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Vibrations of an Inhomogeneous Rectangular Plate

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Low-frequency vibrations of a thin multifibrous plate are analysed. Asymptotic homogenization and finite element methods are used to get the vibration frequencies. Approximate formulas for the lowest frequencies of thin inhomogeneous rectangular plate are found. The comparison of numerical and asymptotic results is performed.

1 Introduction

Lately many industries including nano-technologies have been more and more interested in composite inhomogeneous materials construction. These materials, in comparison with pure homogeneous materials, have advanced mechanical properties. A fundamental vibration frequency is an important characteristic of thin-walled structure. A simple way to increase the fundamental frequency and avoid resonance is increasing thickness of the structure. However in this case the mass of the structure also increases. An optimal design of thin-walled structure provides raising its the frequency without increase of its mass (see Bauer (1993)). The optimal design of an inhomogeneous plate is fairly difficulty problem. The method based on an asymptotic approach provides the construction of simple algorithms for the calculation of the optimal parameters (see Naumova and Ivanov (2007)).

2 Basic equations

Consider a square plate with length L and thickness h. The thickness of plate is small in comparison with its sizes in the plan $\left(\frac{h}{L} < 0.1\right)$. Consider a cartesian coordinate system $OX_1X_2X_3$ on the middle surface of the plate, as shown in Figure 1. For constructing mathematical model of the plate we suppose two basic hypotheses. The first hypothesis (Kirchhoff) assumes that a normal to middle surface of the plate remains a normal to it after deformation. The second hypothesis asserts that the stress state in plate's points is biaxial, i.e. normal and tangential components of the pressure in the platforms perpendicular to axes z can be neglected.

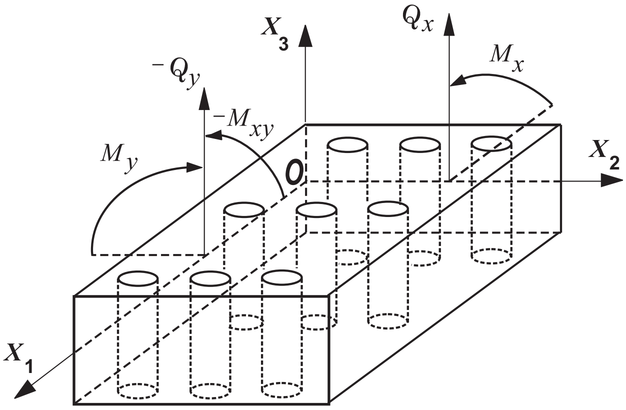

Figure 1: Part of inhomogeneous plate.

Following A. (1970), R. (1991), the free lateral vibrations of plate element can be expressed by means of the equilibrium equation on axis z

$$\frac{\partial Q_{x_1}}{\partial x_1} + \frac{\partial Q_{x_2}}{\partial x_2} - \lambda\, w = 0, \tag{1}$$

and the elasticity relations

$$\frac{\partial M_{x_1}}{\partial x_1} + \frac{\partial M_{x_1x_2}}{\partial x_2} + Q_{x_1} = 0, \quad \frac{\partial M_{x_1x_2}}{\partial x_1} + \frac{\partial M_{x_2}}{\partial x_2} + Q_{x_2} = 0. \tag{2}$$

$$M_{x_1} = D_0\left(\frac{\partial^2 w}{\partial {x_1}^2} + \mu\frac{\partial^2 w}{\partial {x_2}^2}\right), \quad M_{x_2} = D_0\left(\frac{\partial^2 w}{\partial {x_2}^2} + \mu\frac{\partial^2 w}{\partial {x_1}^2}\right), \tag{3}$$

$$M_{x_1x_2} = D_0(1-\mu)\,\frac{\partial^2 w}{\partial x_1 \partial x_2}, \quad D_0 = \frac{Eh^3}{12(1-\mu^2)}, \quad \lambda = \frac{\rho\omega^2 L^4 h}{D_0}.$$

In these equations, w is the transversal deflection, Q_{x_1}, Q_{x_2} are the shear stress-resultant, M_{x_1}, M_{x_2}, $M_{x_1x_2}$ are the stress-couples, E is Young's modulus, μ is Poisson's ratio, ρ is the mass density, ω is the vibration frequency.

3 Investigation of plate vibration frequencies and modes.

The solution of system (1) — (3) can be expressed as

$$Q_{x_1} = \widehat{Q}_{x_1}(x_1,\, x_2)\cdot \sin\omega t,$$

$$Q_{x_2} = \widehat{Q}_{x_2}(x_1,\, x_2)\cdot \sin\omega t,$$

$$w = \widehat{w}(x_1,\, x_2)\cdot \sin\omega t.$$

The dimensionless variables $\widetilde{x}$, $\widetilde{y}$, $\widetilde{w}$, $\widetilde{Q}_{x_1}$, $\widetilde{Q}_{x_2}$, $\widetilde{M}_{x_1}$, $\widetilde{M}_{x_1x_2}$, $\widetilde{M}_{x_2}$ are given by

