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Thermodynamic Consistency of Two-mechanism Models in the Non-isothermal Case

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This note investigates two-mechanism models (= 2M models) in the case of plastic behavior. 2M models (or, generally, multi-mechanism models) are a useful tool for modelling of complex material behavior. They have been studied and applied for the last twenty years. We prove thermodynamic consistency for some classes of 2M models, and we derive new coupled evolution equations for the back stresses. Moreover, a coupling in the evolution equations of the internal variables is presented. Finally, a comparison between a 2M model and a modified Chaboche model is presented in order to illustrate the possibilities and problems in modelling of complex material behavior like ratcheting.

1 Introduction

1) Two-mechanism (or, generally, multi-mechanism) models have been studied for the last twenty years. Their characteristic trait is the additive decomposition of the inelastic (i.e., plastic or visco-plastic, e.g.) strain into two (or more) parts (sometimes called "mechanisms") in the case of small deformations. In comparison with rheological models (cf. Palmov (1998), e.g.), there is an interaction between these mechanisms (see Figure 1). This interaction allows to describe important observable effects, but it requires additional efforts in modelling and simulation. Each inelastic strain part may exhibit plastic or general inelastic behavior. The (thermo-)elastic strain is usually not considered as an own mechanism. Each mechanism has its own internal variables with corresponding evolution equations. Moreover, each mechanism may have its own yield criterion, or there may be a common yield criterion for several mechanisms. Thus, in the case of two mechanisms, there are models of the type 2M1C and 2M2C ("2 mechanisms with 1 (yield-)or 2 (yield) criteria", see Figure 2). A mechanism without yield criterion like creep can be formally treated as a mechanism with its own criterion with zero yield stress.

If the inelastic strain is seen as one mechanism (as it was historically first), one refers to a "unified model" (or "Chaboche" model) (cf. the survey by Chaboche (2008) and the references cited therein). In this case plastic and viscous components are considered together in the same variable. As explained in Contesti and Cailletaud (1989) and Cailletaud and Saï (1995), there are experimentally observable effects (inverse strain-rate sensibility, e.g.) which can be qualitatively correctly described by the two-mechanism approach.

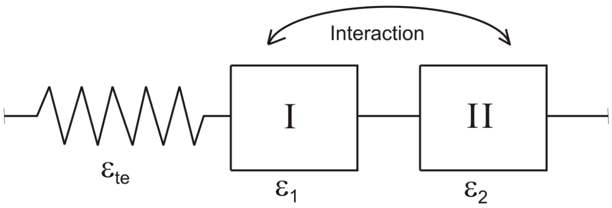


Figure 1: Scheme of a two-mechanism model. The two inelastic mechanisms 1 and 2 have their own evolution equations. But they are not independent from each other. The thermoelastic strain ε_{te} is usually not regarded as a mechanism.

2) Up to now, there are only relatively few publications dealing directly with multi-mechanism models. We refer to Contesti and Cailletaud (1989), Saï (1993), Cailletaud and Saï (1993), Cailletaud and Saï (1995), Blaj and Cailletaud (2000), Besson et al. (2001), Saï et al. (2004), Aeby-Gautier and Cailletaud (2004), Taleb et al. (2006), Velay et al. (2006), Saï and Cailletaud (2007), Wolff and Taleb (2008), Chaboche (2008), Wolff et al. (2008), Hassan et al. (2008), Taleb and Hauet (2009), Taleb and Cailletaud (2010), Wolff et al. (2010), Saï (2011). In contrary to this manageable number, there is a large variety of papers dealing with complex material behavior of metals, soils, composites, biological tissues etc. in which the inelastic strain is decomposed into several parts. But, as a rule, multi-mechanism models are not directly addressed. We give some examples below.

To our knowledge, a first systematic formulation and investigation of two mechanism models was given by Contesti and Cailletaud (1989). Besides, the papers by Cailletaud and Saï (1995), by Saï and Cailletaud (2007), and, by Taleb and Cailletaud (2010) give overviews and show applications. In Saï (2011), one can find the current state of art of 2M models. The report Wolff et al. (2010) contains detailed explanations of 2M models and accents the mathematical and continuum-mechanical framework. Moreover, we refer to the thesis of Saï (1993) and to the book by Besson et al. (2001). The survey article by Chaboche (2008) contains comments concerning multi-mechanism models, too.

