

Asymptotic Analysis of Thin Interface in Composite Materials with Coated Boundary

I.V. Andrianov, G.A. Starushenko, D. Weichert

This paper considers the problem of thin interface in a fibre reinforced composite material. Using the singular asymptotic procedure, the authors obtain simplified relations known as spring model. Phenomenon of edge effect is also studied using the Papkovitch-Fadle approach. The singularities of the limit problem are analysed.

1 Introduction

Thin coatings at the interfaces of the constituents of a composite material can make a substantial difference in the functional characteristics and reliability of composites. The optimum use of stiffness and strength properties of composites directly depends on the effectiveness of the transfer of load from the inclusions to the matrix, proceeding through the coatings. Furthermore, in the heterogeneous materials the greatest concentrations of local stresses occur, as a rule, on the interfaces between the constituents and, thus, the strength of coatings is one of the key factors, determining the load bearing capacity of composite as a whole. The fracture of coatings leads to the development of dislocations and cracks, which in the majority of the cases entails the rapid destruction of entire material.

The problems of analysis of the composites with the coatings were examined by many authors, see, e.g., Achenbach and Zhu, 1990; Chen and Liu, 2001; Hashin, 2002; Jasiuk and Kouider, 1993; Lagache et al., 1994; Lucas da Silva et al., 2008 a, b; Milton, 2002; Van Fo Fy, 1971. The analysis of the limiting cases of soft and rigid coatings is given by Benveniste and Miloh, 2001.

It should be noted, that interaction between the neighbouring fibres can cause a significant variation of physical fields in the composite on the microlevel. Increase in the rigidity of fibers and their volume fraction (i.e., the decrease of distances between the neighbouring fibres) leads to an increase in the local stresses on the interface of constituents. In this case the application of many known analysis methods can be limited by the difficulties of computational nature. Thus, analytical approaches based on representing stress fields in the form of expansions in various infinite series, can experience a deficiency in the convergence. Numerical methods require an increase in the mesh density and, accordingly, a significant increase in computing time (Mishuris and Öchsner, 2005). Mentioned difficulties justify the introduction of a model of the interface which simplifies the computation of the solution and furnishes a good approximation. The interface between fibre and matrix can play an important role in determining the properties of the composite material. Usually, stresses are continuous across the interface, while the displacements may be continuous or discontinuous. In the former case the interface is called “strong”, whereas in the latter case it is called “weak”. We deal with a weak interface described by the spring-layer model, which assumes that the interfacial stress is a function of the gap in the displacements. This model was initially proposed by Golland and Reissner (1944). Asymptotic justification of spring-layer model was proposed by many authors, e.g., Geymonat et al., 1999; Klarbring, 1991; Krasucki and Lenci, 2000 a, b; Lenci, 1999. As a rule they dealt with infinite domains, but for real composite materials it is very important behaviour near the boundaries. In the present work we propose an asymptotic analysis of the interface taking into account edge effects near the boundary for dilute fibre composite materials with coated boundary.

2 Governing relations

We will consider the case of a single fibre weakly bonded to a surrounding half-space (Fig. 1). The fibre is loaded by uniformly distributed across its cross-section load P .

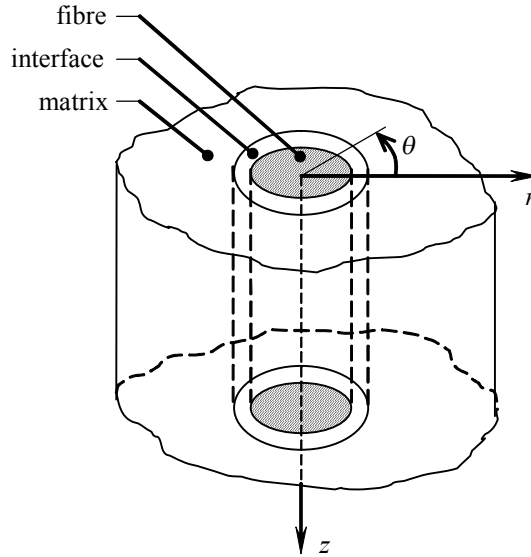


Fig. 1. A single fibre embedded in an elastic half-space with thin unterface.