$$w(x_1,x_2) = h\widetilde{w}\left(L\widetilde{x_1},\, L\widetilde{x_2}\right), \quad Q_{x_1} = \frac{D_0 h}{L^3}\widetilde{Q}_{x_1}, \quad Q_{x_2} = \frac{D_0 h}{L^3}\widetilde{Q}_{x_2},$$

$$M_{x_1} = \frac{D_0 h}{L^2}\widetilde{M}_{x_1}, \quad M_{x_2} = \frac{D_0 h}{L^2}\widetilde{M}_{x_2}, \quad M_{x_1x_2} = \frac{D_0 h}{L^2}\widetilde{M}_{x_1x_2}.$$

$$D_0 = Dd$$

The coefficients D_0 and d have the dimension $\mathrm{N}\cdot\mathrm{m}^2$. It is assumed that $d = 1\mathrm{N}\cdot\mathrm{m}^2$, and D is a dimensionless function. Note also that the symbol " $\sim$ " introduced for dimensionless variables is omitted and is used for other purposes.

Further we use the multiple scales method (Argatov, 2004; Bahvalov, 1984). Combined with the variables $x_1,\, x_2$ we consider so-called *fast* variables $\xi_1,\ \xi_2$ $\left(\xi_1 = \frac{x_1}{\varepsilon},\ \xi_2 = \frac{x_2}{\varepsilon}\right)$. Thus an elementary cell, a parallelepiped with the sizes $(0,\varepsilon)\times(0,\varepsilon)\times(0,h)$ transforms into a parallelepiped $(0,1)\times(0,1)\times(0,h)$ in variables $\xi_1,\ \xi_2$, and each of the unknown functions, dependent on the variables $x_1,\ x_2$, becomes formally dependent also on the variables $\xi_1,\ \xi_2$:

$$f(x_1,\, x_2) = \widetilde{f}(x_1,\, x_2,\, \xi_1,\, \xi_2).$$

Asymptotic expansions for the functions $\widetilde{w}$ and λ have the form

$$\begin{aligned} &\widetilde{w} = \widetilde{w}_0(x_1,\, x_2,\, \xi_1,\, \xi_2) + \varepsilon\widetilde{w}_1(x_1,\, x_2,\, \xi_1,\, \xi_2) + \\ &+ \varepsilon^2\widetilde{w}_2(x_1,\, x_2,\, \xi_1,\, \xi_2) + \ \varepsilon^3\widetilde{w}_3(x_1,\, x_2,\, \xi_1,\, \xi_2) + \ldots \\ &\lambda = \lambda_0(\xi_1,\, \xi_2) + \varepsilon\lambda_1(\xi_1,\, \xi_2) + \ldots. \end{aligned} \tag{4}$$

Taking into account (4) and the composite function differentiation rule

$$\frac{d\widetilde{f}}{dx_1} = \frac{\partial\widetilde{f}}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial\widetilde{f}}{\partial\xi_1}, \quad \frac{d^2\widetilde{f}}{dx_1^2} = \left(\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial\xi_1}\right)^2\widetilde{f},$$

we obtain the following expressions for the stress-couples and the shear stress-resultant

$$\begin{aligned} &M_{x_1} = \varepsilon^{-2}(M_{x_1 0} + \varepsilon M_{x_1 1} + \varepsilon^2 M_{x_1 2} + \cdots + \varepsilon^k M_{x_1 k} + \ldots), \\ &M_{x_1x_2} = \varepsilon^{-2}(M_{x_1x_2 0} + \varepsilon M_{x_1x_2 1} + \varepsilon^2 M_{x_1x_2 2} + \cdots + \varepsilon^k M_{x_1x_2 k} + \ldots), \\ &M_{x_2} = \varepsilon^{-2}(M_{x_2 0} + \varepsilon M_{x_2 1} + \varepsilon^2 M_{x_2 2} + \cdots + \varepsilon^k M_{x_2 k} + \ldots), \\ &Q_{x_1} = \varepsilon^{-3}(Q_{x_1 0} + \varepsilon Q_{x_1 1} + \varepsilon^2 Q_{x_1 2} + \cdots + \varepsilon^k Q_{x_1 k} + \ldots), \\ &Q_{x_2} = \varepsilon^{-3}(Q_{x_2 0} + \varepsilon Q_{x_2 1} + \varepsilon^2 Q_{x_2 2} + \cdots + \varepsilon^k Q_{x_2 k} + \ldots), \quad k = 1,\ 2,\ 3, \ldots. \end{aligned} \tag{5}$$

