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A Necessary and Sufficient Condition for Solving a Rigid Body Problem

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In this paper, the motion of a rigid body about a fixed point under the influence of a Newtonian force field is investigated. The Euler-Poisson equations are used to represent that motion. Three first integrals of these equations are well known. The exact solutions of these equations require, in general, a fourth algebraic first integral. The necessary and sufficient condition for some functions to be a fourth first integral of the governing equations is obtained.

1 Introduction

The problem of the motion of a rigid body about a fixed point in a uniform force field or in a Newtonian one has attracted the interest of many researchers during the last two centuries (Arkhangel'skii 1963; Kharlamov 1964; Keis 1964; Gorr et al. 2002; Gao 2003; Burov 2003; Burov 2005; Kuleshov 2006). This motion is governed by six non-linear differential equations with three first integrals, Arkhangel'skii (1963). Many attempts were made by outstanding scientists to find the solution of these equations but they have not found it in its full generality, except for three special cases (Euler-Poinsot, Lagrange-Poisson and Kovalevskaya). These cases have certain restrictions on the location of the body's centre of mass and on the values of the principal moments of inertia (Arkhangel'skii 1963; Kharlamov 1964; Keis 1964). Arkhangel'skii (1963) showed that this fourth algebraic integral exists only in two special cases analogous to those of Euler and Lagrange, and that other cases with single-valued integrals are not independent cases but it can be reduced to previous cases. Gorr et al. (2002) obtained new ways of integrating Poisson's equations, which correspond to the case when a fractionally linear first integral exists in these equations. Gao (2003) extended Kovalevskaya's work in terms of hyperelliptic functions. In Burov 2003; Burov 2005, the problem of the existence of integrable cases for the motion of a heavy rigid body is studied. The author considered the integrable cases of the Euler and Lagrange types and some particular cases. An explicit form of two first integrals of the equations of motion of the gyrostat is presented in Kuleshov (2006). In this work, we obtain a necessary and sufficient condition for some functions $F(p, q, r, \gamma, \gamma', \gamma'')$ to be a first integral for the Euler- Poisson equations when the motion of a rigid body is acted upon by a central Newtonian force field.

2 Formulation of the Problem

Consider a rigid body of mass M, its center of mass is c, a point O fixed in that body, whose ellipsoid of inertia is arbitrary and its center of mass does not coincide with the fixed point. Let the body be subjected to a Newtonian potential field exerted by an attracting center located on a fixed vertical downward OZ - axis at distance R from the point O. Choosing the axes OX, OY and OZ to represent a fixed frame in space and the axes Ox, Oy and Oz to represent the principal axes of the ellipsoid of inertia constructed for the body at the fixed point O, which rotate in space with the same angular velocity of the body. Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ be the unit vectors in the direction of the moving axes Ox, Oy and Oz, respectively, and $\underline{\hat{K}}$ be the unit vector in the direction of the fixed vertical OZ - axis, its direction cosines with respect to the moving axes are γ, γ' and γ'', respectively. So one writes

$$\underline{\hat{K}} = \gamma\,\underline{i} + \gamma'\,\underline{j} + \gamma''\,\underline{k}. \tag{1}$$

The position vector of the center of mass is defined as follow

$$\underline{r_c} = x_c\, \underline{i} + y_c\, \underline{j} + z_c\, \underline{k}. \tag{2}$$

Let $\underline{\omega}$ be the angular velocity vector of the body with components p, q and r, expressed with respect to the basis $\underline{i}, \underline{j}$ and $\underline{k}$,

$$\underline{\omega} = p\ \underline{i} + q\, \underline{j} + r\, \underline{k}. \tag{3}$$

To describe the motion of the rigid body about the fixed point O, the Euler - Poisson equations are used (Arkhangel'skii 1977)

$$\begin{aligned} A\frac{dp}{dt} &= (B-C)qr + (h_2 - H_2)\gamma'' - (h_3 - H_3)\gamma', \\ \frac{d\gamma}{dt} &= r\gamma' - q\gamma'', \end{aligned} \tag{4}$$

where

$$\begin{aligned} &h_1 = M\,g\,x_G, \qquad h_2 = M\,g\,y_G, \qquad h_3 = M\,g\,z_G, \\ &H_1 = \alpha\, A\gamma, \qquad H_2 = \alpha\, B\gamma', \qquad H_3 = \alpha\, C\gamma'', \qquad \alpha = \frac{3g}{R}. \end{aligned}$$