Wolff and Taleb (2008) proved thermodynamic consistency of two-mechanism models dealt with in Taleb et al. (2006). The question about thermodynamic consistency is not trivial, if one leaves the class of "generalized standard models" (cf. Besson et al. (2001), e.g.). This is the case for important model modifications (cf. Taleb et al. (2006), Saï and Cailletaud (2007)). Additionally, there is the typical *mutual influence of mechanisms* (in particular via the back stresses). Thus, generally, a separate investigation of thermodynamic consistency with respect to each mechanism is not successful. This is a substantial difference to rheologic models (cf. Palmov (1998), e.g.). Generally, the material parameters depend on temperature. Most of the papers about multi-mechanism models cited above only consider the isothermal case, as ratcheting experiments, up to now, are only performed under constant temperature. In the current paper we will also address the *non-isothermal* case. This leads to more complex equations at some places.

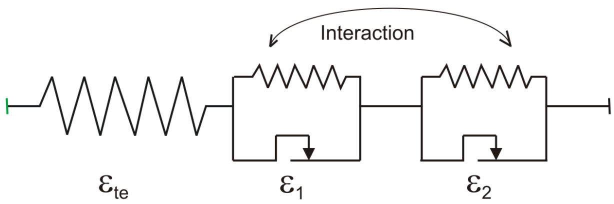


Figure 2: 2M2C model with two plastic mechanisms with kinematic hardening.

3) An important application of two-mechanism models is cyclic plasticity including ratcheting. There are many papers dealing with ratcheting both in modelling as well as in simulation and comparison with experimental data. For general modelling and simulation we exemplarily refer to Portier et al. (2000), Bari and Hassan (2002), Taleb et al. (2006), Kang (2008), Jiang and Zhang (2008), Hassan et al. (2008), Abdel-Karim (2009), Taleb and Hauet (2009), Krishna et al. (2009), Abdel-Karim (2010) and the references therein. In the majority of the literature ratcheting is dealt within the framework of one-mechanism models. Investigations of ratcheting with the aid of two-mechanism models can be found in Cailletaud and Saï (1995), Blaj and Cailletaud (2000), Saï et al. (2004) [using a 2M2C model], Taleb et al. (2006), Velay et al. (2006), Saï and Cailletaud (2007), Hassan et al. (2008), Taleb and Hauet (2009), Taleb and Cailletaud (2010), Saï (2011). Finally, experiments and simulations must decide, in which situation which model delivers the better approximation of the reality. In Hassan et al. (2008), a direct comparison between a modified Chaboche model and a 2M model has been performed (See Section 7).

4) Another important application of two-mechanism models lies in modelling of complex material behavior of steel under phase transformations. The two-mechanism approach directly used in Videau et al. (1994) and Wolff et al. (2008) allows a good description of interactions between classical and transformation-induced plasticity. On the other hand, in Leblond et al. (1986a), Leblond et al. (1986b), Leblond et al. (1989), Leblond (1989), Fischer et al. (1998), Fischer et al. (2000), Devaux et al. (2000), Taleb and Sidoroff (2003), the transformation-induced plasticity itself is the focus, and the two-mechanism approach arises in a natural way without a special reference. More recent experiments and simulations (cf. Taleb and Petit (2006), e.g.) show that, in some cases, the transformation-induced plasticity after a pre-deformation of austenite *cannot* be qualitatively correctly described with the aid of the model developed in Leblond et al. (1986a), Leblond et al. (1986b), Leblond et al. (1989), Leblond (1989), Devaux et al. (2000), Taleb and Sidoroff (2003). However, the consistent access via the two-mechanism model allows for a qualitatively correct description of this phenomenon (cf. Wolff et al. (2008), Wolff et al. (2009)).

Contrary to Videau et al. (1994), Wolff et al. (2008), Mahnken et al. (2009) and others, in Aeby-Gautier and Cailletaud (2004) the material behavior of steel is described by a multi-mechanism model at the macro level as well as at the meso level (sometimes called micro level), whereas the proof of thermodynamic consistency still remains open. Furthermore, it should be noted that some authors combine classical and transformation-induced plasticity in one model ("unified transformation-thermoplasticity", cf. Inoue and Tanaka (2006)).

5) The complex material behavior of important materials (such as visco-plastic materials, shape-memory alloys, soils, granular materials, composites, biological tissues) leads to multi-mechanism models, when taking the additive decomposition of the strain tensor into account. However, in most cases, the concrete application is *not* set in the framework of multi-mechanism models in the sense of Cailletaud and Saï (1995). We give some examples.