Substituting (5) into (1)—(3) and equating the coefficients at consecutive degrees of the parameter ε to zero we come to the following recurrent equations chain (6)—(8)

$$\begin{aligned}
&\frac{\partial Q_{x_1 0}}{\partial \xi_1} + \frac{\partial Q_{x_2 0}}{\partial \xi_2} = 0\\
&\frac{\partial M_{x_1 0}}{\partial \xi_1} + \frac{\partial M_{x_1 x_2 0}}{\partial \xi_2} + Q_{x_1 0} = 0\\
&\frac{\partial M_{x_1 x_2 0}}{\partial \xi_1} + \frac{\partial M_{x_2 0}}{\partial \xi_2} + Q_{x_2 0} = 0
\end{aligned} \tag{6}$$

for $k = 1,\ 2,\ 3:$

$$\begin{aligned}
&\frac{\partial Q_{x_1 k}}{\partial \xi_1} + \frac{\partial Q_{x_2 k}}{\partial \xi_2} + \frac{\partial Q_{x_1 k-1}}{\partial x_1} + \frac{\partial Q_{x_2 k-1}}{\partial x_2} = 0\\
&\frac{\partial M_{x_1 k}}{\partial \xi_1} + \frac{\partial M_{x_1 x_2 k}}{\partial \xi_2} + \frac{\partial M_{x_1 k-1}}{\partial x_1} + \frac{\partial M_{x_1 x_2 k-1}}{\partial x_2} + Q_{x_1 k} = 0\\
&\frac{\partial M_{x_1 x_2 k}}{\partial \xi_1} + \frac{\partial M_{x_2 k}}{\partial \xi_2} + \frac{\partial M_{x_1 x_2 k-1}}{\partial x_1} + \frac{\partial M_{x_2 k-1}}{\partial x_2} + Q_{x_2 k} = 0
\end{aligned} \tag{7}$$

for $k = 4:$

$$\begin{aligned}
&\frac{\partial Q_{x_1 k}}{\partial \xi_1} + \frac{\partial Q_{x_2 k}}{\partial \xi_2} + \frac{\partial Q_{x_1 k-1}}{\partial x_1} + \frac{\partial Q_{x_2 k-1}}{\partial x_2} - \lambda_0 \widetilde{w}_0 = 0\\
&\frac{\partial M_{x_1 k}}{\partial \xi_1} + \frac{\partial M_{x_1 x_2 k}}{\partial \xi_2} + \frac{\partial M_{x_1 k-1}}{\partial x_1} + \frac{\partial M_{x_1 x_2 k-1}}{\partial x_2} + Q_{x_1 k} = 0\\
&\frac{\partial M_{x_1 x_2 k}}{\partial \xi_1} + \frac{\partial M_{x_2 k}}{\partial \xi_2} + \frac{\partial M_{x_1 x_2 k-1}}{\partial x_1} + \frac{\partial M_{x_2 k-1}}{\partial x_2} + Q_{x_2 k} = 0.
\end{aligned} \tag{8}$$

In this notations the equations (6)—(8) are five systems, and every system consists of three differential equations. For convenience of further mathematical transformations and numerical calculations, the system (6)—(8) can be written concerning transversal deflection as (9)

$$\begin{aligned}
&L_0 \widetilde{w}_0 = 0,\\
&L_1 \widetilde{w}_0 + L_0 \widetilde{w}_1 = 0,\\
&L_2 \widetilde{w}_0 + L_1 \widetilde{w}_1 + L_0 \widetilde{w}_2 = 0,\\
&L_3 \widetilde{w}_0 + L_2 \widetilde{w}_1 + L_1 \widetilde{w}_2 + L_0 \widetilde{w}_3 = 0,\\
&(L_4 - \lambda_0)\, \widetilde{w}_0 + L_3 \widetilde{w}_1 + L_2 \widetilde{w}_2 + L_1 \widetilde{w}_3 + L_0 \widetilde{w}_4 = 0.
\end{aligned} \tag{9}$$

We introduce the following notations for the partial derivatives

$$p_{x_1} = \frac{\partial}{\partial x_1},\ p_{x_2} = \frac{\partial}{\partial x_2},\ p_{\xi_1} = \frac{\partial}{\partial \xi_1},\ p_{\xi_2} = \frac{\partial}{\partial \xi_2}.$$

In the system (9) differential operators L_0, L_1, L_2, L_3, L_4 are given by

$$\begin{aligned}
L_0 =& p_{\xi_1}(p_{\xi_1}(D(p_{\xi_1}^2 + \mu p_{\xi_2}^2)) + p_{\xi_2}(D(1-\mu)p_{\xi_1}p_{\xi_2}))+\\
&+ p_{\xi_2}(p_{\xi_2}(D(p_{\xi_2}^2 + \mu p_{\xi_1}^2)) + p_{\xi_1}(D(1-\mu)p_{\xi_1}p_{\xi_2})),\\
L_1 =& L_{11}(p_{\xi_1}, p_{\xi_2})p_{x_1} + L_{11}(p_{\xi_2}, p_{\xi_1})p_{x_2},\\
L_2 =& L_{21}(p_{\xi_1}, p_{\xi_2})p_{x_1}^2 + L_{22}(p_{\xi_1}, p_{\xi_2})p_{x_1}p_{x_2} + L_{21}(p_{\xi_2}, p_{\xi_1})p_{x_2}^2,\\
L_3 =& L_{31}(p_{\xi_1}, p_{\xi_2})p_{x_1}^3 + L_{32}(p_{\xi_1}, p_{\xi_2})p_{x_1}^2 p_{x_2} + L_{32}(p_{\xi_2}, p_{\xi_1})p_{x_2}^2 p_{x_1}+\\
&+ L_{31}(p_{\xi_2}, p_{\xi_1})p_{x_2}^3,\\
L_4 =& L_{41}(p_{\xi_1}, p_{\xi_2})p_{x_1}^4 + L_{42}(p_{\xi_1}, p_{\xi_2})p_{x_1}^3 p_{x_2} + L_{43}(p_{\xi_2}, p_{\xi_1})p_{x_1}^2 p_{x_2}^2+\\
&+ L_{42}(p_{\xi_2}, p_{\xi_1})p_{x_1}p_{x_2}^3 + L_{41}(p_{\xi_2}, p_{\xi_1})p_{x_2}^4,
\end{aligned}$$