The matrix material is assumed to be isotropic and linear elastic with elastic constants E and ν . The axial Young's modulus of the circular fibre with radius R is denoted by E_1 . We will use a circular cylindrical coordinate system (r, θ, z) ; axis of the fibre coincides with the z -axis. The problem is axially symmetric. The axial displacement of the fibre is denoted by $U_f(z)$ and the radial and longitudinal displacement of the matrix by $U_r(r, z)$ and $U_z(r, z)$, respectively. We also denote stresses in the matrix by $\sigma_r(r, z)$, $\sigma_z(r, z)$, $\sigma_\theta(r, z)$, $\tau_{rz}(r, z)$. Now let us suppose that the matrix is coated by thin elastic layer with the small thickness, rigidly bonded to the elastic half-space.

Let us discuss the possible nature of this coating. As it is known that in elastic bodies the natural surface nonuniformity takes place. This effect is closely connected with the interaction of particles. In the bulk of the body there is a statistical symmetry of the forces with which the particles interact, but the particles at the surface experience a one-sided action from other particles. This leads to considerable nonuniformity of the mechanical properties near the boundary and to surface tension. The last factor is small for solid bodies. On the other hand, experimental and theoretical investigations of polymers show that the modulus of the very thin surface layer can exceed the modulus in the bulk of the body by a factor 2-3 (Alexandrov et al., 1981; Alexandrov and Mkhitarian, 1983, Chapt. VI). On the other hand, the boundary of composite materials very often is coated by thin strengthening layer. For example, for a polymer material often apply a metal coating (Alexandrov and Mkhitarian, 1983, Chapt. VI). A thin coating can be treated as an inextensible membrane ideally bonded to the matrix at the half-space boundary (Kovalenko, 2001).

3 Analysis of interface far from half-space boundary

Let us analyze in more detail the stress in the interface. We suppose now the interface as cylinder of small thickness h (Fig. 1), i.e. $h/R \ll 1$. Then we can replace the governing axially symmetric problem to the plane strain one (Fig. 2) and use the lubrication approach (Christensen, 2005). It allows us dramatically simplify the governing equilibrium Eqs. for the interface

$$\frac{\partial^2 U_{ir}(z)}{\partial r^2} = 0, \quad (3.1)$$

$$\frac{\partial^2 U_{iz}(z)}{\partial r^2} = 0. \quad (3.2)$$

Here and further we use a subscript "i" for all displacements and stresses related to the interface.

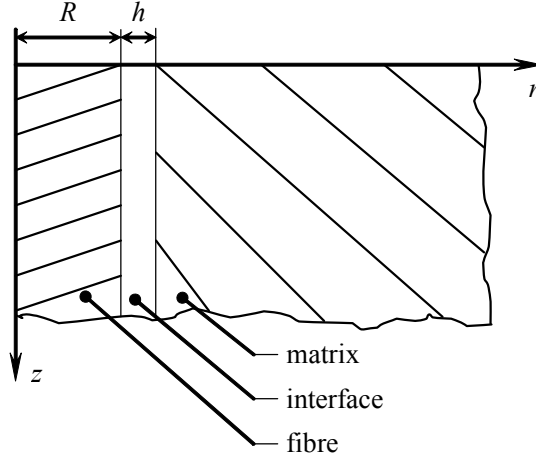


Fig. 2. Reduction of the axially symmetric problem to the plane strain one.

Now let us apply BCs. We consider the fibre as a continuum without transversal deformation and suppose perfect adherence of the fibre and the interface and the interface and the matrix. From these assumptions one obtains the following BCs

