TECHNISCHE MECHANIK, 32, 6, (2012), 684 – 697
submitted: December 09, 2011

Multi-objective Motion Synthesis of a Rolling Sphere in a Rectilinear Chute-conveyor with Transformed Dry Friction

B. Cheshankov, V. Bozdouganova, I. Ivanov, V. Vitliemov

A mathematical model of the dynamics of a rolling rigid sphere in a rectilinear chute-conveyor with transformed dry friction is developed. It includes two types of rolling: with slipping and without slipping. A multi-objective optimization problem of a passively controlled motion is formulated for the model and it has been solved. Salukvadze optimal solution and ranked Pareto optimal solutions are determined.

1 Introduction

The constrained motion modeling of a rigid homogenous sphere is a classical task of mechanics (Routh, 1905; Grigoryan and Fradlin, 1982). Initially, the interest in this problem was motivated by the billiard game and later – by the bowling game. The dynamics of a spherical body in its motion on a fixed and rotating surface is a basic modeling problem of the nonholonomic mechanics (Chaplygin, 1949; Neimark and Fufaev, 1972). This problem continues to be an objective of mainly theoretical research, for example in (Veselov and Veselova, 1989; Kilin, 2001, Borisov et al., 2002; Rozenblat, 2006; Johnson, 2007).

The passively controlled sphere motion is studied in Narayanan (2008) with application in the gravitational transportation of spheroidal fruits. It is known that the rolling with slipping ensures a greater speed than the rolling without slipping which leads to a greater productivity in transportation of spherical bodies at a given distance. In some cases, for example, when dug out tubers are being transported, the rolling without slipping is preferable because in this case their skin is more difficult to be damaged. Such a motion is also advisable when the surface of the bodies in transportation should be irradiated or disinfected.

The efficiency of the gravitational transportation depends on the design parameters of the chute-conveyor and on the energy consumption for its realization. In a lot of technological processes, the gravitational transportation is combined with vibration of the conveyor; with a translational or rotational motion of the contact constraints (Gudushauri and Panovko, 1988). In all these cases, an effect of dry friction linearization appears (pseudo-transformation to viscous friction) without real changes in its physical nature. This allows us to control the transformed friction and to synthesize motions with desirable parameters.

The rigid body gravitational sliding in a rectilinear grove with a moving plane-support is studied in (Andronov, 1988; Andronov and Zhuravlev, 2010) as a basic modeling problem of a mechanical system with transformed dry friction. A multi-criterion optimization problem has been solved in Vitliemov et al. (2006) using the developed model.

A mathematical model of the gravitational transportation of a homogeneous rigid sphere in a rectilinear grove with transformed dry friction is developed with a lot of simplifications. An optimization problem of multi-objective parametric synthesis of passively controlled motion of the sphere using the developed model has been solved in this study.

2 Mathematical Model

The nonimpact motion of a homogeneous sphere with radius R in transportation at a given horizontal distance L is under consideration. The sphere is a rigid body of mass m, moving on an inclined at angle α to the horizontal level plane, which moves translationally with constant velocity $\mathbf{V_P}$ (Figure 1). The sphere motion with respect to the absolute coordinate system *OXYZ* is constrained by fixed planes – the guide-walls of the chute, which are orthogonal to the moving plane and to the vector of its velocity $\mathbf{V_P}$. The surface of the guide-walls is assumed frictionless. There is a play between the constraints and the sphere. The sphere motion is loosely constrained by the guide-walls and the sphere could be in a contact with only one of them at the same time.

Let us assume that only four forces are exerted on the sphere: the gravity force, **G;** the normal reaction of the moving plane, $\mathbf{N_A}$; the friction force, $\mathbf{T_A}$, having components $\mathbf{T_{AX}}$, $\mathbf{T_{AY}}$; and the normal reaction of the active frictionless constraint, $\mathbf{N_B}$, (Figure 1).

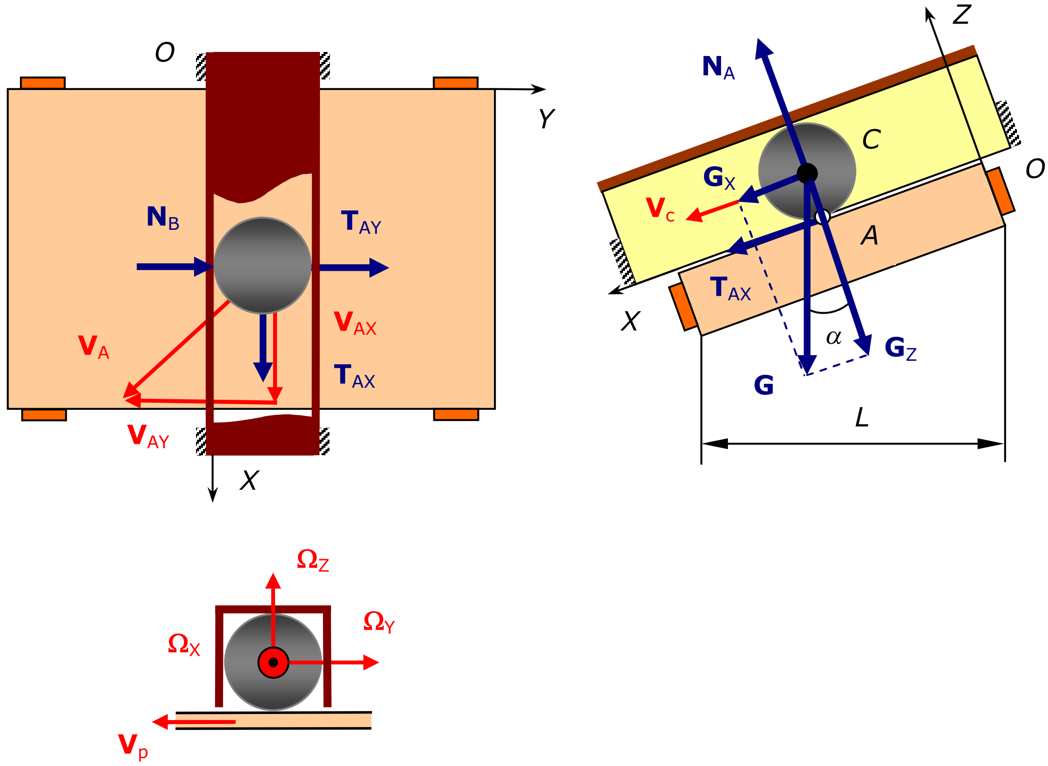


Figure 1. A sphere in a gravitational motion into a rectilinear chute with transformed dry friction

The friction force is modeled by Coulomb's law

$$\mathbf{T_A} = -\,\mu|\mathbf{N_A}|\mathbf{V_A}\,/\,|\mathbf{V_A}|\,,\quad |\mathbf{T_A}| = (T_{AX}{}^2 + T_{AY}{}^2\,)^{1/2} \tag{2.1}$$

if the velocity $\mathbf{V_A}$ of sphere point A has magnitude $|\mathbf{V_A}| = (V_{AX}{}^2 + V_{AY}{}^2)^{1/2} \neq 0$ and

$$|\mathbf{T_A}{}^{\circ}| \leq \mu|\mathbf{N_A}|, \tag{2.2}$$

if $|\mathbf{V_A}| = 0$. In these relationships, μ is the friction coefficient, which has equal values in rolling with and without slipping.

The conditions for permanent contact between the sphere and the constraints are the inequalities imposed on the algebraic values of the normal reactions: $N_A \geq 0$, and $N_B \geq 0$. It is supposed that the initial conditions do not cause impact interaction between the sphere and the contact surfaces, as well as the motion is without bouncing.