where the nonzero operators are equal

$$\begin{aligned}
L_{11}(p_{\xi_1},p_{\xi_2}) &= 2p^2_{\xi_2}(2D+2(D\mu))p_{\xi_1}+p_{\xi_1}p_{\xi_2}(-2D+D\mu)p_{\xi_2}+\\
&+6p_{\xi_1}Dp^2_{\xi_1}+4p_{\xi_1}Dp_{\xi_1}p_{\xi_2}+2p_{\xi_1}Dp^2_{\xi_2}+\\
&+4p^3_{\xi_1}Dp^3_{\xi_1}+4p_{\xi_1}p^2_{\xi_2}Dp_{\xi_1}p^2_{\xi_2},\\
L_{21}(p_{\xi_1},p_{\xi_2}) &= p^2_{\xi_2}((D\mu)+p^2_{\xi_1}D)+6p_{\xi_1}Dp_{\xi_1}+2p_{\xi_2}Dp_{\xi_2}+\\
&+6Dp^2_{\xi_1}+2Dp^2_{\xi_2},\\
L_{22}(p_{\xi_1},p_{\xi_2}) &= 2p_{\xi_1}p_{\xi_2}((D\mu)-D)+4p_{\xi_2}Dp_{\xi_1}+\\
&+4p_{\xi_1}Dp_{\xi_2}+8Dp_{\xi_1}p_{\xi_2},\\
L_{31}(p_{\xi_1},p_{\xi_2}) &= 2p_{\xi_1}D+4Dp_{\xi_1},\ L_{32}=2p_{\xi_2}D+4Dp_{\xi_2},\\
L_{41}(p_{\xi_1},p_{\xi_2}) &= D,\ L_{43}=2D.
\end{aligned}$$

Assuming $x_1,\ x_2$ and $\xi_1,\ \xi_2$ as independent variables the system (9) can be considered as a recurrent chain of the differential equations of variables ξ_1 and ξ_2 with unknown function $w_i(x_1,\ x_2,\ \xi_1,\ \xi_2)$ and parameters $x_1,\ x_2$. The unknown functions $w_0,\ w_1,\ w_2,\ w_3$, according to Bahvalov (1984) can be expressed as

$$\begin{aligned}
&\widetilde{w}_0 = v_0(x_1,\ x_2),\quad \widetilde{w}_1 = N_1 p_{x_1} v_0(x_1,\ x_2) + N_2 p_{x_2} v_0(x_1,\ x_2),\\
&\widetilde{w}_2 = M_1 p^2_{x_1} v_0(x_1,\ x_2) + M_2 p_{x_1} p_{x_2} v_0(x_1,\ x_2) + M_3 p^2_{x_2} v_0(x_1,\ x_2),\\
&\widetilde{w}_3 = F_1 p^3_{x_1} v_0 + F_2 p^2_{x_1} p_{x_2} v_0 + F_3 p_{x_1} p^2_{x_2} v_0 + F_4 p^3_{x_2} v_0,
\end{aligned}\tag{10}$$

where $N_1,\ N_2,\ M_1,\ M_2,\ M_3,\ F_1,\ F_2,\ F_3,\ F_4$ are functions depending only on variables ξ_1 and ξ_2. Substituting (10) into (9) we get the system of differential equations of variables $\xi_1,\ \xi_2$ concerning unknown functions $N_1,\ N_2$, $M_1,\ M_2,\ M_3,\ F_1,\ F_2,\ F_3,\ F_4$:

