 1, 2 \tag{2.34}$$

the Clausius-Planck inequality becomes

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_1 + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_2 - \sum_{i=1}^{2}\boldsymbol{X}_i : \dot{\boldsymbol{\varepsilon}}_i = \sum_{i=1}^{2}\boldsymbol{V}_i^{-1}(\boldsymbol{\sigma} - \boldsymbol{X}_i) : (\boldsymbol{\sigma} - \boldsymbol{X}_i) \geq 0. \tag{2.35}$$

Clearly, due to the positive definiteness of V_i (and thus of V_i^{-1}), the inequality (2.35) is always fulfilled. In this example, the two Kelvin-Voigt bodies are not coupled, they form a rheological model (and a simple 2M model). If the forming elements of a rheological model (here the two KV bodies) are thermodynamically consistent, the whole model is so (Palmov (1998)).

Case of Coupling – Two-Mechanism Model

Now, we allow a possible coupling between the KV bodies leading to an "authentic" 2M model (see Figure 2 (right)). As a consequence, the stress-strain relations (2.30) become more general. Therefore, we start with the assumption of the free energy. Instead of (2.33), we assume

$$\psi = \frac{1}{2\varrho}\big(\boldsymbol{E}_{11}\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_1 + 2\boldsymbol{E}_{12}\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_2 + \boldsymbol{E}_{22}\,\boldsymbol{\varepsilon}_2 : \boldsymbol{\varepsilon}_2\big). \tag{2.36}$$

From now on, for convenience we write E_{ii} instead of E_i. The fourth-order tensors E_{11}, E_{12} and E_{22} fulfil the symmetry conditions (2.13). Additionally, without loss of generality, we may assume that E_{12} is self-adjoint in the following sense (see remark 2.2 (i), (ii) for reasoning)

$$(E_{12})_{ijkl}(\theta) = (E_{12})_{klij}(\theta) \qquad \text{for all } i, j, k, l \in \{1, 2, 3, 4\}. \tag{2.37}$$

It is reasonable to require that the stored energy is bounded below as a function of the (inelastic) strains (see remark 2.2 (iii)). Setting formally $\boldsymbol{E}_{21} := \boldsymbol{E}_{12}$ for convenience, we ensure this by assuming

$$\sum_{i,j=1}^{2}\boldsymbol{E}_{ij}\,\boldsymbol{\varepsilon}_i : \boldsymbol{\varepsilon}_j \geq 0 \quad \text{for all symmetric second-order tenors } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2. \tag{2.38}$$

Note that it follows from (2.38) that $\boldsymbol{E}_{11}$ and $\boldsymbol{E}_{22}$ are positive semi-definite in the sense of (2.31). Now the back-stress relations are given by

$$\boldsymbol{X}_1 := \varrho\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_1} = \boldsymbol{E}_{11}\,\boldsymbol{\varepsilon}_1 + \boldsymbol{E}_{12}\,\boldsymbol{\varepsilon}_2, \quad \boldsymbol{X}_2 := \varrho\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_2} = \boldsymbol{E}_{12}\,\boldsymbol{\varepsilon}_1 + \boldsymbol{E}_{22}\,\boldsymbol{\varepsilon}_2. \tag{2.39}$$

Moreover, we assume the material laws to be similar as in (2.32), but written with more general back stresses

$$\dot{\boldsymbol{\varepsilon}}_i = \boldsymbol{H}_i(\boldsymbol{\sigma} - \boldsymbol{X}_i), \qquad i = 1, 2. \tag{2.40}$$

As now we start with (2.40), we write $\boldsymbol{H}_i$ instead of $\boldsymbol{V}_i^{-1}$ as in (2.32). The Clausius-Planck inequality is the same as in case without coupling (i.e., (2.35) with $\boldsymbol{H}_i$ instead of $\boldsymbol{V}_i^{-1}$). Now, the positive semi-definiteness of $\boldsymbol{H}_i$ ensures thermodynamical consistency.

As we will see in Section 3, the coupling in the free energy in (2.36) (and thus in (2.39)) may lead to qualitatively new effects in comparison with the rheological model without such coupling.

Remarks 2.2.