Let us assume that in the time interval $T \in [0, T_1]$, $T_1 > 0$, the sphere slides the moving plane and one of the guide-walls. Using the equation (2.1), the equation of sphere's mass center, C, motion, and the equations for sphere's angular momentum rate of change about its center (Kraige and Meriam, 2002), the following equations are obtained

$$\begin{gathered} m\mathrm{d}V_C/\mathrm{d}T = T_{AX} + mg\sin\alpha, \\ (I/R)\mathrm{d}\Omega_X/\mathrm{d}T = T_{AY}\,,\quad (I/R)\,\mathrm{d}\Omega_Y/\mathrm{d}T = -\,T_{AX}, \\ (I/R)\,\mathrm{d}\Omega_Z/\mathrm{d}T = 0,\quad N_A - mg\cos\alpha = 0,\quad N_B + T_{AY} = 0, \end{gathered} \tag{2.3}$$

where

$$|\mathbf{T_A}| = \mu|\mathbf{N_A}| = (T_{AX}{}^2 + T_{AY}{}^2)^{1/2},\quad |\mathbf{T_{AX}}| = |\mathbf{T_{AY}}|\,|V_{AX}/V_{AY}|,$$

$$I = (2/5)mR^2, \quad T_{AX} = -|\mathbf{T_{AX}}|\sigma_{AX}, \quad T_{AY} = -|\mathbf{T_{AY}}|\sigma_{AY}, \tag{2.4}$$

$$V_{AX} = V_C - R\Omega_Y, \quad V_{AY} = R\Omega_X - V_P,$$

$$\sigma_{AX} \equiv \mathrm{sgn}(V_{AX}), \quad \sigma_{AY} \equiv \mathrm{sgn}(V_{AY}).$$

From the equations (2.3)-(2.4), the following equations are derived

$$|\mathbf{T_{AY}}| = \mu mg\cos\alpha /[1 + (V_C - R\Omega_Y)^2/(R\Omega_X - V_P)^2]^{1/2},$$

$$|\mathbf{T_{AX}}| = |\mathbf{T_{AY}}|\, |(V_C - R\Omega_Y)/(R\Omega_X - V_P)|, \tag{2.5}$$

$$|\mathbf{N_A}| = mg\cos\alpha, \quad |\mathbf{N_B}| = |\mathbf{T_{AY}}|,$$

$$|\mathbf{V_A}| = (V_{AX}{}^2 + V_{AY}{}^2)^{1/2} \neq 0.$$

Taking into account the relationships (2.3)-(2.5), the following model is obtained, which describes the state of the sphere

$$m\mathrm{d}V_C/\mathrm{d}T = -|\mathbf{T_{AX}}|\sigma_{AX} + mg\sin\alpha,$$

$$(I/R)\mathrm{d}\Omega_X/\mathrm{d}T = -|\mathbf{T_{AY}}|\sigma_{AY}, \tag{2.6}$$

$$(I/R)\,\mathrm{d}\Omega_Y/\mathrm{d}T = |\mathbf{T_{AX}}|\sigma_{AX},$$

$$(I/R)\,\mathrm{d}\Omega_Z/\mathrm{d}T = 0,$$

where

$$V_C(0) = V_{C0}, \quad V_{C0} \neq R\Omega_{Y0},$$

$$\Omega_X(0) = \Omega_{X0}, \quad \Omega_{X0} \neq V_P/R, \quad \Omega_Y(0) = \Omega_{Y0}, \quad \Omega_Z(0) = \Omega_{Z0}. \tag{2.7}$$

Let us assume that in the time interval $T \in (T_1, T_2]$, where T_2 is the smallest root of the equation $X(T_2)/L - (1 + \tan^2\alpha)^{1/2} = 0$, the sphere is rolling without slipping on the moving plane. In this case, the following equalities could be written

$$|\mathbf{V_A}| = 0, \quad V_{AX} = V_C - R\Omega_Y = 0, \quad V_{AY} = R\Omega_X - V_P = 0,$$

$$\mathrm{d}V_C/\mathrm{d}T = R\mathrm{d}\Omega_Y/\mathrm{d}T, \quad \Omega_X = V_P/R, \quad \mathrm{d}\Omega_X/\mathrm{d}T = 0. \tag{2.8}$$

Taking into account (2.2), the following system of equations is obtained

$$m\mathrm{d}V_C/\mathrm{d}T = T_{AX}{}^\circ + mg\sin\alpha,$$

$$(I/R)\mathrm{d}\Omega_X/\mathrm{d}T = T_{AY}{}^\circ, \quad (I/R)\,\mathrm{d}\Omega_Y/\mathrm{d}T = -T_{AX}{}^\circ, \quad (I/R)\,\mathrm{d}\Omega_Z/\mathrm{d}T = 0, \tag{2.9}$$

$$N_A - mg\cos\alpha = 0, \quad N_B + T_{AY}{}^\circ = 0,$$

where

$$|\mathbf{T_A}{}^\circ| = (T_{AX}{}^{\circ 2} + T_{AY}{}^{\circ 2})^{1/2} \leq \mu\, |\mathbf{N_A}|. \tag{2.10}$$

Plugging the equations (2.8) in (2.9) we could derive

$$T_{AX}{}^\circ = -(I/R^2)mg\sin\alpha/(m + I/R^2), \quad T_{AY}{}^\circ = 0, \quad N_B = 0, \quad N_A = mg\cos\alpha. \tag{2.11}$$

Substituting $I = (2/5)mR^2$ into equations (2.11) and then in the inequality (2.10) we arrive at the inequality

$$\tan\alpha \leq 3.5\mu\,. \tag{2.12}$$

The differential equations, which model the sphere's rolling without slipping in the time interval $T \in (T_1, T_2]$, take the form

$$
\begin{gathered}
\mathrm{d}V_C/\mathrm{d}T = mg\sin\alpha/(m + I/R^2),\\
\mathrm{d}\Omega_X/\mathrm{d}T = 0,\\
\mathrm{d}\Omega_Y/\mathrm{d}T = mg\sin\alpha/[R(m + I/R^2)],\\
\mathrm{d}\Omega_Z/\mathrm{d}T = 0,
\end{gathered}
\tag{2.13}
$$

where

$$
\begin{gathered}
V_C(T_1) = V_C(T_1),\quad \Omega_X(T_1) = V_P/R,\\
\Omega_Y(T_1) = V_C(T_1)/R,\quad \Omega_Z(T_1) = \Omega_Z(T_1).
\end{gathered}
\tag{2.14}
$$

The equations (2.5)-(2.7), (2.11)-(2.14) describe a compound dynamical model, which includes both cases of rolling with and without slipping. It contains the following parameters: R, μ, α, V_P, V_{C0}, Ω_{X0}, Ω_{Y0}, Ω_{Z0}.
In order to simplify and generalize the model, we introduce dimensionless variables and parameters

$$
\begin{gathered}
r = R/L,\quad x = X/L,\quad x = \eta(1 + \tan^2\alpha)^{1/2},\quad \omega = \Omega(L/g)^{1/2},\\
\omega_X = \Omega_X(L/g)^{1/2},\quad \omega_Y = \Omega_Y(L/g)^{1/2},\quad \omega_Z = \Omega_Z(L/g)^{1/2},\\
i = I/mL^2 = (2/5)r^2,\quad t = T/(L/g)^{1/2},\quad c = V_C(0)/(Lg)^{1/2},\\
v_C = V_C/(Lg)^{1/2},\quad v_{Cf} = V_C(T_2)/(Lg)^{1/2},\quad w = V_C^2/Lg,\\
v_P = V_P/(Lg)^{1/2},\quad v_A = V_A/(Lg)^{1/2},\\
v_{AX} = V_{AX}/(Lg)^{1/2},\quad v_{AY} = V_{AY}/(Lg)^{1/2},\\
\mathbf{n_A} = \mathbf{N_A}/mg,\quad \mathbf{n_B} = \mathbf{N_B}/mg,\\
\mathbf{f_A} = \mathbf{T_A}/mg,\quad \mathbf{f_{AX}} = \mathbf{T_{AX}}/mg,\quad \mathbf{f_{AY}} = \mathbf{T_{AY}}/mg,
\end{gathered}
\tag{2.15}
$$

and we denote: $u_1 \equiv \tan\alpha$, $u_2 \equiv v_P$, $S_{AX} \equiv \mathrm{sgn}(v_{AX})$, $S_{AY} \equiv \mathrm{sgn}(v_{AY})$.
Applying the relationships (2.15) the dynamical model (2.5)-(2.7), (2.11)-(2.14) is converted into dimensionless form.
If the following conditions are satisfied