$$\begin{aligned}
&L_0N_1=0,\quad L_0N_2=0,\\
&L_0M_1+L_{11}(p_{\xi_1},p_{\xi_2})N_1+L_{21}(p_{\xi_1},p_{\xi_2})1=0,\\
&L_0M_2+L_{12}(p_{\xi_1},p_{\xi_2})N_2+L_{12}(p_{\xi_2},p_{\xi_1})N_1+L_{22}(p_{\xi_1},p_{\xi_2})1=0,\\
&L_0M_3+L_{11}(p_{\xi_2},p_{\xi_1})N_2+L_{21}(p_{\xi_2},p_{\xi_1})1=0,\\
&L_0F_1+L_{11}(p_{\xi_1},p_{\xi_2})M_1+L_{21}(p_{\xi_1},p_{\xi_2})N_1+L_{31}(p_{\xi_1},p_{\xi_2})1=0,\\
&L_0F_2+L_{11}(p_{\xi_2},p_{\xi_1})M_1+L_{11}(p_{\xi_1},p_{\xi_2})M_2+\\
&+L_{21}(p_{\xi_1},p_{\xi_2})N_2+L_{22}(p_{\xi_1},p_{\xi_2})N_1+L_{32}(p_{\xi_1},p_{\xi_2})1=0,\\
&L_0F_3+L_{11}(p_{\xi_1},p_{\xi_2})M_2+L_{11}(p_{\xi_1},p_{\xi_2})M_3+\\
&+L_{21}(p_{\xi_2},p_{\xi_1})N_1+L_{22}(p_{\xi_1},p_{\xi_2})N_2+L_{32}(p_{\xi_2},p_{\xi_1})1=0,\\
&L_0F_4+L_{11}(p_{\xi_2},p_{\xi_1})M_3+L_{21}(p_{\xi_2},p_{\xi_1})N_2+L_{31}(p_{\xi_2},p_{\xi_1})1=0.
\end{aligned}\tag{11}$$

Solutions of two first equations in system (11) satisfying the condition of periodicity have the form

$$N_1=C_1,\ N_2=C_2,\quad \text{where}\quad C_1,C_2=\text{const.}\tag{12}$$

In Bahvalov and P. (1984) periodicity of the solution is studied for the elliptical equation of second order. In our case it is not hard to prove this fact of the solution periodicity for each equation of system (9).
Following the Asymptotic Homogenization Scheme to obtain the coefficients of the averaged vibrations equation (15) it is enough to find only functions $M_1,\ M_2,\ M_3$. Now we must only prove this. For this aim we consider the first equation of (8)

$$p_{\xi_1}Q_{x_14}+p_{\xi_2}Q_{x_24}+p_{x_1}Q_{x_13}+p_{x_2}Q_{x_23}-\lambda_0\widetilde{w}_0=0$$

and integrate it on a cell, taking periodicity conditions into account

$$\iint\limits_s (p_{x_1}Q_{x_13}+p_{x_2}Q_{x_23}-\lambda_0\widetilde{w}_0)d\xi_1 d\xi_2=0.$$

Moreover, we change Q_{x_13} and Q_{x_23} to $-(p_{x_1}M_{x_12}+p_{x_2}M_{x_1x_22})$ and $-(p_{x_2}M_{x_22}+p_{x_1}M_{x_1x_22})$, respectively, using corresponding equations of (7) for $k=3$

$$\begin{aligned}
&\iint\limits_s((p_{x_1}(p_{x_1}M_{x_12}+p_{x_2}M_{x_1x_22})+\\
&+p_{x_2}(p_{x_2}M_{x_22}+p_{x_1}M_{x_1x_22})+\lambda_0\widetilde{w}_0)d\xi_1 d\xi_2=0.
\end{aligned}\tag{13}$$

Recall that $M_{x_12},\ M_{x_22},\ M_{x_1x_22}$ are the third terms of the expansion (5) for stress-couples.

The decomposition of $M_{x_12},\ M_{x_22},\ M_{x_1x_22}$ combining with (4) and (12) yields that

$$\begin{array}{l} M_{x_12} = D(p_{x_1}^2\widetilde{w}_0 + p_{\xi_1}^2\widetilde{w}_2 + \mu(p_{x_2}^2\widetilde{w}_0 + p_{\xi_2}^2\widetilde{w}_2)), \\ M_{x_22} = D(p_{x_2}^2\widetilde{w}_0 + p_{\xi_2}^2\widetilde{w}_2 + \mu(p_{x_1}^2\widetilde{w}_0 + p_{\xi_1}^2\widetilde{w}_2)), \\ M_{x_1x_22} = D(1-\mu)(p_{x_1}p_{x_2}\widetilde{w}_0 + p_{\xi_1}p_{\xi_2}\widetilde{w}_2). \end{array} \tag{14}$$

Substituting (14) into (13) and taking into account (10), we get the averaged vibrations equation of considered plate

$$\begin{array}{l} A_1p_{x_1}^4\widetilde{w}_0 + A_2p_{x_1}^3p_{x_2}\widetilde{w}_0 + A_3p_{x_1}^2p_{x_2}^2\widetilde{w}_0 + \\ +A_4p_{x_2}^3p_{x_1}\widetilde{w}_0 + A_5p_{x_2}^4\widetilde{w}_0 - \lambda_{aver}\widetilde{w}_0 = 0. \end{array} \tag{15}$$