(i) Regarding a fourth-order tensor E as a linear operator acting on the linear space $\mathbb{R}^9$ (i.e. the space of all 3×3 matrices), the adjoint E^T of E is defined by

$$\boldsymbol{E}\boldsymbol{\varepsilon} : \boldsymbol{\alpha} = E_{ijkl}\varepsilon_{kl}\alpha_{ij} = E^T_{ijkl}\alpha_{kl}\varepsilon_{ij} = \boldsymbol{\varepsilon} : \boldsymbol{E}^T\boldsymbol{\alpha} \quad \text{for all second-order tensors } \boldsymbol{\varepsilon}, \boldsymbol{\alpha}. \tag{2.41}$$

(Summation convention is used, ":" is the scalar product in $\mathbb{R}^9$). Therefore, one has

$$E^T_{ijkl} = E_{klij} \qquad \text{for all } i, j, k, l \in \{1, 2, 3, 4\}, \tag{2.42}$$

and the relation (2.37) means that $\boldsymbol{E}$ is self-adjoint.

(ii) Due to the obvious relation

$$\boldsymbol{E}_{12}\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_2 + \boldsymbol{E}_{21}\,\boldsymbol{\varepsilon}_2 : \boldsymbol{\varepsilon}_1 = \left(\boldsymbol{E}_{12} + \boldsymbol{E}^T_{21}\right)\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_2 = 2\,\tilde{\boldsymbol{E}}_{12}\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_2 \tag{2.43}$$

with $\tilde{\boldsymbol{E}}_{12} := \frac{1}{2}(\boldsymbol{E}_{12} + \boldsymbol{E}^T_{21})$, the fourth-order tensor $\boldsymbol{E}_{12}$ can be assumed to be self-adjoint (in the sense of (2.37)) without any loss of generality.

(iii) Due to the bi-linearity, the property (2.38) is equivalent to the convexity of the ψ given in (2.36) as well as to boundedness from below.

(iv) The general approach in (2.36) covers the scalar case. Letting $\boldsymbol{E}_{ij} = c_{ij}\boldsymbol{I}$ ($i, j = 1, 2$, $\boldsymbol{I}$ unity fourth-order tensor) with real numbers c_{ij}, we obtain from (2.36):

$$\psi = \frac{1}{2\varrho}\left(c_{11}\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_1 + 2c_{12}\,\boldsymbol{\varepsilon}_1 : \boldsymbol{\varepsilon}_2 + c_{22}\,\boldsymbol{\varepsilon}_2 : \boldsymbol{\varepsilon}_2\right). \tag{2.44}$$

Moreover, the condition (2.38) means that the matrix c is positive semi-definite. A sufficient and necessary criterion for a symmetric matrix c to be positive semi-definite is (see Wolff and Taleb (2008), e.g.)

$$c_{12}^2 \le c_{11}c_{22}. \tag{2.45}$$

Obviously, the condition (2.45) is sufficient and necessary for the matrix c to be bounded from below as well as convex. A violation of condition (2.45) may lead to physically unreasonable results (we refer to Wolff and Taleb (2008) for corresponding simulations in case of plasticity.).

2.3 General Two-Mechanism Models in Thermo-Viscoelasticity

Now we want to generalize the example of two coupled KV bodies in subsection 2.2 to a complex 2M model with visco-elastic mechanisms. Based on (2.5) and (2.22), we start with the additive split of the bulk strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{in} = \boldsymbol{\varepsilon}_{te} + A_1\,\boldsymbol{\varepsilon}_1 + A_2\,\boldsymbol{\varepsilon}_2, \quad A_1, A_2 > 0. \tag{2.46}$$

The thermo-elastic part ψ_{te} is given by (2.10). Thus, the thermo-elastic strain $\boldsymbol{\varepsilon}_{te}$ is related to the stress $\boldsymbol{\sigma}$ by (2.17) due to the first potential relation in (2.15). In the case of isotropy, this leads to equation (2.28).
We define the inelastic part of the free energy by

$$\psi_{in}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \theta) = \frac{1}{2\varrho}\Big(\boldsymbol{E}_{11}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2\boldsymbol{E}_{12}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + \boldsymbol{E}_{22}(\theta)\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 + \\ - 2(\theta - \theta_0)\boldsymbol{G}_1(\theta) : \boldsymbol{\alpha}_1 - 2(\theta - \theta_0)\boldsymbol{G}_2(\theta) : \boldsymbol{\alpha}_2\Big). \tag{2.47}$$