$$
\eta \in [0,\eta_1],\quad 0 \le \eta_1 \le 1,\quad |v_{AX}| = |w^{1/2} - r\omega_Y| > 0,\quad |v_{AY}| = |r\omega_X - v_P| > 0,
\tag{2.16}
$$

then the sphere moves by rolling with slipping and its motion is modeled by the following equations

$$
\begin{gathered}
\mathrm{d}t/\mathrm{d}\eta = [(1 + u_1^2)/w]^{1/2},\quad t(0) = 0,\\
\mathrm{d}w/\mathrm{d}\eta = 2(1 + u_1^2)^{1/2}(-|\mathbf{f_{AX}}|S_{AX} + \sin\alpha),\quad w(0) = c^2,\\
\mathrm{d}\omega_X/\mathrm{d}\eta = -|\mathbf{f_{AY}}|S_{AY}(1 + u_1^2)^{1/2}/(iw^{1/2}/r),\quad \omega_X(0) = \omega_{X0},\\
\mathrm{d}\omega_Y/\mathrm{d}\eta = |\mathbf{f_{AX}}|S_{AX}(1 + u_1^2)^{1/2}/(iw^{1/2}/r),\quad \omega_Y(0) = \omega_{Y0},\\
\mathrm{d}\omega_Z/\mathrm{d}\eta = 0,\quad \omega_Z(0) = \omega_{Z0},
\end{gathered}
\tag{2.17}
$$

where

$$
\begin{gathered}
|\mathbf{f_{AY}}| = \mu\cos\alpha/[1 + (v_{AX}/v_{AY})^2]^{1/2},\quad |\mathbf{f_{AX}}| = |\mathbf{f_{AY}}||v_{AX}/v_{AY}|,\\
|\mathbf{n_A}| = \cos\alpha,\quad |\mathbf{n_B}| = |\mathbf{f_{AY}}|.
\end{gathered}
\tag{2.18}
$$

If the conditions

$$
\eta \in (\eta_1,1],\quad |v_{AX}| = 0,\quad |v_{AY}| = 0,\quad \tan\alpha \le 3.5\mu,
\tag{2.19}
$$

are satisfied then the sphere moves by rolling without slipping and its motion is modeled by the following equations

$$\begin{gathered}
\mathrm{d}t/\mathrm{d}\eta = [(1 + u_1^2)/w]^{1/2}, \quad t(\eta_1) = t_1, \\
\mathrm{d}w/\mathrm{d}\eta = 2(1 + u_1^2)^{1/2}\sin\alpha/(1 + i/r^2), \quad w(\eta_1) = w_1, \\
\mathrm{d}\omega_X/\mathrm{d}\eta = 0, \quad \omega_X(\eta_1) = v_P/r, \\
\mathrm{d}\omega_Y/\mathrm{d}\eta = [(1 + u_1^2)/w]^{1/2}\sin\alpha/[rw^{1/2}(1+i/r^2)], \\
\omega_Y(\eta_1) = w_1^{1/2}/r, \\
\mathrm{d}\omega_Z/\mathrm{d}\eta = 0, \quad \omega_Z(\eta_1) = \omega_{Z1},
\end{gathered} \tag{2.20}$$

where

$$\begin{gathered}
f_{AX}{}^\circ = -(i/r^2)\sin\alpha/(1 + i/r^2), \quad f_{AY}{}^\circ = 0, \\
|\mathbf{n_A}| = \cos\alpha, \quad |\mathbf{n_B}| = 0.
\end{gathered} \tag{2.21}$$

We denote the mathematical model described by (2.16)-(2.21) with the abbreviation MM for the sake of brevity.

2 Criteria of Mechanical Performance

A basic performance criterion of the considered gravitational chute-conveyor is the dimensionless time for the transportation of a sphere at a given horizontal distance

$$f_1 = t(1), \tag{3.1}$$

which also characterizes its productivity and its perfection.
The energy losses of the mechanical friction are important characteristic of the system. Using the balance of the energy and the work of the sphere forces, we determine a performance criterion for the energy dissipation

$$f_2 = |\, \tfrac{1}{2}\{w(1) - w(0) + i[\omega(1)^2 - \omega(0)^2]\} - u_1 \,|, \tag{3.2}$$

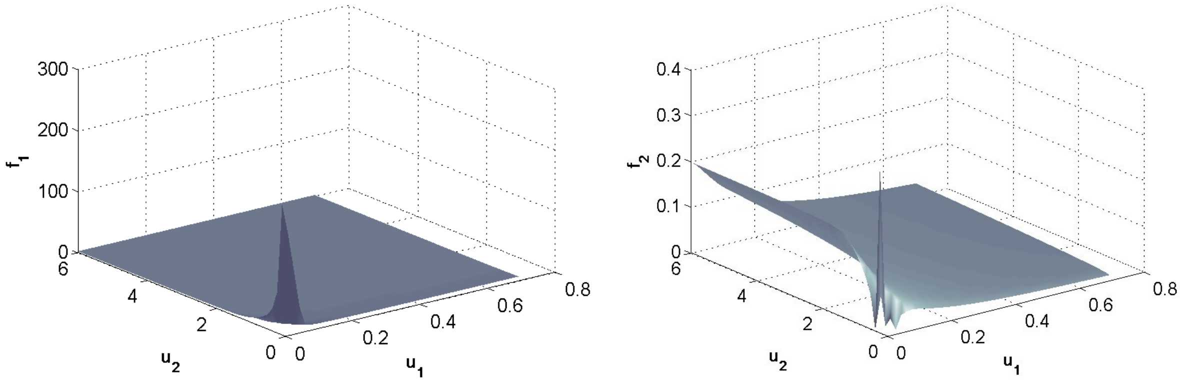
where $\omega = (\omega_X^2 + \omega_Y^2 + \omega_Z^2)^{1/2}$.
In a geometrical aspect the gravitational chute-conveyor is evaluated by the maximal dimensionless height on which the sphere's mass center, C, drops

$$f_3 = u_1 \equiv \tan\alpha. \tag{3.3}$$

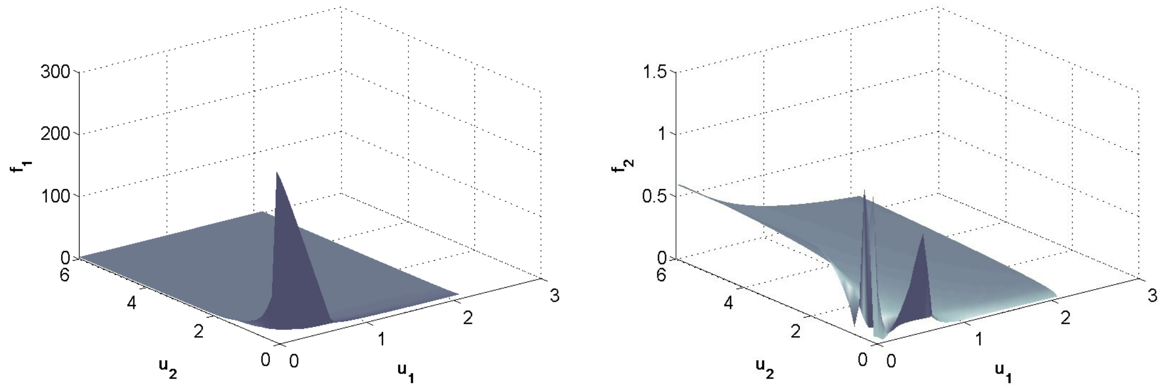
In the kinematical and energy-saving aspects, the transporting device is characterized by the magnitude of the dimensionless velocity of the moving plane

$$f_4 = u_2 \equiv v_P. \tag{3.4}$$

The performance criterion vector components $\mathbf{f}(\mathbf{u}) = [f_1, f_2, f_3, f_4]$, where $\mathbf{u} = [u_1, u_2]$, have to be subjected to Pareto minimization.