$$\text{for } r = R: \quad U_{ir} = 0, \quad (3.3)$$

$$U_{iz} = U_f, \quad (3.4)$$

$$\text{for } r = R + h: \quad U_{ir} = U_r, \quad (3.5)$$

$$U_{iz} = U_z, \quad (3.6)$$

$$\sigma_{ir} = \sigma_r, \quad (3.7)$$

$$\tau_{irz} = \tau_{rz}. \quad (3.8)$$

Solutions of Eqs. (3.1), (3.2) can be written as follows

$$U_{ir} = C_1 + C_2(r - R), \quad (3.9)$$

$$U_{iz} = C_3 + C_4(r - R). \quad (3.10)$$

From BCs (3.3), (3.4) and (3.5) - (3.8) one obtains $C_1 = 0$, $C_3 = U_f$,

$$\beta_i U_r(R + h, z) = \sigma_r, \quad (3.11)$$

$$k \left[U_z(R, z) - U_f \right] = \tau_{rz}, \quad (3.12)$$

where $\beta_i = \frac{E_i(1-\nu_i)}{(1+\nu_i)(1-2\nu_i)h}$, $k = \frac{E_i}{2h(1+\nu_i)}$.

Eqs. (3.11), (3.12) coincide with the equations obtained by Geymonat et al. (1999, p. 209).

BCs for $z = 0$ can be written as follows

$$U_{ir} = 0 \quad \text{for } z = 0, \quad (3.13)$$

$$\sigma_{iz} = 0 \quad \text{for } z = 0. \quad (3.14)$$

Solution (3.9) satisfies BC (3.13). Substitute solution (3.10) to BC (3.14) one obtains

$$\sigma_{iz} = P_1 \neq 0 \quad \text{for } z = 0, \quad (3.15)$$

where $P_1 = \frac{2G_i P}{\pi R^2 E_f}$.

For compensation of the discrepancy in BC (3.14) one must construct the boundary layer solution.

4 Boundary layer in the interface

For solving a plane strain problem for a half-strip we use Papkovich-Fadle bi-orthogonal eigenfunctions expansions (Papkovich, 1940; Fadle, 1940) following Little (1969). Let us introduce new variables $\rho = 2(r - R - 0,5h)/h$, $\psi = 2z/h$, then $\psi \geq 0$, $|\rho| \leq 1$. BCs in new variables can be written as follows

$$\sigma_{i\psi} = P_1, U_{i\rho} = 0 \text{ for } \psi = 0. \quad (4.1)$$

We deal with the boundary layer state; therefore we are interested in self-equilibrating solution. We add to the uniform load $(-P_1)$ equilibrating reactions $P_1 \left[\delta(\rho+1) + \delta(\rho-1) \right]$, where $\delta(\dots)$ is the Dirac's delta-function.

Then BCs (4.1) can be written as follows

$$\sigma_{i\psi} = Q(\rho), U_{i\rho} = 0 \text{ for } \psi = 0, \quad (4.2a)$$

$$\text{where } Q(\rho) = P_1 \left[-1 + \delta(\rho+1) + \delta(\rho-1) \right], \int_{-1}^1 Q(\rho) d\rho = \int_{-1}^1 \rho Q(\rho) d\rho = 0.$$

Searching solution must decay for $\psi \rightarrow \infty$

$$U_{i\rho}, U_{i\psi} \rightarrow 0 \text{ for } \psi \rightarrow \infty. \quad (4.2b)$$

For weak interface one has $E_i \ll E$, $E_i \ll E_1$, then one can approximately suppose

$$U_{i\rho} = U_{i\psi} = 0 \text{ for } \rho = \pm 1. \quad (4.3)$$

Equilibrium and compatibility equations are

$$\frac{\partial \sigma_{i\psi}}{\partial \psi} + \frac{\partial \tau_{i\rho\psi}}{\partial \rho} = 0; \quad \frac{\partial \sigma_{i\rho}}{\partial \rho} + \frac{\partial \tau_{i\rho\psi}}{\partial \psi} = 0; \quad \frac{\partial^2 \sigma_{i\psi}}{\partial \rho^2} + \frac{\partial^2 \sigma_{i\rho}}{\partial \psi^2} + 2 \frac{\partial^2 \sigma_{i\psi}}{\partial \rho \partial \psi} = 0. \quad (4.4)$$

We also introduce a function $\Omega(\rho, \psi)$ as follows

$$\frac{\partial \Omega}{\partial \psi} = \frac{\partial \sigma_{i\psi}}{\partial \rho}.$$