The quantities $\boldsymbol{\alpha}_i$ ($i = 1, 2$) are symmetric tensorial internal variables of strain type having an own evolution (see (2.52)). The quantities $\boldsymbol{E}_{ij}$ are assumed to be fourth-order tensor functions fulfilling (2.38) for each temperature θ (see also remark 2.4 (i)). Besides, $\boldsymbol{E}_{12}$ is self-adjoint (for each temperature) in the sense of (2.37). The symmetric second-order tensor functions $\boldsymbol{G}_i$ take possible thermo-stress effects into account. From (2.47) we get the back stress relations

$$\boldsymbol{X}_1 := \varrho\,\frac{\partial\psi_{in}}{\partial\boldsymbol{\alpha}_1} = \boldsymbol{E}_{11}(\theta)\,\boldsymbol{\alpha}_1 + \boldsymbol{E}_{12}(\theta)\,\boldsymbol{\alpha}_2 - (\theta - \theta_0)\boldsymbol{G}_1(\theta), \tag{2.48}$$

$$\boldsymbol{X}_2 := \varrho\,\frac{\partial\psi_{in}}{\partial\boldsymbol{\alpha}_2} = \boldsymbol{E}_{12}(\theta)\,\boldsymbol{\alpha}_1 + \boldsymbol{E}_{22}(\theta)\,\boldsymbol{\alpha}_2 - (\theta - \theta_0)\boldsymbol{G}_2(\theta). \tag{2.49}$$

We assume the following evolution laws for the inelastic (visco-elastic) strains $\boldsymbol{\varepsilon}_i$, generalizing (2.40)

$$\dot{\boldsymbol{\varepsilon}}_i = \Big\{ \big(\frac{\|\boldsymbol{\sigma}_i - \boldsymbol{X}_i\|}{D_i}\big)^{m_i-1} \frac{1}{D_i} s_i^{k_i} \Big\} \boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i - \boldsymbol{X}_i), \qquad i = 1, 2. \tag{2.50}$$

$\|\boldsymbol{A}\| := \sqrt{\boldsymbol{A} : \boldsymbol{A}}$ is the Euclidian norm of a second-order tensor $\boldsymbol{A}$, $\boldsymbol{\sigma}_i$ are the partial stresses defined by (2.23). $\boldsymbol{H}_i$ are fourth-order tensor functions. $m_i > 0$ and k_i are material functions possibly depending on temperature θ, s_i are the inelastic accumulations defined by (2.25). D_i are scalar drag stresses (cf. Chaboche (2008)). In simple cases, D_i are positive constants. Generally, they have their evolution equations. For instance, one can assume

$$\dot{D}_i = d_i^{(1)} - \sum_{j=1}^{2} d_{ij}^{(2)} D_j, \qquad i = 1, 2. \tag{2.51}$$

$d_i^{(1)}, d_{ij}^{(2)}$ are material parameters. We assume the following evolution equations for the internal variables $\boldsymbol{\alpha}_i$

$$\dot{\boldsymbol{\alpha}}_i = \dot{\boldsymbol{\varepsilon}}_i - \sum_{j=1}^{2} \boldsymbol{B}_{ij}(\theta)\, \boldsymbol{X}_j, \qquad i = 1, 2 \tag{2.52}$$

with given fourth-order tensor functions $\boldsymbol{B}_{ij}$. Note that generally $\boldsymbol{B}_{12} \neq \boldsymbol{B}_{21}$. We put the usual initial conditions for $\boldsymbol{\varepsilon}_i$, D_i and $\boldsymbol{\alpha}_i$:

$$\boldsymbol{\varepsilon}_i(0) = 0, \qquad D_i(0) = 1, \qquad \boldsymbol{\alpha}_i = 0, \quad i = 1, 2. \tag{2.53}$$

Obviously, the drag stresses D_i must be positive for all times. Sufficient conditions for this are:

$$d_i^{(1)} \geq 0, \qquad i = 1, 2, \qquad\qquad \sum_{i,j=1}^{2} d_{ij}^{(2)} a_i a_j \geq 0 \quad \text{for all } a_1, a_2 \in \mathbb{R}. \tag{2.54}$$

The Clausius-Planck inequality (2.24) can be brought into the form

$$(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) \geq 0. \tag{2.55}$$

Taking (2.50) and (2.52) into account, this inequality can be re-written as

$$\sum_{i=1}^{2} \Big\{ \big(\frac{\|\boldsymbol{\sigma}_i - \boldsymbol{X}_i\|}{D_i}\big)^{m_i-1} \frac{1}{D_i} s_i^{k_i} \Big\} \boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i - \boldsymbol{X}_i) : (\boldsymbol{\sigma}_i - \boldsymbol{X}_i) + \sum_{j=1}^{2} \boldsymbol{B}_{ij}(\theta)\, \boldsymbol{X}_j : \boldsymbol{X}_i \geq 0. \tag{2.56}$$

Clearly, for thermodynamical consistency of the model developed above it is sufficient to ensure the inequality (2.56). The following theorem is easy to prove.