(a)



(b)

Figure 2. Objective functions $f_1(\mathbf{u})$ and $f_2(\mathbf{u})$: (a) $\mu = 0.2$; (b) $\mu = 0.6$

The objective functions $f_1(\mathbf{u})$ and $f_2(\mathbf{u})$, $\mathbf{u}\in\mathbf{\Pi} \equiv \{\mathbf{u} \in\mathbf{E}^2\!: \mathbf{u}^- \le \mathbf{u} \le \mathbf{u}^+\}$, when $\mathbf{u}^- = [0, 0.1]$, $\mathbf{u}^+ = [3.5\mu, 6]$, $c = 0.3193$, $r = 0.1$ for $\mu = 0.2$ and $\mu = 0.6$ are presented in Figures 2(a) and 2(b). The pseudo-criteria $f_3(\mathbf{u})$ and $f_4(\mathbf{u})$ do not depend on the change of the parameter μ.

The obtained results in a graphical form show weak parametric sensitivity of the performance criterion $f_1(\mathbf{u})$ in the almost entire domain $\mathbf{\Pi}$ with the exception of a small neighborhood of the boundary point $\mathbf{u}^-$. In this neighborhood, the objective function $f_2(\mathbf{u})$ changes erratically.

4 Optimization Problem

In the case of a given parametric vector $\mathbf{p} = [c, \omega_{X0}, \omega_{Y0}, \omega_{Z0}, \mu, r]$ and an admissible design variable vector $\mathbf{u}\in\mathbf{\Pi}$, we assume that the differential equations of the mathematical model MM describe the problem with desirable accuracy. This is a reason to formulate the optimization problem as a standard multi-objective optimization problem

$$\mathrm{Pmin}_{\,\mathbf{u}\in\mathbf{\Pi}}\ \mathbf{f}(\mathbf{u},\mathbf{p}), \quad \mathbf{\Pi} \equiv \{\mathbf{u} \in\mathbf{E}^2\!: \mathbf{u}^- \le \mathbf{u} \le \mathbf{u}^+\}, \tag{4.1}$$

where “Pmin” is the operator for determination of global Pareto minimal values of the performance criterion vector $\mathbf{f}$.

The solution technique for the problem (4.1) is based on the Pareto optimality principle (Miettinen, 1999). The solution consists of two sets $\mathbf{D}^* \subseteq \mathbf{\Pi}$ and $\mathbf{P}^* \subseteq \mathbf{P}$ from Pareto optimal points: $\mathbf{u}^*\in\mathbf{D}^* \equiv \{\mathbf{u}^*\!: \mathbf{u}^* = \arg\mathrm{Pmin}_{\,\mathbf{u}\in\mathbf{\Pi}}\ \mathbf{f}(\mathbf{u},\mathbf{p})\}$; $\mathbf{f}^*\in \mathbf{P}^* \equiv \{\mathbf{f}^*\!: \mathbf{f}^* = \mathbf{f}(\mathbf{u}^*,\mathbf{p})\}$. The ultimate choice of only one solution from all compromised solutions $\mathbf{D}^*$ and $\mathbf{P}^*$ can be essentially facilitated if they are contracted to several preferentially ranked subsets.

5 Numerical Procedure

A two-stage procedure, described in Cheshankov et al. (2004), is utilized for the solving of the problem (4.1). In the first stage, the sets **D*** and **P*** are constructed by the Parameter Space Investigation (PSI) method for investigation of multi-dimensional parametric domains with uniformly distributed sequences of Sobolev's test points (Statnikov and Matusov, 2002; Sobol' and Statnikov, 2006). This stage contains the following generalized steps:

- Generation of a given number Q Sobolev's test points in the domain $\mathbf{\Pi} \equiv \{\mathbf{u}: \mathbf{u}^{-} \leq \mathbf{u} \leq \mathbf{u}^{+}\}$.
- Determination of discrete set **P**.
- Selection of Pareto optimal sets $\mathbf{D}^* \subseteq \mathbf{\Pi}$ and $\mathbf{P}^* \subseteq \mathbf{P}$.

In the second stage, the subsets $\mathbf{M}_{Re} \subset \mathbf{P}^*$ are determined and sorted by their rank of efficiency $Re \in \{6, 5, \ldots, 1\}$ using three geometrical criteria $\mu_k°$, $k = 1, 2, 3$ evaluating the distance: between each compromise point $\mathbf{f}^*$ and the positive utopian point ($\mathbf{f}^U \equiv [f_j^U], f_j^U = \min_{\mathbf{u}\in\mathbf{D}^*} f_j(\mathbf{u},\mathbf{p}), j = 1, 2, \ldots, 4$); between each compromise point $\mathbf{f}^*$ and the hyper-line UN, connecting the positive U and the negative N ($\mathbf{f}^N \equiv [f_j^N]$, $f_j^N = \max_{\mathbf{u}\in\mathbf{D}^*} f_j(\mathbf{u},\mathbf{p})$, $j = 1, 2, \ldots, 4$) utopian points; and between the projection of $\mathbf{f}^*$ onto hyper-line UN and the utopian point U.

The set $\mathbf{M}_{Re=6}$ with the highest rank of efficiency is the Salukvadze optimal solution ($\mathbf{u}^S$, $\mathbf{f}^S$), (Salukvadze, 1979) if it consists of only one point. It reveals the potential of the transportation device to be improved if all the partial criteria f_j are uniformly approaching their perfect utopian values f_j^U. The ultimate compromise decision is taken interactively through an analysis of the sorted by their rank of efficiency Pareto subsets $\mathbf{M}_{Re}$.

6 Results

The problem (4.1) has been solved using the program PSIMS (Cheshankov et al., 2004) with the following data: $r = 0.1$; $c = 0.3193$; $\mu = 0.2, 0.4, 0.6$; $\omega_{X0} \equiv \omega_{Y0} \equiv \omega_{Z0} \approx 0$; $\mathbf{u}^{-} = [0, 0.1]$; $\mathbf{u}^{+} = [3.5\mu, 6]$.

The domain $\mathbf{\Pi}$ is sounded by means of the PSI-method with $Q = 4096$ Sobolev's test points. The results for $\mu = 0.4$ are illustrated in Figures from 3 to 8. The utopian points in the $\boldsymbol{\mu}°$-space of the geometrical criteria $\mu_k°$, $k = 1, 2, 3$ are designated with the symbol " ● ". The point with the highest rank of efficiency $Re = 6$ is designated with the symbol " ■ ", and the lower rank of efficiency points $Re = 5, 4, \ldots, 1$ – with symbols " ●, ▲, ▶, ◀, ♦ " respectively. The unselected Pareto optimal solutions in Figure 3 are designated by the symbol " + ", and in Figure 4 – with " × ".

The obtained Salukvadze optimal solutions ($\mathbf{u}^S$, $\mathbf{f}^S$) are given in Table 6.1. Since the optimal point $\mathbf{u}^S$ is close to the boundary point $\mathbf{u}^{-}$ and the robustness is not guaranteed it is worthy to analyze the next in the rank of efficiency Pareto optimal subset $\mathbf{M}_{Re=5}$, too.

Table 6.1. Salukvadze optimal solutions

μ	f_1^S	f_2^S	f_3^S	f_4^S
0.2	1.9829	0.0532	0.3074	0.1130
0.4	2.0454	0.0476	0.2635	0.1072
0.6	2.0454	0.0476	0.2635	0.1144

A part of the compromise solutions, for which the performance criterion f_4 has lower values than those in Table 6.1, are shown in Table 6.2. However, if $\mu = 0.6$ then the set $\mathbf{M}_{Re=5}$ is empty and the set $\mathbf{M}_{Re=4}$ does not contain solutions competing with the Salukvadze optimum with respect to the performance criterion f_4.