The other equations for q, r and γ', γ'' are obtained from those for p and γ by cyclic exchange of $(A, B, C), (p, q, r), (\gamma, \gamma', \gamma''), (H_1, H_2, H_3)$ and (h_1, h_2, h_3). Here, A, B, C are the principal moments of inertia and g is the acceleration due to gravity. System (4) is a simple autonomous system of six non-linear ordinary differential equations. The three well known first integrals of this system are the conservation of energy, the constant kinetic moment about the vertical axis and the geometrical constraint on the direction cosines $(\gamma, \gamma', \gamma'')$

$$\begin{aligned} L_1 &= \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) - Mg[x_G\gamma + y_G\gamma' + z_G\gamma''] + \frac{\alpha}{2}(A\gamma^2 + B\gamma'^2 + C\gamma''^2) = C_1, \\ L_2 &= Ap\gamma + Bq\gamma' + Cr\gamma'' = C_2, \\ L_3 &= \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \end{aligned} \tag{5}$$

where C_1 and C_2 are constants.

System (4) has the Jacobi last multiplier $M \equiv 1$. For that case holds: If four integrals are known, the fifth one follows by a quadrature see, Leimanis (1965). Many attempts were made by outstanding scientists during the last two centuries to realize such the fourth first integral. These attempts required operations of the results which could not be expressed explicitly in the general case except for two known special cases namely the Euler case $(x_c = y_c = z_c = 0)$ and the Lagrange one $(A = B,\ x_c = y_c = 0)$. These cases have certain restrictions on the location of the body's center of mass c and on the values of the principal moments of inertia, Arkhangel'skii (1963).

3 General identity for the first integrals

Let the function $F(p, q, r, \gamma, \gamma', \gamma'') = const.$ be a first integral of system (4). So one can write

$$\frac{dF}{dt}=\frac{\partial F}{\partial p}\frac{dp}{dt}+\frac{\partial F}{\partial q}\frac{dq}{dt}+\frac{\partial F}{\partial r}\frac{dr}{dt}+\frac{\partial F}{\partial \gamma}\frac{d\gamma}{dt}+\frac{\partial F}{\partial \gamma'}\frac{d\gamma'}{dt}+\frac{\partial F}{\partial \gamma''}\frac{d\gamma''}{dt}=0. \tag{7}$$

Furthermore, the geometric integral can be rewritten in the following way

$$\gamma^2+\gamma'^2+\gamma''^2=C_3, \tag{8}$$

where C_3 is an arbitrary positive constant. This assumption does not introduce any discrepancies into the equations since the derivative of (8) is null.

Equation (8) allows the initial conditions $(\gamma_0, \gamma_0', \gamma_0'')$ to be arbitrary. This fact will be used below. However, having found the fourth first integral, nothing stands in the way of substituting $C_3=1$. Equation (7) has to be considered as an identity with respect to the variables p, q, r, γ, γ' and γ'' (Goursat 1959). Indeed it has to be satisfied for each solution of equations (4) at any moment in time and at the initial moment t_0, as well. However, at the initial moment, the values $p_0, q_0, r_0, \gamma_0, \gamma_0'$ and γ_0'' are entirely arbitrary. Hence equation (7) is indeed an identity. Thus, the partial derivation of (7) with respect to each of variables p, q, r, γ, γ' and γ'' is legitimate. In fact, Equation (7) is a partial differential equation for the function $F(p, q, r, \gamma, \gamma', \gamma'')$. This equation has five independent solutions. Three of them are the functions (5). To find a fourth first integral means to find a solution of the partial differential equation (7) independent of the solutions (5). We can give new partial differential equations for the fourth first integral alone. The functions (5) are not solutions of these equations, i.e., the three first integrals are eliminated from equation (7).