Theorem 2.3. The visco-elastic 2M model defined by (2.47), (2.50), (2.51), (2.52) and (2.53) is thermodynamically consistent, if

(i) the fourth-order tensor functions H_i are positive semi-definite in the sense of (2.31) for each fixed admissible temperature,

(ii) the fourth-order tensor functions B_{ij} fulfil (2.38) for each fixed admissible temperature,

(iii) the conditions (2.54) are fulfilled.

The heat-conduction equation (2.21) can be specialized, substituting the inelastic mechanical dissipation by the left-hand side of (2.56) and using the expressions for the back stresses (2.48), (2.49). We conclude this subsection with commentating and extending remarks.

Remarks 2.4.

(i) The material parameters (functions) $\boldsymbol{H}_i$, m_i, k_i, $d_i^{(1)}$, $d_{ij}^{(2)}$, $\boldsymbol{B}_{ij}$ $(i, j = 1, 2)$ occurring in the evolution equations (2.50) – (2.52) may depend on temperature and further quantities like s_i or invariants of the stress tensors $\boldsymbol{\sigma}_i$. This dependence does not change the structure of the inequality (2.56). Thus, Theorem 2.3 remains valid, if the stated conditions are fulfilled for all admissible arguments.

(ii) The tensor functions $\boldsymbol{E}_{ij}$ are assumed to fulfil the condition (2.38) ensuring that the inelastic free energy is a convex (and bounded from below too, see Remark 2.2 (iii), (iv)) function of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ for $\theta = \theta_0$. To ensure the boundedness below for all admissible temperatures, one has to demand

$$\boldsymbol{E}_{11}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2\boldsymbol{E}_{12}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + \boldsymbol{E}_{22}(\theta)\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2 - 2(\theta-\theta_0)\boldsymbol{G}_1(\theta) : \boldsymbol{\alpha}_1 + \\ - 2(\theta-\theta_0)\boldsymbol{G}_2(\theta) : \boldsymbol{\alpha}_2 \geq c > -\infty \quad \forall\,\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \tag{2.57}$$

and for all admissible temperatures $\theta > 0$. Formally, the condition (2.57) does not influence the thermodynamic consistency. Thus, it is not an assumption in theorem 2.3.

(iii) Evolution of inelastic strains $\boldsymbol{\varepsilon}_i$: For $\boldsymbol{H}_i = h_i\boldsymbol{I}$ ($h_i > 0, \boldsymbol{I}$ - unity tensor) (i.e., "$\boldsymbol{H}_i$ being real"), the evolution equations (2.50) are similar to creep behavior. The parameters k_i define the "creep" stages: $k_i < 0$ corresponds to primary creep, $k_i = 0$ and $k_i > 0$ correspond to secondary and tertiary one, respectively (cf. Naumenko and Altenbach (2007), e.g.).

(iv) Evolution of internal variables $\boldsymbol{\alpha}_i$:

- The equations (2.52) generalize an approach due to Robinson in visco-plasticity (for only one index)

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}_{in} - b\,\boldsymbol{X} \tag{2.58}$$

  with a scalar $b > 0$ (cf. Arya and Kaufman (1989) for further references).

- To the authors knowledge, the idea of coupling through back stresses via a matrix (B_{ij}) within the evolution equations (2.52) for the internal variables of strain type has been firstly introduced in Wolff et al. (2011).

- The approach in (2.52) is similar to non-linear kinematic hardening models in plasticity (Armstrong-Frederick approach and extensions by many authors, see Chaboche (2008), Abdel-Karim (2010), e.g.). The difference is that in models concerning plasticity the rate of plastic accumulation occurs in the recovery terms

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}_{in} - b\,\boldsymbol{X}\,\dot{s}_{in}. \tag{2.59}$$

  Of course, the proposal in (2.52) could be changed by

$$\dot{\boldsymbol{\alpha}}_i = \dot{\boldsymbol{\varepsilon}}_i - \sum_{j=1}^{2} \boldsymbol{B}_{ij}\,\boldsymbol{X}_j\,\sqrt{\dot{s}_i\dot{s}_j}, \qquad i = 1, 2 \tag{2.60}$$

  without any influence on thermodynamic consistency. A reason for this changing may be a better approximation of reality. Similar considerations can be made with respect to the evolution equations (2.51) of the drag stresses D_i.