Table 6.2. The selected compromise solutions of rank of efficiency $Re = 5$

μ	Test point	f_1	f_2	f_3	f_4
0.2	258	1.9330	0.0585	0.3514	0.1115
0.2	1286	1.8611	0.0683	0.4392	0.1029
0.2	2314	1.8930	0.0635	0.3953	0.1101
0.4	1356	1.8611	0.0683	0.4392	0.1043

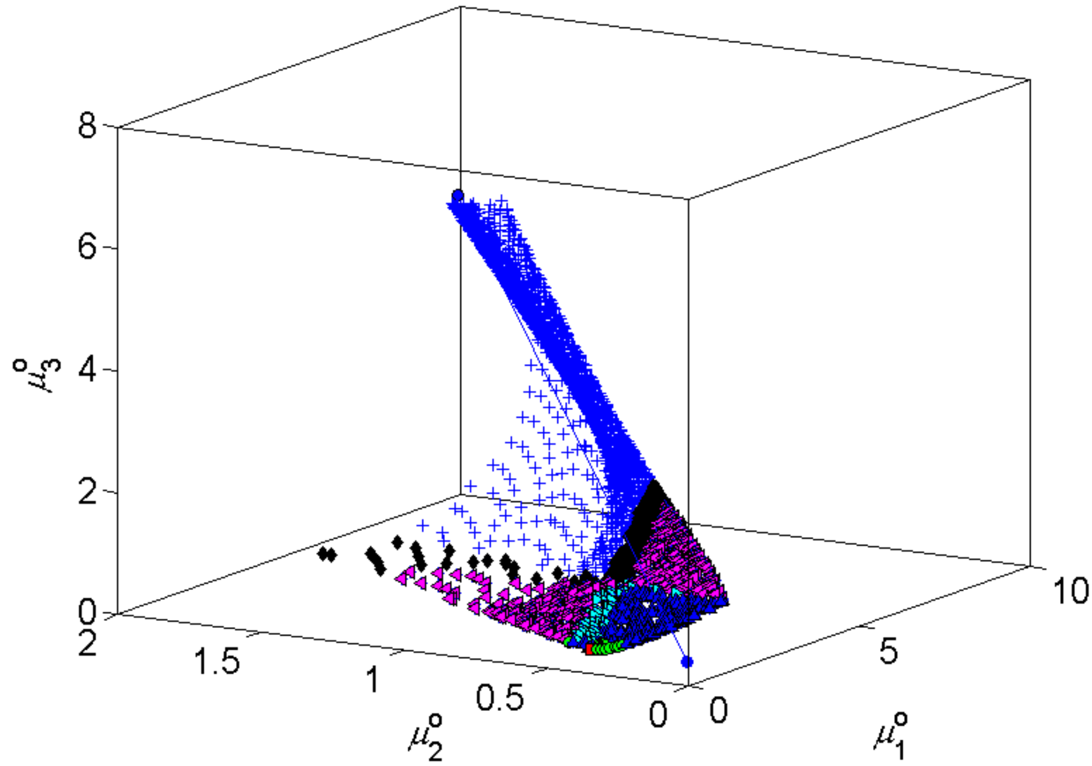


Figure 3. Pareto optimal points in the $\mathbf{\mu}^o$-space