4 Formulation of the Theorem

Theorem: A necessary and sufficient condition for the function $F=F(p,q,r,\gamma,\gamma',\gamma'')$ to be a new first integral of system (4) is the following system of linear inhomogeneous partial differential equations

$$\begin{aligned}
U_1&=\Omega_1-AD[\Gamma_1(p\mu_1+\gamma\delta_1)+\Gamma_2(p\mu_2+\gamma\delta_2)+\Gamma_3(p\mu_3+\gamma\delta_3)],\\
U_2&=\Omega_2-BD[\Gamma_1(q\mu_1+\gamma'\delta_1)+\Gamma_2(q\mu_2+\gamma'\delta_2)+\Gamma_3(q\mu_3+\gamma'\delta_3)],\\
U_3&=\Omega_3-CD[\Gamma_1(r\mu_1+\gamma''\delta_1)+\Gamma_2(r\mu_2+\gamma''\delta_2)+\Gamma_3(r\mu_3+\gamma''\delta_3)],
\end{aligned} \tag{9}$$

where Ω_i and Γ_i $(i=1, 2, 3)$ are the corresponding partial derivatives of the function F, i.e.,

$$\Omega_1=\frac{\partial F}{\partial p},\quad \Omega_2=\frac{\partial F}{\partial q},\quad \Omega_3=\frac{\partial F}{\partial r},\quad \Gamma_1=\frac{\partial F}{\partial \gamma},\quad \Gamma_2=\frac{\partial F}{\partial \gamma'},\quad \Gamma_3=\frac{\partial F}{\partial \gamma''}. \tag{10}$$

Here, the functions U_i $(i=1, 2, 3)$ form a non-zero solution of the following system

$$\begin{aligned}
\frac{dU_1}{dt}&=\frac{A-C}{B}rU_2+\frac{B-A}{C}qU_3\\
&\quad -AD[F_1(p\mu_1+\gamma\delta_1)+F_2(p\mu_2+\gamma\delta_2)+F_3(p\mu_3+\gamma\delta_3)],\\
\frac{dU_2}{dt}&=\frac{B-A}{C}pU_3+\frac{C-B}{A}rU_1\\
&\quad -BD[F_1(q\mu_1+\gamma'\delta_1)+F_2(q\mu_2+\gamma'\delta_2)+F_3(q\mu_3+\gamma'\delta_3)],\\
\frac{dU_3}{dt}&=\frac{C-B}{A}qU_1+\frac{A-C}{B}pU_2\\
&\quad -CD[F_1(r\mu_1+\gamma''\delta_1)+F_2(r\mu_2+\gamma''\delta_2)+F_3(r\mu_3+\gamma''\delta_3)],
\end{aligned} \tag{11}$$

and satisfying the condition

$$U_1 \frac{dp}{dt} + U_2 \frac{dq}{dt} + U_3 \frac{dr}{dt} = 0, \tag{12}$$

where

$$F_1 = S_1(\dot{d}_1 + \frac{A}{B} H_2 r - \frac{A}{C} H_3 q) + S_2(\frac{A}{B} H_2 \gamma'' - \frac{A}{C} H_3 \gamma') + \frac{a_2}{C} U_3 - \frac{b_3}{B} U_2,$$
$$F_2 = S_1(\dot{d}_2 + \frac{B}{C} H_3 p - \frac{B}{A} H_1 r) + S_2(\frac{B}{C} H_3 \gamma - \frac{B}{A} H_1 \gamma'') + \frac{a_3}{A} U_1 - \frac{b_1}{C} U_3,$$
$$F_3 = S_1(\dot{d}_3 + \frac{C}{A} H_1 q - \frac{C}{B} H_2 p) + S_2(\frac{C}{A} H_1 \gamma' - \frac{C}{B} H_2 \gamma) + \frac{a_1}{B} U_2 - \frac{b_2}{A} U_1,$$
$$a_1 = h_1 + (\frac{C}{A} - 1) H_1, \qquad a_2 = h_2 + (\frac{A}{B} - 1) H_2, \qquad a_3 = h_3 + (\frac{B}{C} - 1) H_3,$$
$$b_1 = h_1 + (\frac{B}{A} - 1) H_1, \qquad b_2 = h_2 + (\frac{C}{B} - 1) H_2, \qquad b_3 = h_3 + (\frac{A}{C} - 1) H_3,$$
$$d_1 = h_1 - H_1, \qquad d_2 = h_2 - H_2, \qquad d_3 = h_3 - H_3,$$
$$\mu_1 = Bq\gamma'' - Cr\gamma', \qquad \mu_2 = Cr\gamma - Ap\gamma'', \qquad \mu_3 = Ap\gamma' - Bq\gamma,$$
$$\delta_1 = d_2\gamma'' - d_3\gamma', \qquad \delta_2 = d_3\gamma - d_1\gamma'', \qquad \delta_3 = d_1\gamma' - d_2\gamma,$$
$$D = -1/(d_1\mu_1 + d_2\mu_2 + d_3\mu_3).$$