- A further extension in (2.52) is possible, using an approach stemming from Burlet and Cailletaud (1987):

$$\dot{\boldsymbol{\alpha}}_i = \dot{\boldsymbol{\varepsilon}}_i - \sum_{j=1}^{2} \boldsymbol{B}_{ij}\{\eta_j\boldsymbol{X}_j + (1-\eta_j)(\boldsymbol{X}_j : \boldsymbol{n}_j)\boldsymbol{n}_j\} \qquad i = 1, 2 \tag{2.61}$$

  with $\boldsymbol{n}_j := \frac{\boldsymbol{\sigma}_j - \boldsymbol{X}_j}{\|\boldsymbol{\sigma}_j - \boldsymbol{X}_j\|}$ and $0 \leq \eta_j \leq 1$. This approach has been introduced for a better modelling of plastic behavior in case of non-proportional loading. We refer to Taleb et al. (2006), Wolff and Taleb (2008) for discussion and further extensions.

- Another extension of (2.52) consists in

$$\dot{\boldsymbol{\alpha}}_i = \overline{\boldsymbol{A}}_i\dot{\boldsymbol{\varepsilon}}_i - \sum_{j=1}^{2} \boldsymbol{B}_{ij}\boldsymbol{X}_j \qquad i = 1, 2 \tag{2.62}$$

  with $\overline{\boldsymbol{A}}_j$ being scalars or fourth-order tensors. In this case, one needs smallness conditions on the parameters involved in the model in order to ensure thermodynamic consistency (see Wolff et al. (2011)).

(v) Generalized Armstrong-Frederick relations: If the tensors $\boldsymbol{E}_{ij}$ in (2.48) and (2.49) *do not depend* on the temperature θ, the internal variables $\boldsymbol{\alpha}_i$ can be eliminated, using (2.52). This leads to a coupled system of ordinary differential equations for the back stresses $\boldsymbol{X}_i$:

$$\dot{\boldsymbol{X}}_1 = \boldsymbol{E}_{11}\dot{\boldsymbol{\varepsilon}}_1 - \boldsymbol{E}_{11}\big\{\sum_{j=1}^{2}\boldsymbol{B}_{1j}(\theta)\boldsymbol{X}_j\big\} + \boldsymbol{E}_{12}\dot{\boldsymbol{\varepsilon}}_2 - \boldsymbol{E}_{12}\big\{\sum_{j=1}^{2}\boldsymbol{B}_{2j}(\theta)\boldsymbol{X}_j\big\} + \\ - \dot{\theta}\big(\boldsymbol{G}_1(\theta) - (\theta-\theta_0)\frac{\mathrm{d}\boldsymbol{G}_1}{\mathrm{d}\theta}(\theta)\big), \tag{2.63}$$

$$\dot{\boldsymbol{X}}_2 = \boldsymbol{E}_{12}\dot{\boldsymbol{\varepsilon}}_1 - \boldsymbol{E}_{12}\big\{\sum_{j=1}^{2}\boldsymbol{B}_{1j}(\theta)\boldsymbol{X}_j\big\} + \boldsymbol{E}_{22}\dot{\boldsymbol{\varepsilon}}_2 - \boldsymbol{E}_{22}\big\{\sum_{j=1}^{2}\boldsymbol{B}_{2j}(\theta)\boldsymbol{X}_j\big\} + \\ - \dot{\theta}\big(\boldsymbol{G}_2(\theta) - (\theta-\theta_0)\frac{\mathrm{d}\boldsymbol{G}_2}{\mathrm{d}\theta}(\theta)\big). \tag{2.64}$$

In the case of plastic (or visco-plastic) behavior, similar relations are called Armstrong-Frederick relations. The case of temperature-depending $\boldsymbol{E}_{ij}$ is more complicated. It has been dealt with in Wolff et al. (2011) for plastic mechanisms.