Eqs. (4.4) can be written in the following matrix form

$$\frac{\partial \mathbf{f}}{\partial \rho} + \mathbf{A} \frac{\partial \mathbf{f}}{\partial \psi} = 0, \quad (4.5)$$

$$\text{where } \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \sigma_{i\rho} \\ -\tau_{i\rho\psi} \\ \sigma_{i\psi} \\ \Omega \end{pmatrix}; \quad \mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

BCs (4.2a), (4.2b) and (4.3) can be rewritten as follows

$$f \rightarrow 0 \quad \text{for } \psi \rightarrow \infty ; \quad (4.6)$$

$$f_1 = v_i P_1; \quad f_3 = P_1 \quad \text{for } z = 0 ; \quad (4.7)$$

$$f_3 - v_i f_1 = 0; \quad (2 + v_i) f_2 + f_4 = 0 \quad \text{for } \rho = \pm 1 . \quad (4.8)$$

Let us suppose

$$f(\rho, \psi) = \sum_{n=1} C_n \varphi_n(\rho) \exp(i\beta_n \psi) . \quad (4.9)$$

Substituting Eq. (4.9) to Eqs. (4.5) and BCs (4.8) one obtains

$$\frac{d\varphi_n}{d\rho} + i\beta_n \mathbf{A} \varphi_n = 0, \quad i = \sqrt{-1} ; \quad (4.10)$$

$$\varphi_n^{(3)} = v_i \varphi_n^{(1)}; \quad \varphi_n^{(4)} = -(2 + v_i) \varphi_n^{(2)} . \quad (4.11)$$

Solutions of the boundary value problem (4.10), (4.11) can be written as follows

$$\varphi_n^{(1)} = -\beta_n^2 [\beta_n \rho \operatorname{ch} \beta_n \operatorname{sh}(\beta_n \rho) - (\beta_n \operatorname{sh} \beta_n + 2(1 - v) \operatorname{ch} \beta_n) \operatorname{ch}(\beta_n \rho)];$$

$$\varphi_n^{(2)} = i\beta_n^2 [\beta_n \rho \operatorname{ch} \beta_n \operatorname{ch}(\beta_n \rho) - (\beta_n \operatorname{sh} \beta_n + (1 - 2v) \operatorname{ch} \beta_n) \operatorname{sh}(\beta_n \rho)];$$

$$\varphi_n^{(3)} = \beta_n^2 [\beta_n \rho \operatorname{ch} \beta_n \operatorname{sh}(\beta_n \rho) - (\beta_n \operatorname{sh} \beta_n - 2v \operatorname{ch} \beta_n) \operatorname{ch}(\beta_n \rho)];$$

$$\varphi_n^{(4)} = -i\beta_n^2 [\beta_n \rho \operatorname{ch} \beta_n \operatorname{ch}(\beta_n \rho) - (\beta_n \operatorname{sh} \beta_n + (1 + 2v) \operatorname{ch} \beta_n) \operatorname{sh}(\beta_n \rho)];$$

where β_n are the roots of the following transcendental equation

$$\operatorname{sh} 2\beta = \frac{2}{3 - 4v_i} \beta . \quad (4.12)$$

Eq. (4.12) is solved numerically. In tables 1, 2 we show first ten roots of Eq. (4.12) for $v = 0$ and $v = 0,3$.