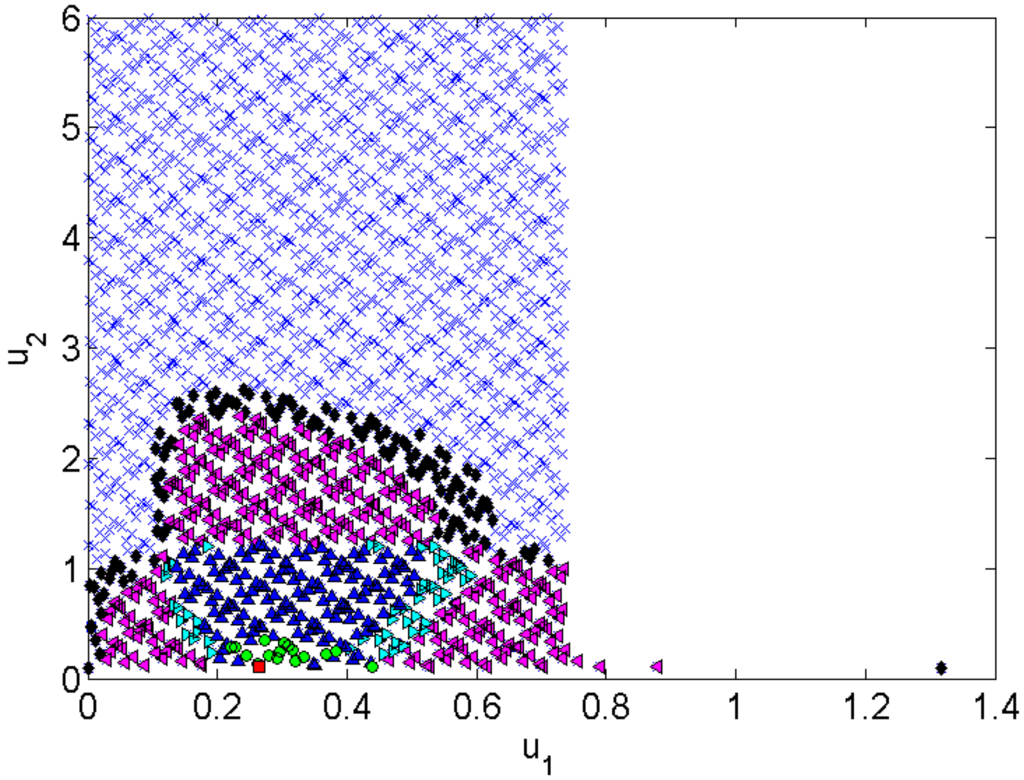


Figure 4. Pareto optimal points in the domain $\mathbf{\Pi}$

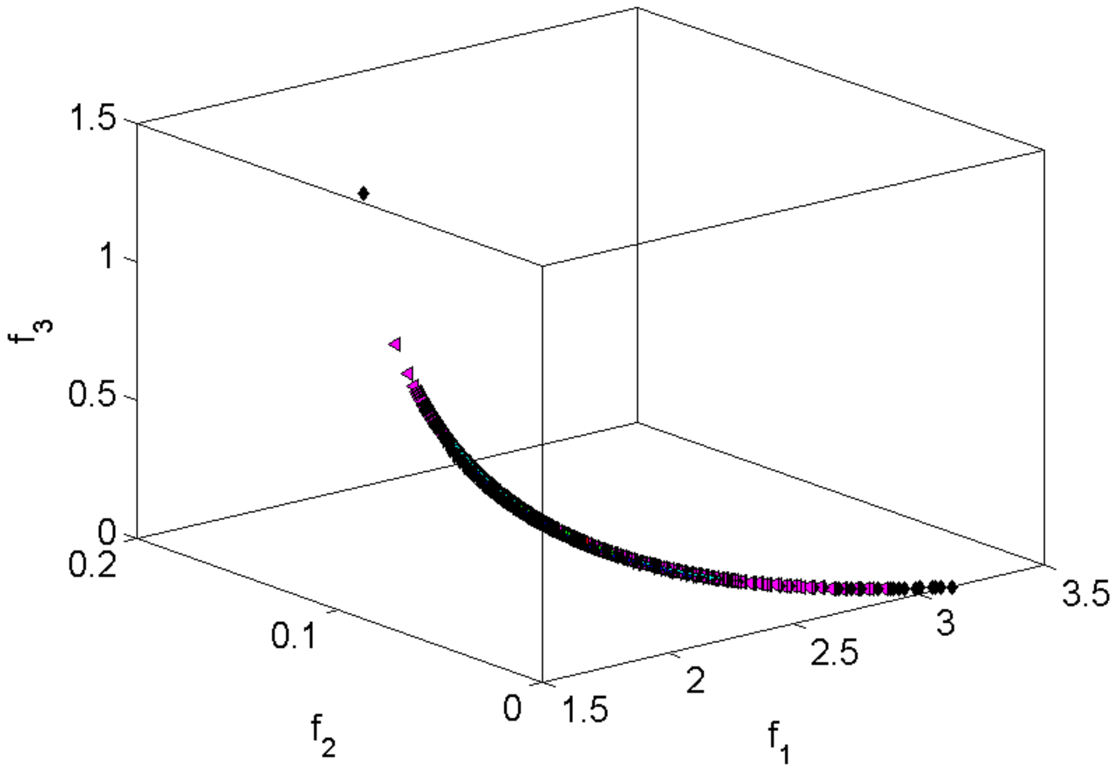


Figure 5. Pareto optimal points in the objective-space of f_j, $j = 1, 2, 3$

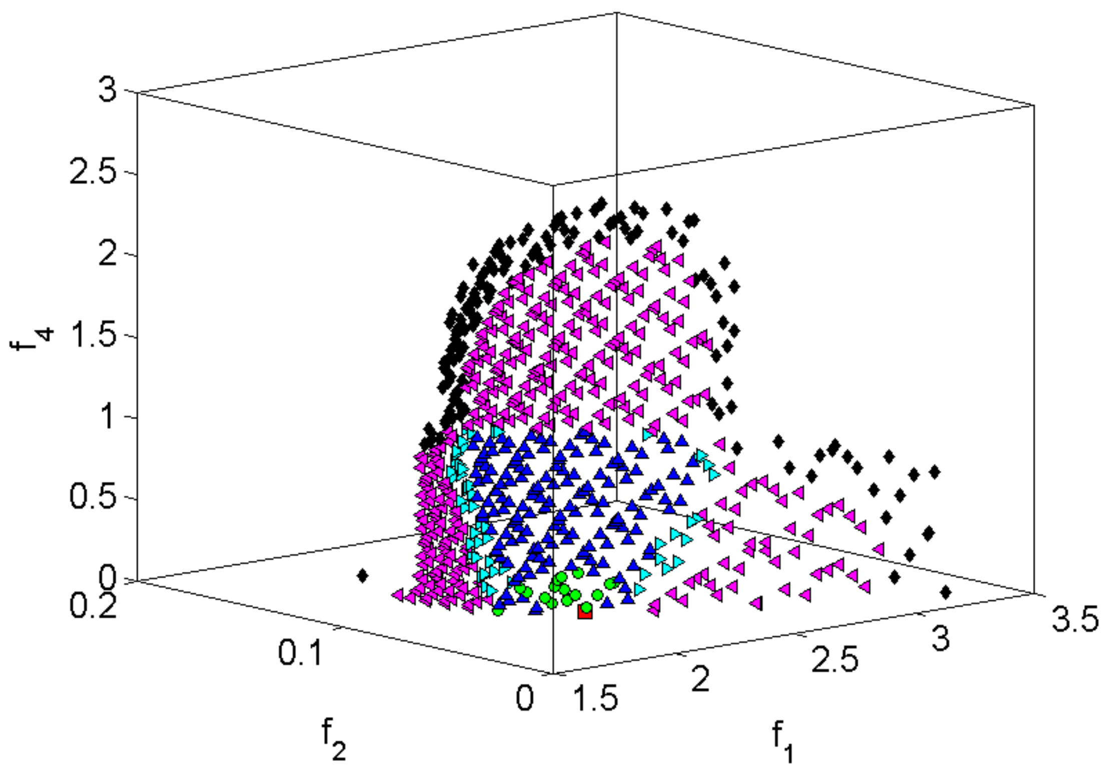
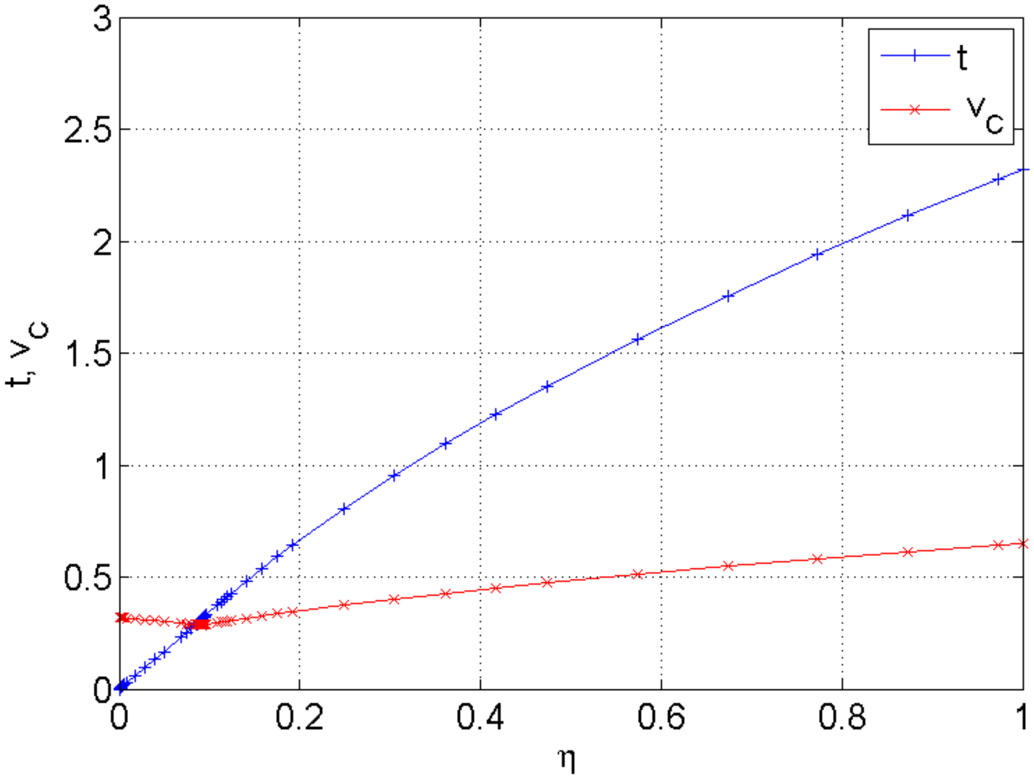
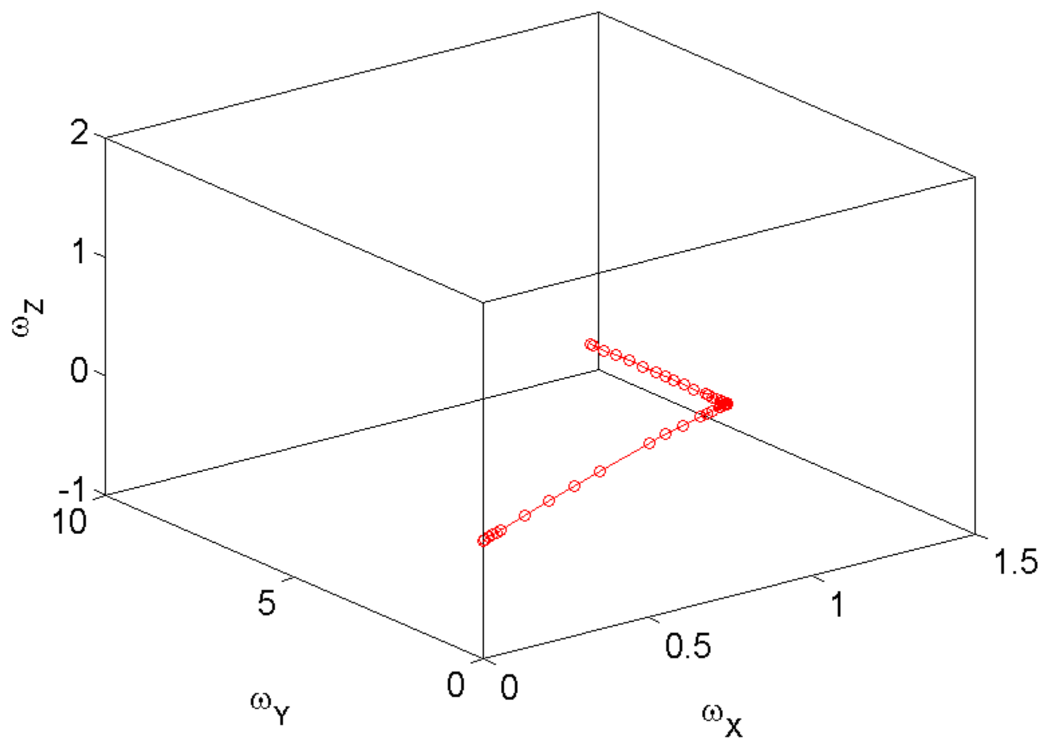