Proof:

Let the function $F = F(p, q, r, \gamma, \gamma', \gamma'')$ be a new first integral of system (4), then using partial differential equation (7), we get

$$\Omega_1 \frac{dp}{dt} + \Omega_2 \frac{dq}{dt} + \Omega_3 \frac{dr}{dt} + \Gamma_1 \frac{d\gamma}{dt} + \Gamma_2 \frac{d\gamma'}{dt} + \Gamma_3 \frac{d\gamma''}{dt} = 0. \tag{13}$$

This equation is considered as an identity with respect to the variables p, q, r, γ, γ' and γ'' and can be differentiated with respect to these variables. Partial derivation of (13) with respect to p, taking (4) into account, leads to

$$\begin{aligned} &\frac{\partial \Omega_1}{\partial p} \frac{dp}{dt} + \frac{\partial \Omega_2}{\partial p} \frac{dq}{dt} + \frac{\partial \Omega_3}{\partial p} \frac{dr}{dt} + \frac{\partial \Gamma_1}{\partial p} \frac{d\gamma}{dt} + \frac{\partial \Gamma_2}{\partial p} \frac{d\gamma'}{dt} + \frac{\partial \Gamma_3}{\partial p} \frac{d\gamma''}{dt} = \frac{A - C}{B} r \Omega_2 \\ &+ \frac{B - A}{C} q \Omega_3 - \Gamma_2 \gamma'' + \Gamma_3 \gamma'. \end{aligned} \tag{14}$$

Because of the equality between the second mixed derivatives of the function F, equation (14) can be written down as follows

$$\begin{aligned} &\frac{\partial \Omega_1}{\partial p} \frac{dp}{dt} + \frac{\partial \Omega_1}{\partial q} \frac{dq}{dt} + \frac{\partial \Omega_1}{\partial r} \frac{dr}{dt} + \frac{\partial \Omega_1}{\partial \gamma} \frac{d\gamma}{dt} + \frac{\partial \Omega_1}{\partial \gamma'} \frac{d\gamma'}{dt} + \frac{\partial \Omega_1}{\partial \gamma''} \frac{d\gamma''}{dt} = \frac{A - C}{B} r \Omega_2 \\ &+ \frac{B - A}{C} q \Omega_3 - \gamma'' \Gamma_2 + \gamma' \Gamma_3. \end{aligned} \tag{15}$$

The left hand side of this equality is $d\Omega_1/dt$. Partial derivation of (13) in the same way with respect to the other variables, leads to the following system of six partial differential equations with respect to the functions Ω_i and F_i $(i = 1, 2, 3)$