Here $\lambda_{aver} = \frac{1}{S}\iint\limits_s \lambda_0(\xi_1,\ \xi_2)d\xi_1 d\xi_2$ and coefficients A_1, A_2, A_3, A_4, A_5 are defined by formulas

$$\begin{aligned} A_1 =& \frac{1}{S}\iint\limits_s (D + 6p_{\xi_1}Dp_{\xi_1}M_1 + M_1p_{\xi_1}^2D + 6Dp_{\xi_1}^2M_1 + 2p_{\xi_2}M_1p_{\xi_2}D + \\ &+ 2M_1p_{\xi_2}Dp_{\xi_2}\mu + M_1\mu p_{\xi_2}^2D + 2Dp_{\xi_2}^2M_1 + DM_1p_{\xi_2}^2\mu)d\xi_1\xi_2, \end{aligned}$$

$$\begin{aligned} A_2 =& \frac{1}{S}\iint\limits_s (2p_{\xi_2}M_2p_{\xi_2}D + 2M_2p_{\xi_2}Dp_{\xi_2}\mu + M_2p_{\xi_2}^2D\mu + 2Dp_{\xi_2}^2M_2 + \\ &+ DM_2p_{\xi_2}^2\mu + 4p_{\xi_1}Dp_{\xi_2}M_1 - 2p_{\xi_1}Dp_{\xi_2}\mu M_1 + 4p_{\xi_2}Dp_{\xi_1}M_1 + \\ &+ 6p_{\xi_1}Dp_{\xi_1}M_2 - 2M_1p_{\xi_2}Dp_{\xi_1}\mu + 2M_1p_{\xi_2}p_{\xi_1}D - 2M_1p_{\xi_2}p_{\xi_1}D\mu + \\ &+ 8Dp_{\xi_1}p_{\xi_2}M_1 - 2M_1Dp_{\xi_2}p_{\xi_1}\mu + M_2p_{\xi_1}^2D + 6Dp_{\xi_1}^2M_2)d\xi_1d\xi_2, \end{aligned}$$

$$\begin{aligned} A_3 =& \frac{1}{S}\iint\limits_s (2D + 6p_{\xi_2}Dp_{\xi_2}M_1 + 2p_{\xi_2}p_{\xi_2}M_3 - 2p_{xi_2}Dp_{\xi_2}\mu M_3 + \\ &+ 2M_1p_{\xi_2}^2D + p_{\xi_2}^2M_3\mu + 6Dp_{\xi_2}^2M_1 + 2Dp_{\xi_2}M_3 + Dp_{\xi_2}M_3 + \\ &+ 4p_{\xi_1}Dp_{\xi_2}M_2 - 2p_{\xi_1}M_2p_{\xi_2}\mu + 2p_{\xi_1}Dp_{\xi_1}M_1 + 4p_{\xi_2}Dp_{\xi_1}M_2 + \\ &+ 6p_{\xi_1}Dp_{\xi_1}M_3 - 2p_{\xi_2}Dp_{\xi_1}\mu M_2 + 2p_{\xi_1}Dp_{\xi_1}\mu M_1 + \\ &+ 2p_{\xi_1}p\xi_2DM_2 - 2Dp_{\xi_1}p\xi_2\mu M_2 + 4p_{\xi_1}Dp_{\xi_2}M_2 + \\ &+ 8Dp_{\xi_1}p_{\xi_2}M_2 + M_3p_{\xi_1}^2D + p_{\xi_1}^2D\mu M_1 + 2Dp_{\xi_1}^2M_1 + \\ &+ 6Dp_{\xi_1}^2M_3 + DM_1p_{\xi_1}^2\mu + 2M_2p_{\xi_1}p_{\xi_2}D)d\xi_1d\xi_2, \end{aligned}$$

$$\begin{aligned} A_4 =& \frac{1}{S}\iint\limits_s (2p_{\xi_1}M_2p_{\xi_1}D + 2M_2p_{\xi_1}Dp_{\xi_1}\mu + M_2p_{\xi_1}^2D\mu + 2Dp_{\xi_1}^2M_2 + \\ &+ DM_2p_{\xi_1}^2\mu + 4p_{\xi_2}Dp_{\xi_1}M_1 - 2p_{\xi_2}Dp_{\xi_1}\mu M_3 + 4p_{\xi_1}Dp_{\xi_2}M_3 + \\ &+ 6p_{\xi_2}Dp_{\xi_2}M_2 - 2M_1p_{\xi_1}Dp_{\xi_2}\mu + 2M_1p_{\xi_1}p_{\xi_2}D - 2M_3p_{\xi_1}p_{\xi_2}D\mu + \\ &+ 8Dp_{\xi_2}p_{\xi_2}M_3 - 2M_3Dp_{\xi_2}p_{\xi_1}\mu + M_2p_{\xi_2}^2D + 6Dp_{\xi_2}^2M_2)d\xi_1d\xi_2, \end{aligned}$$

$$\begin{aligned} A_5 =& \frac{1}{S}\iint\limits_s (D + 6p_{\xi_2}Dp_{\xi_2}M_3 + M_3p_{\xi_2}^2D + 6Dp_{\xi_2}^2M_3 + 2p_{\xi_1}M_3p_{\xi_2}D + \\ &+ 2M_3p_{\xi_1}Dp_{\xi_1}\mu + M_3\mu p_{\xi_1}^2D + 2Dp_{\xi_1}^2M_3 + DM_3p_{\xi_1}^2\mu)d\xi_1\xi_2. \end{aligned}$$

Continuing this line of reasoning, we see that coefficients A_1, A_2, A_3, A_4, A_5 depend on $M_1,\ M_2,\ M_3$. This completes the proof.