2.4 Special Cases and Extensions

Material Symmetries

The model developed in subsection 2.3 covers the full anisotropic case. In many cases, the materials under consideration have some symmetry properties like isotropy or orthotropic symmetry (see Bertram and Olschewski (1993), Haupt (2002), Naumenko and Altenbach (2007) for explanations, e.g.). It is well-known that in case of isotropy the application of a fourth-order tensor $\boldsymbol{E}$ can be described by two scalars in the form

$$\boldsymbol{E}\boldsymbol{\alpha} = e_1\boldsymbol{\alpha} + e_2\mathrm{tr}(\boldsymbol{\alpha})\boldsymbol{I} \qquad \text{for all symmetric second-order tensors } \boldsymbol{\alpha} \tag{2.65}$$

with $e_1, e_2 \in \mathbb{R}$. Based on (2.65), under full isotropy, one gets the isotropic special case of the model developed in subsection 2.3, substituting the fourth-order tensors in (2.47), (2.48), (2.49), (2.50) and (2.52) in accordance with (2.65). Moreover, the second-order tensors $\boldsymbol{G}_i$ in (2.47), (2.48) and (2.49) should be spherical, and the thermo-elastic behavior should be isotropic as in remark 2.1 (ii). Finally, the two mechanisms may have different symmetry properties.

Isochoric Mechanisms

The model developed in subsection 2.3 can be modified to isochoric mechanisms, i.e. to the case

$$\mathrm{tr}(\boldsymbol{\varepsilon}_i) = 0 \qquad \text{for } i \in \{1,2\}. \tag{2.66}$$

To maintain consistency, the evolution equation (2.50) of an isochoric mechanism $\boldsymbol{\varepsilon}_i$ needs a correction:

$$\dot{\boldsymbol{\varepsilon}}_i = \Big\{\big(\frac{\|\boldsymbol{\sigma}_i - \boldsymbol{X}_i\|}{D_i}\big)^{m_i-1}\,\frac{1}{D_i}\,s_i^{k_i}\Big\}\,\big(\boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*)\big)^*. \tag{2.67}$$

In many cases, the effective partial stress $\boldsymbol{\sigma}_i - \boldsymbol{X}_i$ in the first part of (2.67) is also substituted by its deviator. Taking the relation

$$\big(\boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*)\big)^* : (\boldsymbol{\sigma}_i - \boldsymbol{X}_i) = \big(\boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*)\big)^* : (\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*) = \boldsymbol{H}_i(\theta)(\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*) : (\boldsymbol{\sigma}_i^* - \boldsymbol{X}_i^*) \tag{2.68}$$

into account, the theorem 2.3 about thermodynamic consistency remains valid. If *both* mechanisms are isochoric, it may be reasonable to let the internal variables $\boldsymbol{\alpha}_i$, back stresses $\boldsymbol{X}_i$ and thermo-stress tensors $\boldsymbol{G}_i$ be traceless too (see Wolff et al. (2010) for discussion).

Consideration of further Models in Visco-Elasticity

In a similar way as above, some other rheological models in visco-elasticity can be regarded as two- or multi-mechanism models, possibly after bringing them into a mechanically equivalent form which is suitable for the multi-mechanism approach. For instance, a Burgers body is defined as two Maxwell bodies in parallel and is equivalent to a Maxwell and a KV body in series (see Figure 3). The connection in series of a Burgers body and a Kelvin-Voigt body is mechanically equivalent to a Hooke element, a Newton element and two Kelvin-Voigt bodies all connected in series (see figure 4). Therefore, the whole model can be regarded as a three-mechanism model involving one viscous and two visco-elastic mechanisms. The single Hooke element represents the thermo-elastic strain ε_{te} and is usually not regarded as an own mechanism. Furthermore, a coupling between the two KV elements as above is possible. Additionally, the viscosity of the Newton element may depend on the inelastic accumulation associated with one KV body, e.g.

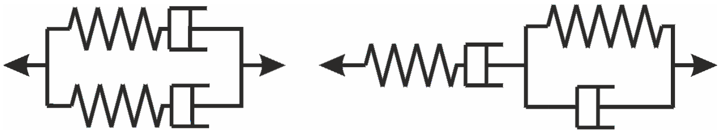

Figure 3: Burgers body in two mechanically equivalent forms.

Difficulties arise when trying to regard rheological models with complex models in parallel as multi-mechanism models. The combination of the multi-mechanism approach and the theory of visco-elasticity based on rate-dependent functionals (cf. Haupt (2002) for explanation and further references, e.g.) is a challenge for future work.