Table 1. Comparison of numerical and asymptotic solutions for $v = 0$

Numerical solution	Asymptotic formula (4.13)
$Re \beta = 0; \quad Im \beta = 1,139431330$	$Re \beta = 0,23058798 \ 50; \quad Im \beta = 0,78539816 \ 35$
$Re \beta = 0,80654096 \ 60; \quad Im \beta = 3,81446498 \ 6$	$Re \beta = 0,82777775 \ 45; \quad Im \beta = 3,92699081 \ 8$
$Re \beta = 1,11679601 \ 0; \quad Im \beta = 6,98753142 \ 5$	$Re \beta = 1,12167108 \ 8; \quad Im \beta = 7,06858347 \ 2$
$Re \beta = 1,30383552 \ 0; \quad Im \beta = 10,1455796 \ 2$	$Re \beta = 1,30553347 \ 8; \quad Im \beta = 10,2101761 \ 3$
$Re \beta = 1,43899834 \ 2; \quad Im \beta = 13,2975313 \ 4$	$Re \beta = 1,43966547 \ 0; \quad Im \beta = 13,3517687 \ 8$
$Re \beta = 1,54507343 \ 3; \quad Im \beta = 16,4463325 \ 6$	$Re \beta = 1,54532001 \ 8; \quad Im \beta = 16,4933614 \ 3$
$Re \beta = 1,63244450 \ 2; \quad Im \beta = 19,5932708 \ 6$	$Re \beta = 1,63249671 \ 1; \quad Im \beta = 19,6349540 \ 9$
$Re \beta = 1,70675075 \ 5; \quad Im \beta = 22,7390068 \ 4$	$Re \beta = 1,70670671 \ 4; \quad Im \beta = 22,7765467 \ 4$
$Re \beta = 1,77140566 \ 2; \quad Im \beta = 25,8839173 \ 3$	$Re \beta = 1,77131258 \ 0; \quad Im \beta = 25,9181394 \ 0$
$Re \beta = 1,82863549 \ 6; \quad Im \beta = 29,0282342 \ 8$	$Re \beta = 1,82851775 \ 5; \quad Im \beta = 29,0597320 \ 5$

Table 2. Comparison of numerical and asymptotic solutions for $v = 0,3$

Numerical solution	Asymptotic formula (4.13)
$Re \beta = 0; Im \beta = 0,8829313320$	$Re \beta = 0,2784716105; Im \beta = 0,7853981635$
$Re \beta = 1,078636789; Im \beta = 3,784602776$	$Re \beta = 1,083190567; Im \beta = 3,926990818$
$Re \beta = 1,377797140; Im \beta = 6,970235340$	$Re \beta = 1,377083899; Im \beta = 7,068583472$
$Re \beta = 1,562123529; Im \beta = 10,13340858$	$Re \beta = 1,560946290; Im \beta = 10,21017613$
$Re \beta = 1,696183660; Im \beta = 13,28814668$	$Re \beta = 1,695078282; Im \beta = 13,35176878$
$Re \beta = 1,801696337; Im \beta = 16,43869847$	$Re \beta = 1,800732830; Im \beta = 16,49336143$
$Re \beta = 1,888739798; Im \beta = 19,58683814$	$Re \beta = 1,887909523; Im \beta = 19,63495409$
$Re \beta = 1,962837644; Im \beta = 22,73344947$	$Re \beta = 1,962119526; Im \beta = 22,77654674$
$Re \beta = 2,027351358; Im \beta = 25,87902603$	$Re \beta = 2,026725391; Im \beta = 25,91813940$
$Re \beta = 2,084480895; Im \beta = 29,02386672$	$Re \beta = 2,083930567; Im \beta = 29,05973205$

The first roots give possibility to estimate decaying of stress-strain state in the interface.

For $n \rightarrow \infty$ Little (1969) proposed the following asymptotic formula

$$\beta_n \approx \frac{1}{2} \left\{ \ln \left[\frac{\pi(4n+1)}{3-4\nu_i} \right] + i\pi(2n+0,5) \right\} \quad (4.13)$$

Numerical results based on formula (4.13) are shown in tables 1, 2. It is obvious that in practical calculation one can use formula (4.13) for all roots of the transcendental equation (4.12).

Only the eigenvalues in the first quadrant are listed, but β_n^* , $-\beta_n$ and $-\beta_n^*$ are also roots, where $(...)^*$ denotes the complex conjugate of $(...)$. Only roots in the upper half plane will be used, due to condition (4.2b).

For satisfying BCs (4.7) one must construct biorthogonal vectors. The adjoint equation of Eq. (4.10) is

$$\frac{d\psi_n}{d\rho} + i\beta_n^* \mathbf{A}^+ \psi_n = 0, \quad (4.14)$$

where $(...)^+$ denotes the complex conjugate transpose of the matrix.