The vibration frequencies of the heterogeneous plate can be easily calculated by formulas (16)—(17).

$$\omega_{aver} = \sqrt{\frac{\lambda_{aver}}{2\pi\rho_{aver}h}}, \quad \rho_{aver} = \frac{\rho_1S_1 + \rho_2S_2}{S_1 + S_2}, \tag{16}$$

$$\lambda_{aver} = \pi^4(A_1k^4/a^4 + A_3k^2/a^2 \cdot l^2/b^2 + A_5l^4/b^4), \tag{17}$$

where k and l are the numbers of waves along directions x_1 and x_2, respectively; $S_1,\ S_2$ are the areas of the matrix and the inclusion within one of the cells, and $S = S_1 + S_2$. In order to get (16)—(17) we assumed the axial symmetry of a cell concerning axes parallel x_1 and x_2 and also a condition of freely supported edges of the plate. In our case for the square plate the boundary conditions can be expressed as

$$\begin{aligned} &\widetilde{w}_0 = 0, \quad \widetilde{M}_{x_1 2} = 0 \quad \text{for} \quad x_1 = 0, \quad x_1 = L, \\ &\widetilde{w}_0 = 0, \quad \widetilde{M}_{x_2 2} = 0 \quad \text{for} \quad x_2 = 0, \quad x_2 = L, \end{aligned} \tag{18}$$

where

$$\widetilde{M}_{x_1 2} = R_{11} p_{x_1}^2 \widetilde{w}_0 + R_{12} p_{x_2}^2 \widetilde{w}_0, \quad \widetilde{M}_{x_2 2} = R_{12} p_{x_1}^2 \widetilde{w}_0 + R_{22} p_{x_2}^2 \widetilde{w}_0, \tag{19}$$

R_{11}, R_{12}, R_{22} are the constant coefficients and $R_{11} = R_{22}$.

On the basis of the above-stated theory the authors came to the conclusion that in order to find the averaged equation coefficients A_1, A_2, A_3, A_4, A_5 of the equation (15), the solution can be represented as

$$w = \alpha_{11} x_1^2 + \alpha_{12} x_1 x_2 + \alpha_{22} x_2^2 + \varepsilon^2 \Psi(\xi_1, \xi_2),$$

where Ψ is a periodic function that can be submitted as the sum $\Psi = \alpha_{11}\Psi_1 + \alpha_{12}\Psi_2 + \alpha_{22}\Psi_3$. Functions Ψ_1, Ψ_2, Ψ_3 do not depend on α_{ij}. Certainly, the solution does not satisfy our boundary conditions, but this fact does not affect coefficients A_1, A_2, A_3, A_4, A_5. The values $\Psi(\xi_1, \xi_2)$, D, μ, w are periodic functions that can be expressed in double Fourier series:

$$\begin{aligned} \Psi^{kl}(\xi_1, \xi_2) = &\sum_{i=0,j=0}^{k,l} \Psi_{ij}^{cc} \cos i\xi_1 \cos j\xi_2 + \sum_{i=0,j=1}^{k,l} \Psi_{ij}^{cs} \cos i\xi_1 \sin j\xi_2 + \\ + &\sum_{i=1,j=0}^{k,l} \Psi_{ij}^{sc} \sin i\xi_1 \cos j\xi_2 + \sum_{i=1,j=1}^{k,l} \Psi_{ij}^{ss} \sin i\xi_1 \sin j\xi_2, \quad \Psi \to D \to \mu. \end{aligned}$$

Instead of infinite Fourier series for $\Psi(\xi_1, \xi_2)$, D, μ let's consider the truncated Fourier series expression for the equation of vibrations and also equate the coefficients at corresponding products of cosinus and sinus to zero. As a result we obtain a linear system of equations concerning $\Psi_{ij}^{cc}, \quad \Psi_{ij}^{cs}, \quad \Psi_{ij}^{sc}, \quad \Psi_{ij}^{ss}$. To get the coefficients of the averaged vibrations equation we substitute

$$w = w_0 + \varepsilon^2 \left(M_1 p_{x_1}^2 w_0 + M_2 p_{x_1} p_{x_2} w_0 + M_3 p_{x_2}^2 w_0 \right)$$

into the equations (1)—(3) using an asymptotic homogenization scheme and consider the expression at $\varepsilon = 0$. It is worth noting that for $w_0 = \alpha_{11} x_1^2 + \alpha_{12} x_1 x_2 + \alpha_{22} x_2^2$, where $2\alpha_{11} = p_{x_1}^2 w_0$, $\alpha_{12} = p_{x_1} p_{x_2} w_0$, $2\alpha_{22} = p_{x_2}^2 w_0$, we obtain $M_1 = \frac{1}{2}\Psi_1$, $M_2 = \Psi_2$, $M_3 = \frac{1}{2}\Psi_3$.