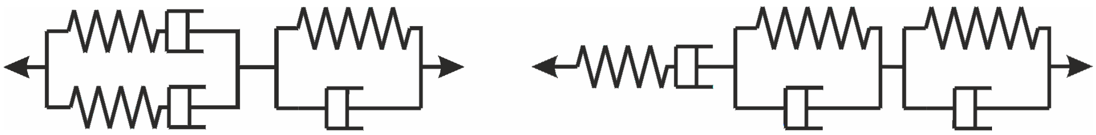

Figure 4: Burgers body and Kelvin-Voigt body in series (left). This model is mechanically equivalent to a serial connection of Hooke element, Newton element and two Kelvin-Voigt bodies (right).

Coupling with Visco-Plastic Mechanisms

In this study, visco-elastic mechanisms are in the focus. However, it is possible to deal with 2M models having a visco-elastic and a plastic (or visco-plastic) mechanism. The plastic mechanism requires a flow function (yield function) involving a corresponding norm (von Mises, e.g.) of the deviator of the effective partial stress. We refer, e.g. to Saï (2011), Wolff et al. (2011) for details Moreover, three- and multi-mechanism models are possible with several visco-elastic and plastic mechanisms.

3 Numerical Simulations for a Rod with Special Viscoelastic Behavior

As a numerical example we present simulations for the longitudinal movement of a visco-elastic rod behaving like two coupled Kelvin-Voigt bodies as in Subsection 2.2. Our aim is to show that simple two-mechanism models with linear material behavior can produce a ratcheting effect. Thus, we consider the (isothermal) spatially one-dimensional case. Moreover, we deal with the scalar case: The fourth-order tensors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{H}_1 := \boldsymbol{V}_1^{-1}$ and $\boldsymbol{H}_2 := \boldsymbol{V}_2^{-1}$ (cf. (2.40)) are equal to scalars times the fourth-order unity tensor. In other words, we assume the following non-dimensional (scalar) parameters:

$$E_1 = 1, \quad E_2 = 2, \quad H_1 = 10, \quad H_2 = 1, \quad \varrho = 60. \tag{3.1}$$

The coupling parameter E_{12} will vary in accordance with

$$-\sqrt{E_1 E_2} = -\sqrt{2} \leq E_{12} \leq \sqrt{E_1 E_2} = \sqrt{2}. \tag{3.2}$$

In Kröger et al. (2012), the $1d$ model has been described in detail and the arising mathematical problem has been solved, using the theory of Sobolev spaces and weak solutions.
Here, for simplicity we let the parameters be constant. The simulations can be performed for varying parameters

without difficulties. The length of the rod is 1. The left end is fixed, the right end undergoes a periodic stress with non-zero mean value:

$$\sigma(t) = 0.2\sin(t - \arcsin(0.25)) + 0.05. \tag{3.3}$$

The simulations have been performed with the package COMSOL®. In Figure 5, the stress-strain curves at the right rod's end are given. While in the regular case of coupling (i.e., $|E_{12}| < \sqrt{E_1 E_2} = \sqrt{2}$) a shakedown occurs, the singular case yields a clear ratcheting effect with unbounded strain. As reported in Cailletaud and Saï (1995), a similar behavior can be observed for a plastic two-mechanism model.

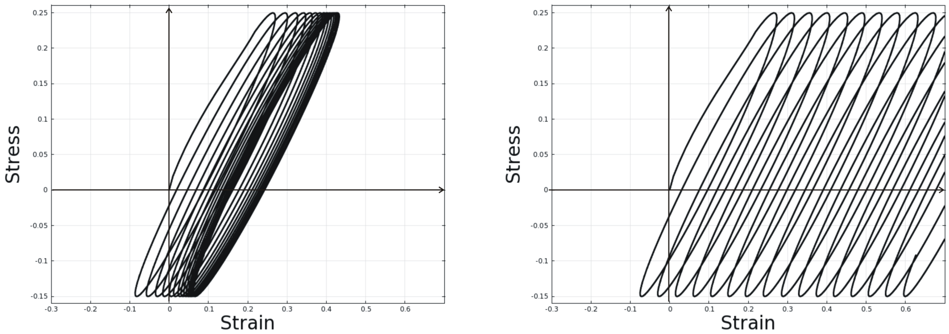

Figure 5: Stress-strain curves at the right end of the rod behaving like two coupled KV bodies: Regular case with $E_{12} = 0.99\sqrt{2}$ leading to a shakedown (left), singular case with $E_{12} = \sqrt{2}$ with ratcheting (right).