We choose BCs on ψ as follows

$$\psi_n^{(3)} = -\nu_i \psi_n^{(1)}; \psi_n^{(4)} = (2 + \nu_i) \psi_n^{(2)}. \quad (4.15)$$

The following biorthogonality conditions holds

$$\int_{-1}^1 \psi_n^+ \mathbf{A} \varphi_k d\rho, \quad n \neq k. \quad (4.16)$$

The components of the biorthogonal vector become

$$\begin{aligned} \psi_n^{(1)} &= \beta_n^* \rho \operatorname{ch} \beta_n^* \operatorname{ch}(\beta_n^* \rho) - (\beta_n^* \operatorname{sh} \beta_n^* + (3-2\nu) \operatorname{ch} \beta_n^*) \operatorname{sh}(\beta_n^* \rho); \\ \psi_n^{(2)} &= i \left[\beta_n^* \rho \operatorname{ch} \beta_n^* \operatorname{sh}(\beta_n^* \rho) - (\beta_n^* \operatorname{sh} \beta_n^* + (2-\nu) \operatorname{ch} \beta_n^*) \operatorname{ch}(\beta_n^* \rho) \right]; \\ \psi_n^{(3)} &= \beta_n^* \rho \operatorname{ch} \beta_n^* \operatorname{ch}(\beta_n^* \rho) - (\beta_n^* \operatorname{sh} \beta_n^* + (1-2\nu) \operatorname{ch} \beta_n^*) \operatorname{sh}(\beta_n^* \rho); \\ \psi_n^{(4)} &= i \left[\beta_n^* \rho \operatorname{ch} \beta_n^* \operatorname{sh}(\beta_n^* \rho) - (\beta_n^* \operatorname{sh} \beta_n^* + 2(1-2\nu) \operatorname{ch} \beta_n^*) \operatorname{ch}(\beta_n^* \rho) \right]. \end{aligned}$$

Using Eq. (4.16) one obtains constants C_n

$$C_n = \frac{1}{M_n} \int_{-1}^1 \psi_n^+ \mathbf{A} \mathbf{f}^{(b)} d\rho,$$

where $M_n = \int_{-1}^1 \psi_n^+ \mathbf{A} \boldsymbol{\varphi}_n d\rho$; $f_1^{(b)} = v_i P_1$, $f_2^{(b)} = \sum_{n=1} C_n \varphi_n^{(2)}(\rho)$, $f_3^{(b)} = P_1$, $f_4^{(b)} = \sum_{n=1} C_n \varphi_n^{(4)}(\rho)$.

The inner biorthogonality conditions are

$$\int_{-1}^1 [\psi_n^{(1)*} \varphi_k^{(2)} + \psi_n^{(3)*} \varphi_k^{(4)}] d\rho = 0,5 M_n \delta_n^k,$$

where δ_n^k is Kronecker's symbol.

Then one obtains constants C_n

$$C_n = \frac{2P_1}{M_n} \int_{-1}^1 [(2 + v_i) \psi_n^{(4)*} - \psi_n^{(2)*}] d\rho.$$

5 Corner singularities in the interface and matrix

For complete understanding of the deformation of the interface we must analyze singularities at the points $\psi = 0$, $\rho = \pm 1$ more detail. As mentioned by Sinclair (2004 a, b), for axially symmetric configurations one expects that, in the local vicinity of greatest interest, a state of plane strain dominates response. So we deal with the plane strain problems of edge-bonded elastic quarter-planes loaded at the boundary. At the point $\psi = 0$, $\rho = -1$ takes place the contact between the fibre and the interface and at the point $\psi = 0$, $\rho = 1$ takes place the contact of the interface and the matrix. Now we can use results obtained by Rössle (2000). At the point $\psi = 0$, $\rho = -1$ we will treat the interface as the elastic quarter-plane hardly clamped of one part of boundary (edge-bonded with the fibre) and simply supported on half-space boundary:

$$\sigma_{i\psi} = 0, U_{i\rho} = 0 \text{ for } \psi = 0.$$

As it is shown by Rössle (2000), presence or absence of stress singularity at the corner points depends on the corner inner angle χ : stress singularity is absent for $\chi < \pi/2$ and is present for $\chi > \pi/2$. So we have some uncertainty, closely connected with our rough assumption about the hard clamping of the interface half-strip on the long sides. In reality on the boundary of the interface and the fibre one has elastic clamping,

$$U_{i\rho} = f_1 \sigma_{i\rho}, U_{i\psi} = f_2 \tau_{i\rho\psi},$$

where f_1, f_2 are some parameters, which characterized degree of elastic clamping, $0 \leq f_i \leq \infty$.