$$\begin{aligned}
\frac{d\Omega_1}{dt} &= \frac{A-C}{B} r\Omega_2 + \frac{B-A}{C} q\Omega_3 - \gamma''\Gamma_2 + \gamma'\Gamma_3, \\
\frac{d\Omega_2}{dt} &= \frac{B-A}{C} p\Omega_3 + \frac{C-B}{A} r\Omega_1 - \gamma\,\Gamma_3 + \gamma''\Gamma_1, \\
\frac{d\Omega_3}{dt} &= \frac{C-B}{A} q\Omega_1 + \frac{A-C}{B} p\Omega_2 - \gamma'\Gamma_1 + \gamma\,\Gamma_2. \\
\frac{d\Gamma_1}{dt} &= \frac{a_2}{C}\Omega_3 - \frac{b_3}{B}\Omega_2 + r\,\Gamma_2 - q\,\Gamma_3, \\
\frac{d\Gamma_2}{dt} &= \frac{a_3}{A}\Omega_1 - \frac{b_1}{C}\Omega_3 + p\,\Gamma_3 - r\,\Gamma_1, \\
\frac{d\Gamma_3}{dt} &= \frac{a_1}{B}\Omega_2 - \frac{b_2}{A}\Omega_1 + q\,\Gamma_1 - p\,\Gamma_2.
\end{aligned} \tag{16}$$

It is obvious that the vectors composed of the corresponding partial derivatives of each autonomous first integral of system (4) are solutions of system (16).

Consider the following substitution in system (16)

$$\begin{aligned}
\Omega_1 &= A\,p\,S_1 + A\gamma\,S_2 + U_1, & \Gamma_1 &= -d_1S_1 + A\,p\,S_2 + \gamma\,S_3, \\
\Omega_2 &= B\,q\,S_1 + B\gamma'\,S_2 + U_2, & \Gamma_2 &= -d_2S_1 + B\,q\,S_2 + \gamma'\,S_3, \\
\Omega_3 &= C\,r\,S_1 + C\gamma''\,S_2 + U_3, & \Gamma_3 &= -d_3S_1 + C\,r\,S_2 + \gamma''\,S_3,
\end{aligned} \tag{17}$$

where $S_1, S_2, S_3, U_1, U_2, U_3$ can be grasped as auxiliary function of the time.

Solving (17) for $S_1, S_2, S_3, U_1, U_2, U_3$ leads by (16) to the time derivatives, we have

$$\begin{aligned}
\frac{dU_1}{dt} &= \frac{A-C}{B} rU_2 + \frac{B-A}{C} qU_3 \\
&\quad - A\,D[F_1(p\mu_1 + \gamma\,\delta_1) + F_2(p\mu_2 + \gamma\,\delta_2) + F_3(p\mu_3 + \gamma\,\delta_3)], \\
\frac{dU_2}{dt} &= \frac{B-A}{C} pU_3 + \frac{C-B}{A} rU_1 \\
&\quad - B\,D[F_1(q\mu_1 + \gamma'\delta_1) + F_2(q\mu_2 + \gamma'\delta_2) + F_3(q\mu_3 + \gamma'\delta_3)], \\
\frac{dU_3}{dt} &= \frac{C-B}{A} qU_1 + \frac{A-C}{B} pU_2 \\
&\quad - C\,D[F_1(r\mu_1 + \gamma''\delta_1) + F_2(r\mu_2 + \gamma''\delta_2) + F_3(r\mu_3 + \gamma''\delta_3)],
\end{aligned} \tag{18.a}$$

and

$$\begin{aligned}
\frac{dS_1}{dt} &= D(F_1\mu_1 + F_2\mu_2 + F_3\mu_3), \\
\frac{dS_2}{dt} &= D(F_1\delta_1 + F_2\delta_2 + F_3\delta_3), \\
\frac{dS_3}{dt} &= D(F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3).
\end{aligned} \tag{18.b}$$

The obtained system (18) can be treated as a system of five equations because the right-hand sides of (18) do not depend on S_3. Actually these five equations are the first five equations of (18).