Figure 6 shows the strain evolution in all points under the given stress (3.3) at the right end. In the regular case, in each point, the strain evolves periodically. Moreover, due to the viscous effects the strain distribution over the rod is *not* homogeneous at the same time, even for constant material parameters (see (3.1)). In the singular case, ratcheting effects can be observed in all points.

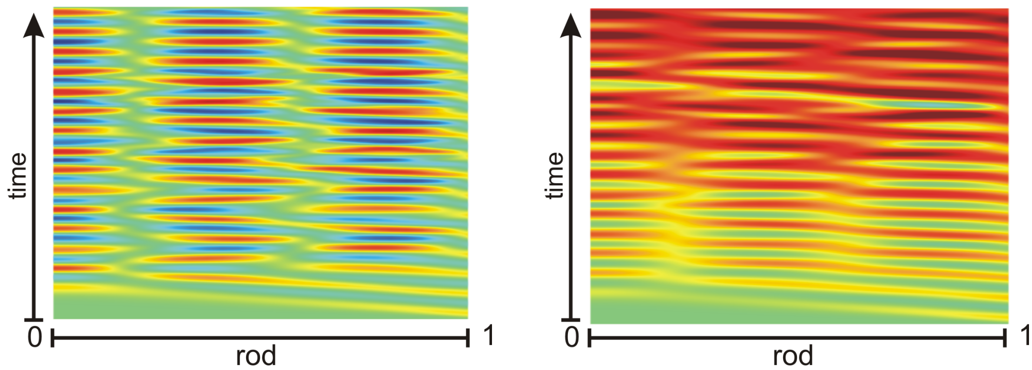

Figure 6: Strain evolution in all rod's points: Regular case with $E_{12} = 0.5\sqrt{2}$, singular case with $E_{12} = \sqrt{2}$ (right).

Remark 3.1. Regarding the Kelvin-Voigt body as a connection in parallel of a spring and a damper, one gets simple mathematical models consisting of ordinary differential equations (ODE). In this case, due to (2.39) and (2.40) the coupling of two KV bodies leads to the following system of ODE with constant coefficients

$$\dot{\varepsilon}_1(t) = H_1(\sigma(t) - E_{11}\varepsilon_1 - E_{12}\varepsilon_2), \tag{3.4}$$
$$\dot{\varepsilon}_2(t) = H_2(\sigma(t) - E_{12}\varepsilon_1 - E_{22}\varepsilon_2). \tag{3.5}$$

In the regular case $|E_{12}| < \sqrt{E_1 E_2}$, the corresponding system matrix has two negative eigenvalues λ_1 and λ_2. Solving the system (3.4), (3.5) for initial values $\varepsilon_i(0) = 0$ $(i = 1, 2)$ and for given stress σ, one gets

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) = \eta_1 e^{\lambda_1 t} \int_0^t \sigma(\tau) e^{-\lambda_1 \tau}\, d\tau + \eta_2 e^{\lambda_2 t} \int_0^t \sigma(\tau) e^{-\lambda_2 \tau}\, d\tau. \tag{3.6}$$

(η_i are constants.) Therefore, bounded stress yields bounded strain. Contrary, in the singular case $|E_{12}| = \sqrt{E_1 E_2}$, one eigenvalue is zero, and the strain evolves in accordance with

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) = \eta_1 \int_0^t \sigma(\tau)\, d\tau + \eta_2 e^{\lambda t} \int_0^t \sigma(\tau) e^{-\lambda \tau}\, d\tau \tag{3.7}$$

($\lambda < 0$ is the second eigenvalue.) Now, a bounded stress may lead to an unbounded strain, for instance for σ given by (3.3).

4 Conclusion

In this study, we have applied the two-mechanism approach to thermo-visco-elasticity with internal variables. Some aspects of two- and multi-mechanism models and references are given in Section 1. Up to now, most applications concern metal plasticity and ratcheting. In Section 2, after providing some thermodynamical basics, a general visco-elastic two-mechanism model has been developed generalizing the material behavior of two Kelvin-Voigt bodies connected in series. In difference to rheological models, the coupling between the internal variables within the free energy gives more possibilities for modeling of complex material behavior.
In Section 3, we have presented simulations for a visco-elastic rod which behaves like two coupled Kelvin-Voigt bodies. In the singular case, an applied periodic stress with non-zero mean value lead to a ratcheting effect similar as in metal plasticity. Some polymers show a material behavior similar to ratcheting (cf. Tao and Xia (2007), Shen et al. (2004)).
There remain further investigations, $3d$ simulations, and comparison with experimental data for future work.

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