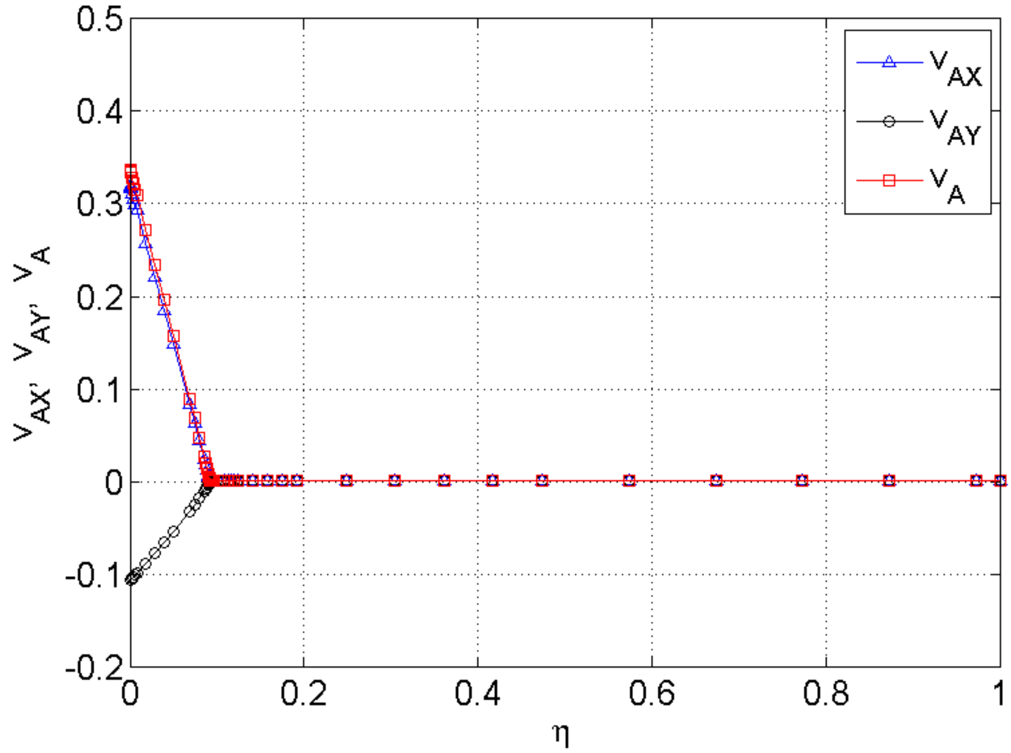
Figure 6. Pareto optimal points in the objective-space of f_j, $j = 1, 2, 4$



(a)

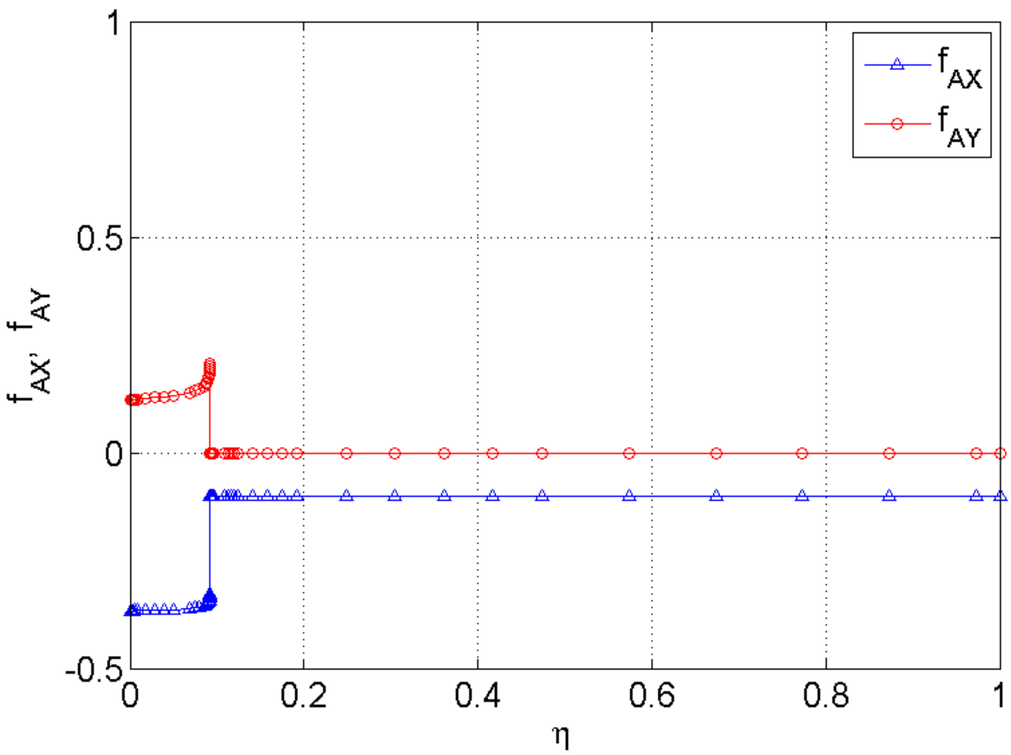


(b)



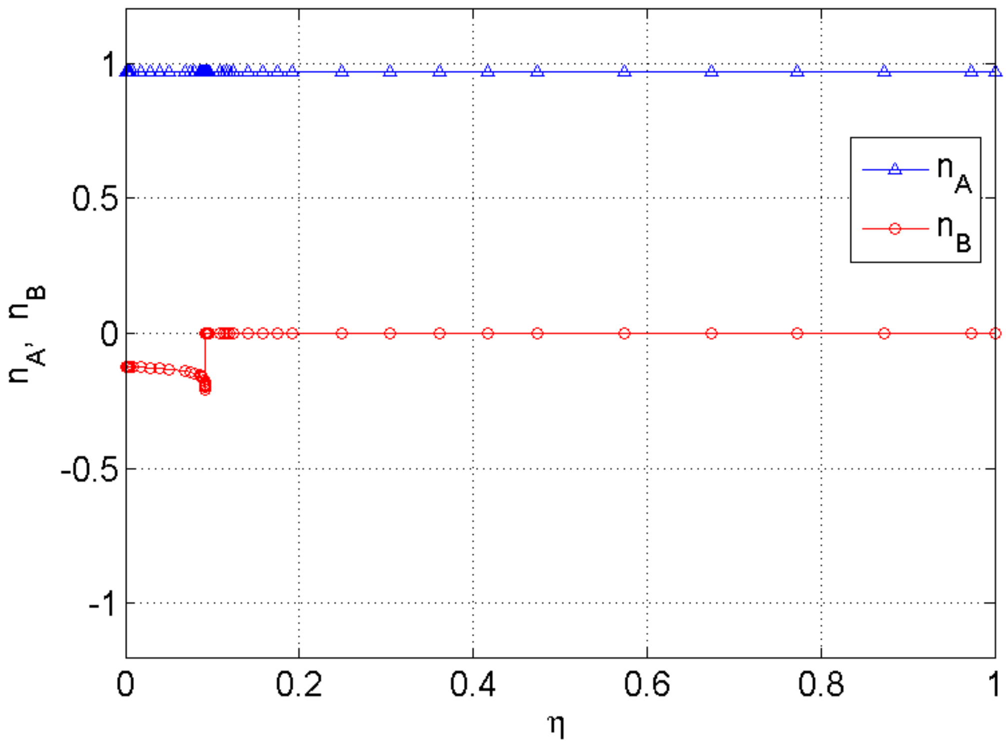
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
0.2
0.4
0.6
0.8
1
η
v_{AX}, v_{AY}, v_A
v_{AX}
v_{AY}
v_A

(c)



1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
η
f_{AX}, f_{AY}
f_{AX}
f_{AY}

(d)



(e)

Figure 7. Kinematical (a, b, c) and dynamical (d, e) characteristics of sphere transportation for Salukvadze optimal design of a gravitational chute-conveyor with $\mu = 0.4$