Let us remark that in the paper (Naumova and Ivanov, 2007) we tried to solve the problem taking into account the boundary conditions on the border of a matrix and a inclusion, but it has not resulted in expected results. In the current paper we apply such solution at which saltus of functions smooths out their expansion in Fourier series. The delta–function having a saltus between a matrix and a inclusion, does not allow corresponding integrals to become zero.

4 Numerical results

The numerical calculations were performed for the square plate such that the length, L, is 1 m, the thickness, h, is 0.01 m and the radius of a inclusion, r, is 0.05 m. Inclusions of the plate arrange in regular intervals and their quantity is 5 along the length and 5 along the width in the first example. We assume the axial symmetry of a cell concerning axes parallel x_1 and x_2, and also a condition of freely supported edges of a plate (18). The material properties (Young's modulus, the mass density, Poisson's ratio) for the considered plates are shown in Table 1.

Material	Young's modulus, E, 10^{11} N/m^2	Mass density, ρ, kg/m^3	Poisson's ratio, ν
Steel	1.93	8030	0.29
Titan	1.02	4850	0.30
Aluminium	0.73	2720	0.33

Table 1. Material properties.

To evaluate the lowest vibrations frequencies for the rectangular composite plate we use formulae (16) — (17) obtained by an Asymptotic Homogenization Scheme and after that we compare asymptotic and finite element method (FEM). The values of the fundamental frequency for the multifibrous plate are shown in Table 2.

Matrix	Inclusion	Asymptotic formulas (16) — (17)	FEM results
Steel	hole	42.260	42.639
Titan	hole	39.752	39.909
Steel	Titan	45.677	45.796
Steel	Aluminium	46.172	46.207
Titan	Steel	43.343	43.760

Table 2. The values of fundamental frequencies, Hz.

The results of the calculations of the fundamental frequency values obtained by the asymptotic formulas (16)—(17) and by means of finite element method (FEM) are listed in the third and forth columns, respectively. About 11000 four-node shell elements are used in FEM calculations. The computation time of the fundamental frequency values by FEM is a few minutes. The calculations by means of the asymptotic formulas execute in three stages. The relative discrepancy in asymptotic and numerical results is less than 5%.
Further we investigate the influence of the quantity of the inclusions on the vibrations frequency. The values of the fundamental frequency for the steel plate with aluminium inclusions are shown in Table 3. The quantity of aluminium inclusions (n along the length and m along the width) is listed in the first column, the other plate parameters have the same values as in the previous examples.

n × m	Asymptotic formulas (16) — (17)	FEM results
3 × 3	46.278	46.423
5 × 5	46.172	46.207
7 × 7	45.610	46.017

Table 3. The values of fundamental frequencies for the steel plate with aluminium inclusions, Hz.

Finally we show (see Table 4) the influence of the quantity of the holes on the vibrations frequency. The quantity of the apertures (n along the length and m along the width) is listed in the first column, other plate parameters have the same values as in the previous examples.

n × m	Asymptotic formulas (16) — (17)	FEM results
3 × 3	44.702	44.704
5 × 5	42.620	42.639
7 × 7	40.341	40.367

Table 4. The values of fundamental frequencies for the steel plate with apertures, Hz.

According to the results presented in Table 3 and Table 4 we conclude that the existence of inclusions and apertures in the plate reduces the values of the vibration frequencies. So, for example, the continuous steel plate, the sizes mentioned above, has the fundamental frequency 48.894 Hz, and the plate that is weakened by inclusions from aluminium (5 × 5) — 46.172 Hz. Increase of inclusions quantity does not influence essentially on vibration modes. The vibration mode plotted by FEM is shown in Figure 2 (top view (left) and side view (right)).

5 Conclusions

The multifibrous plate have been considered as a thin plate with averaged parameters. The approximation asymptotic formulas for the fundamental frequencies values are obtained. In contrast to the previously studied problem (see Naumova and Ivanov (2007)), the problem becomes more difficult. However, the new approach provides obtaining more exact and realistic solutions by means of the Asymptotic Homogenization Scheme. The comparison of asymptotic and FEM results shows the reliability of the presented formulae. It is shown that the replacement of

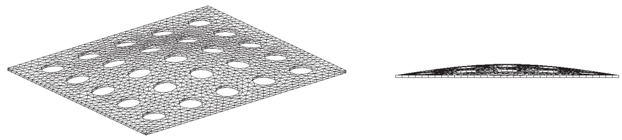

Figure 2: The first vibration mode for a plate with apertures (5×5).

an inhomogeneous rectangular plate by the optimal averaged thin plate with smaller mass can keep the fundamental frequency of a structure.

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