When modelling shape-memory alloys, sometimes, the inelastic part of the strain tensor is decomposed into two parts (into two summands in the case of small deformations). We refer to Helm and Haupt (2003), Helm (2007), Reese and Christ (2008), Kang et al. (2009), e.g. The material behavior of salt in deposits is very complex, and its modelling uses an additive decomposition of inelastic strain into three parts (cf. Munson et al. (1993), e.g.). In Chan et al. (1994), Koteras and Munson (1996), an additional summand is used which is induced by damage. Further references to modelling via several mechanisms can be found in some papers in geomechanics, for instance, for cohesionless soil in Shi and Xie (2002), for clay in Modaressi and L. (1997), for sand in Akiyoshi et al. (1994), Fang (2003) and for granular material in Anandarajah (2008). Similarly, complex material behavior of biologic tissue is modelled using a multi-mechanism approach (cf. Wulandana and Robertson (2005), Doehring et al. (2004), e.g.).

6) The main aims of this note are

- to describe some classes of 2M models, in particular the general non-isothermal case (in Sections 2, 3, 4)
- to prove new results on thermodynamic consistency (in Sections 3, 4)
- to derive new useful general relations for the back stresses generalizing the classical Armstrong-Frederick equations (in Section 5)
- to propose a new additional coupling between the tensorial internal variables leading to non-symmetric Armstrong-Frederick relations (in Section 6)

Note that all arising material parameters (or more precisely material functions) may depend on temperature. Moreover, those parameters which do not occur in the free energy may additionally depend on stress and further quantities. We do not use dissipation potentials (For approaches for 2M models with dissipation potentials we refer to Cailletaud and Saï (1995), Besson et al. (2001)).

2 Description of two-mechanism models

In this section we provide important basic relations for 2M models. At first, there will be common items for models with one and with two yield criteria. After this, we deal separately with 2M models with one and with two criteria.

2.1 General assertions

We restrict ourselves to small deformations. Thus, the equation of momentum, the energy equation and the Clausius-Duhem inequality are given by

$$\varrho\,\ddot{\boldsymbol{u}} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \tag{2.1}$$

$$\varrho\,\dot{e} + \operatorname{div}\boldsymbol{q} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r \tag{2.2}$$

$$-\varrho\,\dot{\psi} - \varrho\,\eta\,\dot{\theta} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{\theta}\boldsymbol{q}\cdot\nabla\theta \geq 0. \tag{2.3}$$

The relations (2.1) - (2.3) have to be fulfilled in the space-time domain $\Omega\times]0,T[$. The notation is standard: ϱ - density in the reference configuration, that means for $t = 0$, $\boldsymbol{u}$ - displacement vector, $\boldsymbol{\varepsilon}$ - linearized Green strain tensor, θ - absolute temperature, $\boldsymbol{\sigma}$ - Cauchy stress tensor, $\boldsymbol{f}$ - volume density of external forces, e - mass density of the internal energy, $\boldsymbol{q}$ - heat-flux density vector, r - volume density of heat supply, ψ - mass density of free (or Helmholtz) energy, η - mass density of entropy. The time derivative is denoted by a dot. $\boldsymbol{\alpha} : \boldsymbol{\beta}$ is the scalar product of the tensors, $\boldsymbol{q}\cdot\boldsymbol{p}$ is the scalar product of the vectors. We note the well-known relations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T), \quad \psi = e - \theta\,\eta. \tag{2.4}$$

In the general case of inelastic material behavior, the full strain $\boldsymbol{\varepsilon}$ is split up via

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{in} \tag{2.5}$$

($\boldsymbol{\varepsilon}_{te}$ - thermoelastic strain, $\boldsymbol{\varepsilon}_{in}$ - inelastic strain). Usually, the inelastic strain is assumed to be traceless, i.e.

$$\mathrm{tr}(\boldsymbol{\varepsilon}_{in}) = 0. \tag{2.6}$$

The accumulated inelastic strain is defined by

$$s_{in}(t) := \int_0^t (\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_{in}(\tau) : \dot{\boldsymbol{\varepsilon}}_{in}(\tau))^{\frac{1}{2}}\, d\tau. \tag{2.7}$$

We drop the dependence on the space variable x. We propose for the free energy ψ the split

$$\psi = \psi_{te} + \psi_{in}. \tag{2.8}$$

The thermoelastic part is given in a standard way. To focus here, we refer to Wolff et al. (2010) for a detailed explanation. We assume that the inelastic part ψ_{in} of ψ has the general form

$$\psi_{in} = \psi_{in}(\xi,\, \theta). \tag{2.9}$$

$\xi = (\xi_1, \ldots, \xi_m)$ (ξ_j - scalars or tensors) represent the internal variables. Further on, these variables will be chosen in accordance with concrete models under consideration. Moreover, they have to fulfil evolution equations which are usually ordinary differential equations (ODE) with respect to the time t. As a rule, one poses zero initial conditions, i.e.