For $f_i = \infty$ singularities take place for $\chi \leq \pi/2$, for $f_i = 0$ singularities take place for $\chi > \pi/2$ (Rössle, 2000). Using the principle of continuity we can conclude that for $f_i < \infty$ stress singularities are absent. The same conclusion is correct for the singularity at the point $\psi = 0$, $\rho = 1$.

For the matrix at the corner point $\psi = 0$, $\rho = 1$ one has the elastic quarter-plane stress-free on part of the boundary (edge-bonded with the interface) and simply supported on half-space boundary. As it is shown by Rössle (2000), in this case the singularity is absent.

So, for the problem under consideration, the weak interface avoids the stress singularities. But if the boundary of half-space is free from the stresses, the stress singularities at the corner points on the lines of contact between the fibre and the interface and the interface and the matrix take place (Geymonat et al., 1999).

6 Conclusion

Composite materials with interfaces between the matrix and the inclusions are referred to as the most actively developed contemporary materials which are widely used in engineering. An adhesion of components usually leads to the appearance of interface cracks which can situate in the interface till the appearance of certain critical conditions which lead to growth and propagation of a crack into the matrix. For the prediction of cracks the appearance and behaviour it is very important to analyze stresses in the interface. Failure of the assemblage starts in the external boundary of the adhesive and then continues along the interface (Adams and Harris, 1987; Lucas da Silva et al., 2009 a, b). However, papers devoted to the composite materials with interfaces, usually deal with the simplest case of domain without boundaries. In the present work we analyze edge effects in the interface near the boundary for the fibre composite materials in the dilute case.

The analysis of singularities at the corner points of the interface is also very important. To increase the resistance of the joint, in practice it is common to avoid (or at least to reduce) the singularity (Adams and Harris, 1987). As it is shown above, a coating of the interface and matrix by the inextensible membrane leads to this aim. So this procedure can be recommended for practical engineers for increasing of the composite materials strengths.

Acknowledgements

This work was supported by the German Research Foundation (Deutsche Forschungsgemeinschaft), grant no. WE 736/25-1 (for I.V. Andrianov).

References

- Achenbach, J.D.; Zhu, H.: Effect of interphases on micro and macromechanical behavior of hexagonal-array fiber composites. *J. Appl. Mech.*, 57, (1990), 956-963.
- Adams, R.D.; Harris, J.A.: The influence of local geometry on the strength of adhesive joints. *Int. J. Adhesion Adhesives*, 7, (1987), 69-80.
- Alexandrov, V.M.; Alexandrova, G.P.; Stepanenko, Yu.P.: The problem of taking into account the scale factor in the mechanics of a solid deformed body. *J. Appl. Mech. Techn. Phys.*, 22 (1), (1981), 128-133.
- Alexandrov, V.M.; Mkhitarjan, S.M.: *Contact Problems for Bodies with Thin Coverings and Inclusions*. Nauka, Moscow (1983).
- Benveniste, Y.; Miloh, T.: Imperfect soft and stiff interfaces in two-dimensional elasticity. *Mech. Mater.*, 33, (2001), 309-324.
- Chen, X.; Liu Y.: Multiple-cell modelling of fiber-reinforced composites with the presence of interphases using the boundary element method. *Comput. Mater. Science*, 21, (2001), 86-94.
- Christensen, R.M.: *Mechanics of Composite Materials*. Dover Publications, Mineola, New York (2005).
- Fadle, J.: Die Selbstspannungs-Eigenwertfunktionen der quadratischen Scheibe. *Öster. Ing.-Arch.*, 11(2), (1940), 125-149.
- Geymonat, G.; Krasucki, F.; Lenci, S.: Mathematical analysis of a bonded joint with a soft thin adhesive. *Math. Mech. Solids*, 4, (1999), 201-225.
- Golland, M.; Reissner, E.: The stresses in cemented joints. *J. Appl. Mech.*, 11, (1944), A17-A27.
- Hashin, Z.: Thin interphase imperfect interface in elasticity with application to coated fiber composites. *J. Mech. Phys. Solids*, 50, (2002), 2509-2537.
- Jasiuk, I.; Kouider, M.W.: The effect of an inhomogeneous interphase on the elastic constants of transversely isotropic composites. *Mech. Mater.*, 15, (1993), 53-63.
- Klarbring, A.: Deviation of a model of adhesively bonded joints by the asymptotic expansion method. *Int. J. Eng. Sci.*, 29, (1991), 493-512.