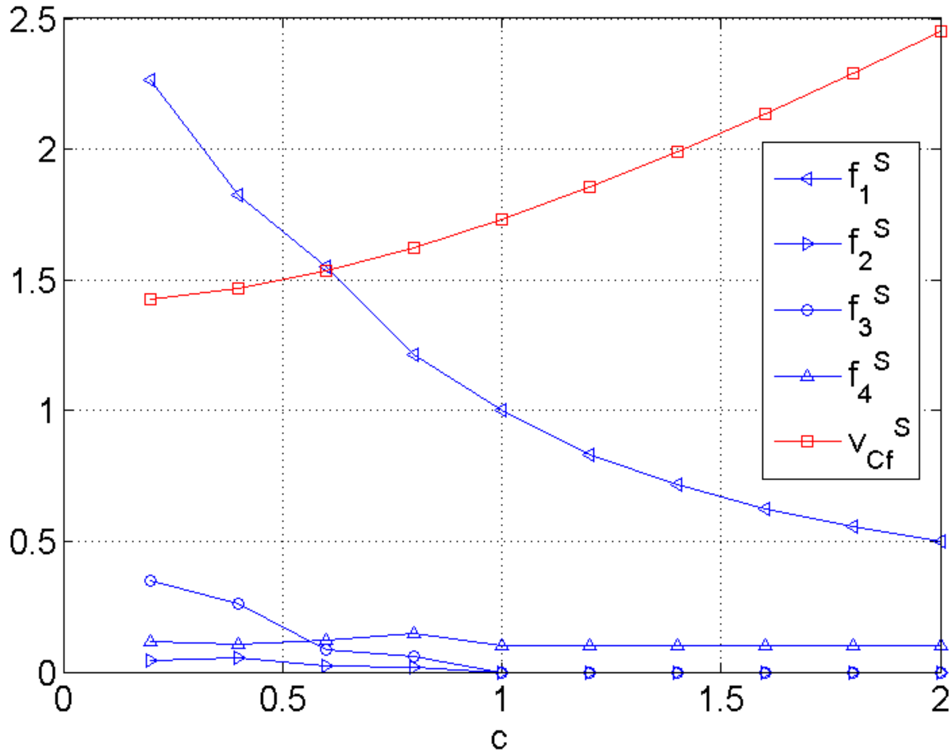


Figure 8. Salukvadze optimal functions $f_j^S(c), j = 1, 2, 3, 4$ and $v_{Cf}^S(c)$ for $\mu = 0.4$

Figure 7 illustrates the basic kinematical and dynamical characteristics of sphere transportation for Salukvadze optimal design of a gravitational chute-conveyor with $\mu = 0.4$. The diagrams show a motion transition from initial rolling with slipping to rolling without slipping at a instant of the transportation. Such a transition is a typical for the optimally designed conveyors and both type of rolling are included. The model reveals both regims of motion and their ratio in the transportation.

The diagrams in Figure 8 are obtained by varying the dimensionless initial velocity of the sphere's mass center, c, and finding the Salukvadze optimal designs of chute-conveyor. They show that such a conveyor could realize a transportation of a sphere with low dimensionless values of the motion duration, f_1^S, as well as high values of the conveyor height, $u_1 \equiv f_3^S$, by almost constant energy losses of friction, f_2^S, and dimensionless velocity magnitude, $u_2 \equiv f_4^S$, of the moving plane-support.

7 Conclusion

The dynamics of a homogeneous rigid sphere, moving without impact in a rectilinear gravitational grove with transformed dry friction, is modeled in this study. The mathematical model considers the basic types of sphere motion – rolling with and without slipping. A multi-objective optimization synthesis problem of passively controlled sphere motion is formulated for the model. It is solved by means of two-stage procedure consisting of uniform sounding the feasible domain of the variables and of selecting Pareto-optimal subsets, ranked by their efficiency. A simple compromise solution, based on the Salukvadze optimal concept, is determined. The obtained results expand the circle of studied problems for modeling mechanical systems with transformed dry friction.

References

Andronov, V. V.: Mechanical systems with transformed dry friction, *Mech. Solids*, 1, (1988), 40-49, (Russ.).

Andronov, V. V.; Zhuravlev, V. Ph.: *Dry Friction in Problems of Mechanics*. RHD, Moskow, Izhevsk (2010), (Russ.).

Borisov, A.V.; Mamaev, I. S;. Killin, A. A.: Rolling a ball on a surface. New integrals and hierachy of dynamics, *Regul. Chaotic Dyn.*, 7 (1), (2002), 1-19.

Chaplygin, S. A.: *Analysis of the Dynamics of Nonholonomic Systems*. Series on Classical Natural Sciences, Moscow (1949), (Russ.).

Cheshankov, B. I.; Ivanov, I. V.; Vitliemov, V. G.; Koev, P. A.: PSI-method multi-criteria optimization contracting the set of trade-off solutions, In: *15th Int. Conf. Sys. Sci.*, Wroclaw, Poland, 1, (2004), 281-288.

Grigoryan, A.T.: Fradlin, B. N.: *A History of Mechanics of the Rigid Body*, Nauka, Moskow (1982), (Russ.).

Gudushauri, E. G.; Panovko, G. Y.: *Theory of Vibro-Technological Processes with Non-Coulomb friction*. Nauka, Moskow (1988), (Russ.).

Johnson, A. D.: The nonholonomy of the rolling sphere, *Amer. Math. Monthly*, 114 (6), (2007), 500-508.

Kilin, A. A.: The dynamics of Chaplygin ball the qualitative and computer analysis, *Regul. Chaotic Dyn*, 6 (3), 2001, 291-306.

Kraige, L. G.; Meriam, J. L.: *Engineering Mechanics: Dynamics*, 5th ed., John Wiley, New York (2002).

Miettinen, K.: *Nonlinear Multiobjective Optimization*, Kluwer, Boston (1999).

Narayanan, P.; Lefcourt, A. M.; Tasch, U.; Rostamian, R.; Grinblat, A.; Kim, M.S.; Chen, Y. R.: Theoretical analysis of stability of axially-symmetric rotating objects with regard to orienting apples, *Trans. ASABE*, 54 (4), (2008), 1353-1364.

Neimark, J. I.; Fufaev, N. A.: *Dynamics of Nonholonomic Systems*, American Mathematical Society, Providence, Rhode Island (1972).

Routh, E. J.: *Advanced Dynamics of a System of Rigid Bodies*, MacMillan Company, London (1905).

Rozenblat, G. M.: On the motion of a two-dimensional rigid body on a rough straight line, *Nonlin. Dyn.*, 2 (3), (2006), 293-306, (Russ.).

Salukvadze, M. E.: *Vector-valued Optimization Problems in Optimal Control Theory*, Academic Press, New York (1979).

Sobol', I. M.; Statnikov R. B.: *Selection of Optimal Parameters in Problems with Multiple Criteria*, Drofa, Moscow (2006), (Russ.).

Statnikov, J. B.; Matusov, R. B.: *Multicriteria Analysis in Engineering*, Kluwer, Dordrecht (2002).

Veselov, A. P.; Veselova, L. E.: Integrable nonholonomic systems on Lie groups, *Math. Notes,* 44, (1989), 810-816.

Vitliemov, V. G.; Kenarov, G. G.; Uzunov, T. I.: Multi-objective synthesis of rectilinear body translation with transformed dry friction, *Mech. Mach.*, 14 (3), (2006), 100-104, (Bulg.).

Addresses: Prof. Dr. Sc. Bojidar Chesankov, Technical University, Department of Stochastics and Optimization, 1157 Sofia, Bulgaria; Assist. Prof. Dr.-Ing. Velina Bozdouganova, Assoc. Prof. Dr.-Ing Ivelin Ivanov, Assoc. Prof. Dr.-Ing Venko Vitliemov, University of Rousse, Department of Technical Mechanics, 7017 Ruse, Bulgaria. *Email*: chesh@tu-sofia.bg; {velina, ivivanov, venvit} @uni-ruse.bg .