$$\xi_j(0) = 0 \qquad \text{for } j = 1, \ldots, m. \tag{2.10}$$

Using standard arguments of thermodynamics (cf. Lemaitre and Chaboche (1990), Maugin (1992), Besson et al. (2001), Haupt (2002), e.g.) and assuming the Fourier law of heat conduction, one obtains the **remaining inequality**

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \varrho \sum_{j=1}^{m} \frac{\partial \psi_{in}}{\partial \xi_j} : \dot{\xi}_j \geq 0. \tag{2.11}$$

Hence, the model under consideration is thermodynamically consistent, if (2.11) is fulfilled.

Up to this point, there is no difference between 1M models ("Chaboche" models) and 2M models. From now on, we deal with 2M models. The general assertions can be extended to multi-mechanism models (in short mM models) without difficulties. In the theory of 2M models the following decomposition of the inelastic strain is crucial:

$$\boldsymbol{\varepsilon}_{in} = A_1\, \boldsymbol{\varepsilon}_1 + A_2\, \boldsymbol{\varepsilon}_2, \tag{2.12}$$

A_1, A_2 are positive real numbers.

Remark 2.1. The parameters A_1 and A_2 open opportunities for further extensions and special applications. We refer to Saï and Cailletaud (2007). A_1 and A_2 can depend on further quantities as, for instance, they can constitute phase fraction in complex materials (steel, shape memory alloys, e.g.). In this sense, here is a bridge from the macro to the meso (or micro) level of modelling.

As usual, the inelastic strains are traceless:

$$\mathrm{tr}(\boldsymbol{\varepsilon}_{in}) = \mathrm{tr}(\boldsymbol{\varepsilon}_1) = \mathrm{tr}(\boldsymbol{\varepsilon}_2) = 0. \tag{2.13}$$

For both $\boldsymbol{\varepsilon}_j$ we introduce *separate* accumulations

$$s_j(t) := \int_0^t (\frac{2}{3}\, \dot{\boldsymbol{\varepsilon}}_j(\tau) : \dot{\boldsymbol{\varepsilon}}_j(\tau))^{\frac{1}{2}}\, d\tau \quad j = 1, 2. \tag{2.14}$$

Note, that s_{in} (as defined in (2.7)) is not the sum of s_1 and s_2. As the roots in (2.7) and (2.14) are norms, one gets useful inequalities

$$|A_1\, \dot{s}_1 - A_2\, \dot{s}_2| \leq \dot{s}_{in} \leq A_1\, \dot{s}_1 + A_2\, \dot{s}_2. \tag{2.15}$$

We introduce the local stresses $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ via

$$\boldsymbol{\sigma}_j := A_j\, \boldsymbol{\sigma} \qquad j = 1, 2 \tag{2.16}$$

From now on, we deal *separately* with 2M1C and 2M2C models (= 2M models with one criterion and 2M models with two criteria, respectively).

2.2 Two-mechanism models with one yield criterion

We specialize the ansatz for the inelastic part of the free energy in (2.9), assuming the internal variables to be given $\xi = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q)$.

$$\psi_{in}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q, \theta) := \frac{1}{3\varrho}\{c_{11}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2\,c_{12}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}(\theta)\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2\} + \frac{1}{2\varrho}Q(\theta)\,q^2, \tag{2.17}$$

The tensorial symmetric internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are related to kinematic hardening, the scalar internal variable q is related to isotropic hardening. All of them are of strain type. $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively.

Remark 2.2. **(i)** For each fixed temperature θ, the inelastic free energy ψ_{in} in (2.17) is a convex function with respect to $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and q, if there hold the conditions

$$c_{11} \geq 0, \qquad\qquad c_{12}^2 \leq c_{11}\,c_{22}, \tag{2.18}$$

$$Q \geq 0. \tag{2.19}$$

We note that the quadratic form related to c_{ij} is positive semi-definite (cf. Wolff and Taleb (2008)). From the physical point of view, it is more precise to require that this part of the free energy is convex.
(ii) In order to focus, we do not consider a possible coupling between kinematic and isotropic hardening in (2.17).