Taking (17) into account, it is not difficult to prove that (9) and (12) are satisfied. Indeed, solving (17) with respect to $U_i \quad (i = 1, 2, 3)$, one obtains (9). After multiplying the left three equations of (17) by $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}$ and the right three equations by $\frac{d\gamma}{dt}, \frac{d\gamma'}{dt}, \frac{d\gamma''}{dt}$ respectively and summing, one gets

$$\Omega_1 \frac{dp}{dt} + \Omega_2 \frac{dq}{dt} + \Omega_3 \frac{dr}{dt} + \Gamma_1 \frac{dr}{dt} + \Gamma_2 \frac{d\gamma'}{dt} + \Gamma_3 \frac{d\gamma''}{dt} = S_1 \frac{dL_1}{dt} + S_2 \frac{dL_2}{dt} + S_3 \frac{dL_3}{dt} + U_1 \frac{dp}{dt} + U_2 \frac{dq}{dt} + U_3 \frac{dr}{dt}. \quad (19)$$

According to (13) the left-hand side of (19) equals zero. The expressions $dL_i/dt \quad (i = 1, 2, 3)$ equal zero according to (5). Hence (12) is satisfied too. To complete the proof of necessity, we have only to prove that $U_i \equiv 0 \quad (i = 1, 2, 3)$ is impossible. We give a proof by contradiction. Let us suppose $U_i = 0 \quad (i = 1, 2, 3)$, then system (9) becomes a homogenous system of linear partial differential equations for the function F, i.e.,

$$\begin{aligned} Z_1(F) &= \Omega_1 - AD[\Gamma_1(p\mu_1 + \gamma\,\delta_1) + \Gamma_2(p\mu_2 + \gamma\,\delta_2) + \Gamma_3(p\mu_3 + \gamma\,\delta_3)] = 0, \\ Z_2(F) &= \Omega_2 - BD[\Gamma_1(q\mu_1 + \gamma'\delta_1) + \Gamma_2(q\mu_2 + \gamma'\delta_2) + \Gamma_3(q\mu_3 + \gamma'\delta_3)] = 0, \\ Z_3(F) &= \Omega_3 - CD[\Gamma_1(r\mu_1 + \gamma''\delta_1) + \Gamma_2(r\mu_2 + \gamma''\delta_2) + \Gamma_3(r\mu_3 + \gamma''\delta_3)] = 0. \end{aligned} \quad (20)$$

The expressions $Z_j[Z_k(F)] - Z_k[Z_j(F)]$ are Poisson brackets for the operators $Z_k(F)$ and $Z_j(F) \quad (j,\ k = 1, 2, 3,\ \ j > k)$. Having composed all kinds of Poisson brackets for system (20), it turns out to be a complete system (Goursat 1959). There exists a theorem that every complete system of k equations in $n-$independent variables has $(n-k)$ independent integrals, and the general integral of the system is an arbitrary function of these $(n-k)$ particular integrals. Applying this theorem, we conclude that system (20) has three independent solutions and any other solution is a function of them. It is not difficult to verify that the first integrals (5) satisfy system (20). Hence the general solution of (20) is represented as an arbitrary function of first integral (5). However, the function F, according to the condition of the theorem, is a new first integral of system (4) and therefore it can not be expressed as a function of the integrals (5). Thus $U_i = 0 \quad (i = 1, 2, 3)$ is impossible, i.e., $U_i = 0 \quad (i = 1, 2, 3)$ form a non- zero solution of system (11). Thus, the necessity of the theorem is proved.

Let equations (9) and (12) be satisfied by a function $F = F(p,\ q,\ r,\ \gamma,\ \gamma',\ \gamma'')$. There exist such functions. Indeed, as we have seen by the proof of the necessity of the theorem, the new first integral of system (4) is such a function. After multiplying equations (9) by $\frac{dp}{dt}, \frac{dq}{dt}$ and $\frac{dr}{dt}$ respectively and summing, one has

$$\Omega_1 \frac{dp}{dt} + \Omega_2 \frac{dq}{dt} + \Omega_3 \frac{dr}{dt} + \Gamma_1 \frac{dr}{dt} + \Gamma_2 \frac{d\gamma'}{dt} + \Gamma_3 \frac{d\gamma''}{dt} = 0. \quad (21)$$

Equation (21) shows that the function F is a first integral of system (4). We will prove that this first integral is the fourth first integral, i.e., that the function F is independent of first integrals (5). We give a proof by contradiction. Let us suppose that the first integral obtained depends on the integral (5), then it will satisfy (9), provided that $U_i = 0 \quad (i = 1, 2, 3)$. This is a contradiction because, according to the condition of the theorem $U_i = 0 \quad (i = 1, 2, 3)$ form a non-zero solution of (11). The proof of the theorem is completed.