- Kovalenko, E.V.: Contact problems for coated solids. In: *Mechanics of Contact Interactions*, eds. Vorovich, I.I. and Alexandrov, V.M. Fizmatlit, Moscow (2001), 459-475.
- Krasucki, F.; Lenci, S.: Analysis of interfaces of variable stiffness. *Int. J. Solids Str.*, 37, (2000, a), 3619-3632.
- Krasucki, F.; Lenci, S.: Yield design of bonded joints. *Eur. J. Mech. A/ Solids*, 19, (2000, b), 649-667.
- Lagache, M.; Agbossou, A.; Pastor, J.; Muller, D.: Role of interphase on the elastic behavior of composite materials: theoretical and experimental analysis. *J. Comp. Mater.*, 28, (1994), 1140-1157.
- Lenci, S.: Bonded joints with nonhomogeneous adhesives. *J. Elasticity*, 53, (1999), 23-35.
- Little, R.W.: Semi-infinite strip problem with built-in edges. *J. Appl. Mech.*, 36 (2), (1969), 320-323.
- Lucas da Silva, F.M.; Paulo das Neves, J.C.; Adams, R.D.; A. Wang A.; Spelt, J.K.: Analytical models of adhesively bonded joints – Part I: Literature survey. *Int. J. Adhesion Adhesives*, 29(3), (2009a), 319-330.
- Lucas da Silva, F.M.; Paulo das Neves, J.C.; Adams, R.D.; A. Wang A.; Spelt, J.K.: Analytical models of adhesively bonded joints – Part II: Comparative study. *Int. J. Adhesion Adhesives*, 29(3), (2009b), 331-341.
- Milton, G.W.: *The Theory of Composites*. Cambridge University Press, New York (2002).
- Mishuris, G.; Öchsner, A.: Edge effects connected with thin interfaces in composite materials. *Comp. Str.*, 68, (2005), 409-417.
- Van Fo Fy, G.A.: *Theory of Reinforced Materials with Coatings*. Naukova Dumka, Kyiv (1971).
- Papkovich, P.F.: On the form of solution of the plane problem of the theory of elasticity for a rectangular strip. *Dokl. Akad. Nauk SSSR*, 27, (1940), 335-339; German cover-to-cover translation: Über eine Form der Lösung des biharmonischen Problem für das Rechteck. *CR (Doklady) Acad. Sci. URSS*, 27, (1940), 334-338.
- Rössle, A.: Corner singularities and regularity of weak solutions for the two-dimensional Lamé equations on domains with angular corners. *J. Elasticity*, 60, (2000), 57-75.
- Sinclair, G.B.: Stress singularities in classical elasticity – I: Removal, interpretation and analysis. *Appl. Mech. Rev.*, 57(4), (2004, a), 251-297.
- Sinclair, G.B.: Stress singularities in classical elasticity – II: Asymptotic identification. *Appl. Mech. Rev.*, 57(5), (2004, b), 385-439.

Addresses: Prof. Dr. Sc. Igor V. Andrianov (corresponding author), Institute of General Mechanics RWTH Aachen University, Templergraben 64, D-52056, Aachen, Germany
email: igor_andrianov@hotmail.com

Dr. Galina A. Starushenko, Dnipropetrovs'k Regional Institute of State Management of National Academy of State Management at the President of Ukraine, 29 Gogolya Str., UA-49631, Dnipropetrovs'k, Ukraine
email: gala_star@km.ru

Prof., Dr.-Ing. Dieter Weichert, Institute of General Mechanics RWTH Aachen University, Templergraben 64, D-52056, Aachen, Germany
email: weichert@iam.rwth-aachen.de