5 Example

As an illustration of this approach, the Euler, Lagrange and kinetic symmetry cases have been examined.

(i) Euler's case $(x_c = y_c = z_c = 0)$ **:**

The following functions satisfy system (18)

$$
\begin{aligned}
U_1 &= 2A[Ap + \alpha D(p(BC\gamma\,\mu_1 + AC\gamma'\mu_2 + AB\gamma''\mu_3) \\
&\quad + \gamma(BC\gamma\,\delta_1 + AC\gamma'\delta_2 + AB\gamma''\delta_3))], \\
U_2 &= 2B[Bq + \alpha D(q(BC\gamma\,\mu_1 + AC\gamma'\mu_2 + AB\gamma''\mu_3) \\
&\quad + \gamma'(BC\gamma\,\delta_1 + AC\gamma'\delta_2 + AB\gamma''\delta_3))], \\
U_3 &= 2C[Cr + \alpha D(r(BC\gamma\,\mu_1 + AC\gamma'\mu_2 + AB\gamma''\mu_3) \\
&\quad + \gamma''(BC\gamma\,\delta_1 + AC\gamma'\delta_2 + AB\gamma''\delta_3))], \\
S_1 &= -2\alpha D(BC\gamma\,\mu_1 + AC\gamma'\mu_2 + AB\gamma''\mu_3), \\
S_2 &= -2\alpha D(BC\gamma\,\delta_1 + AC\gamma'\delta_2 + AB\gamma''\delta_3), \\
S_3 &= -2\alpha D(BC\gamma\,\sigma_1 + AC\gamma'\sigma_2 + AB\gamma''\sigma_3).
\end{aligned} \tag{22}
$$

Using (17), we obtain the partial derivatives of the fourth first integral as follow

$$
\begin{aligned}
&\Omega_1 = 2A^2 p, \qquad &&\Omega_2 = 2B^2 q, \qquad &&\Omega_3 = 2C^2 r, \\
&\Gamma_1 = -2\alpha\, BC\gamma, \qquad &&\Gamma_2 = -2\alpha\, AC\gamma', \qquad &&\Gamma_3 = -2\alpha\, AB\gamma''.
\end{aligned} \tag{23}
$$

Thus, the fourth first integral is

$$F = A^2 p^2 + B^2 q^2 + C^2 r^2 - \alpha\,(BC\gamma^2 + AC\gamma'^2 + AB\gamma''^2). \tag{24}$$

(ii) Lagrange's case $(A = B,\ x_c = y_c = 0)$ **:**
Also, the functions

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = 1, \quad S_1 = 0, \quad S_2 = 0, \quad S_3 = 0, \tag{25}$$

satisfy system (18), then using (17), we obtain the partial derivatives of the fourth first integral as follow

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = 1, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0, \quad \Gamma_3 = 0. \tag{26}$$

Thus, the fourth first integral is

$$F = r. \tag{27}$$

This result agrees with that obtained in Arkhangel'skii (1977).

(iii) Kinetic symmetry case
Using the same manner as cases (i) and (ii), the functions

$$U_1 = x_c, \quad U_2 = y_c, \quad U_3 = z_c, \quad S_1 = 0, \quad S_2 = 0, \quad S_3 = 0, \tag{28}$$

satisfy system (18). Using (17), we obtain the partial derivatives of the fourth first integral as follow

$$\Omega_1 = x_c, \quad \Omega_2 = y_c, \quad \Omega_3 = z_c, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0, \quad \Gamma_3 = 0. \tag{29}$$

The fourth first integral obtained as follow

$$F = x_c p + y_c q + z_c r. \tag{30}$$

6 Conclusion

The equations of motion of a rigid body about a fixed point in a Newtonian force field are investigated. The necessary and sufficient condition for some functions F to be a fourth first integral of the governing equations is obtained. The condition for the fourth first integral is checked by applying it to some special cases (in a uniform field and a Newtonian one). This study is considered as a generalization of some previous studies such as Arkhangel'skii (1963) and Popov (1990).

7 References

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