Assuming additionally

$$c_{11} > 0 \qquad\qquad c_{22} > 0 \qquad\qquad Q > 0, \tag{2.20}$$

we avoid simplifications. The back stresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively, as well as the isotropic hardening R are defined in a usual way via partial derivatives of the free energy with respect to the corresponding internal variables. This leads to

$$\boldsymbol{X}_1 = \varrho\frac{\partial\psi_{in}}{\partial\boldsymbol{\alpha}_1} = \frac{2}{3}\,c_{11}\,\boldsymbol{\alpha}_1 + \frac{2}{3}\,c_{12}\,\boldsymbol{\alpha}_2, \qquad\qquad \boldsymbol{X}_2 = \varrho\frac{\partial\psi_{in}}{\partial\boldsymbol{\alpha}_2} = \frac{2}{3}\,c_{12}\,\boldsymbol{\alpha}_1 + \frac{2}{3}\,c_{22}\,\boldsymbol{\alpha}_2, \tag{2.21}$$

$$R = \varrho\frac{\partial\psi_{in}}{\partial q} = Q\,q. \tag{2.22}$$

(2.11), (2.17), (2.21) and (2.22) imply the following remaining inequality

$$(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) - R\,\dot{q} \geq 0. \tag{2.23}$$

Based on the von Mises stress, we define the quantities

$$J_j := (\frac{3}{2}(\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*) : (\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*))^{\frac{1}{2}} \qquad (j = 1, 2) \tag{2.24}$$

$$J := (J_1^N + J_2^N)^{\frac{1}{N}}. \tag{2.25}$$

The material parameter N has to fulfil

$$N > 1. \tag{2.26}$$

Remark 2.3. The importance of the parameter N in (2.26) for applications consists in the fact, that, if it growths, the two quantities J_1 and J_2 become more and more independent of each other. We refer to Wolff and Taleb (2008), Taleb and Cailletaud (2010) for details.

The yield function is given by

$$f := J - (R + R_0), \tag{2.27}$$

$$R_0 := \sqrt[N]{2}\sigma_0. \tag{2.28}$$

The initial yield stress $\sigma_0 = \sigma_0(\theta)$ can be determined by a standard tension experiment. Since we are dealing only with plastic behavior, we suppose for all 2M1C models the subsequent constraint

$$f(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) \leq 0. \tag{2.29}$$

Based on (2.24), (2.25), (2.27), we define

$$\boldsymbol{n}_j := -\frac{\partial f}{\partial \boldsymbol{X}_j} = \frac{3}{2}\frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{J_j}\left(\frac{J_j}{J}\right)^{N-1} \qquad (j = 1, 2). \tag{2.30}$$

We assume evolution laws for the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ as well as for q:

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda \boldsymbol{n}_j \tag{2.31}$$

($\lambda \geq 0$ - common plastic multiplier for both mechanisms),

$$\dot{q} = r\,\lambda - \frac{b}{Q}R\,\lambda, \tag{2.32}$$

with r and b fulfilling

$$r > 0, \qquad\qquad b > 0, \tag{2.33}$$

(b = 0 corresponds to the simpler case of linear isotropic hardening.) From (2.14), (2.24), (2.25), (2.30) and (2.31) one gets

$$\dot{s}_j = \lambda\,(J_1^N + J_2^N)^{\frac{1}{N}-1} J_j^{N-1}, \tag{2.34}$$

and, after this,

$$\lambda = \left((\dot{s}_1)^{\frac{N}{N-1}} + (\dot{s}_2)^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}}. \tag{2.35}$$

We denote by Λ the primitive of λ, i.e.

$$\Lambda(t) = \int_0^t \lambda(\tau)d\tau. \tag{2.36}$$

It remains the approach for the evolution equation for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. In the next section, we will discuss two variants leading to 2M models which are denoted by 2M1C-a and 2M1C-b, differing by the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

Remark 2.4. Here, in order to focus, we deal with plastic mechanisms. Viscoplastic mechanisms can be dealt without difficulties. Let be f as in (2.27) and $\mathbf{n}_j$ as in (2.30) (for 1C models). Formally, the evolution law for $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ looks like (2.31). Contrary to the plastic case, there is no constraint as in (2.29). The elastic domain is defined by

$$f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) \leq 0. \tag{2.37}$$

In general, the stress is not a-priori bounded. Hence, the viscoplastic multiplier is not determined by flow and consistency conditions, but it must be defined separately, for instance by

$$\lambda := \frac{2}{3\,\eta}\left\langle \frac{1}{D} f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0)\right\rangle^n. \tag{2.38}$$

The McCauley brackets $< \bullet >$ are defined by $< x >:= x$ for $x \geq 0$ and $< x >:= 0$ otherwise. The exponent $n > 0$ and the viscosity $\eta > 0$ generally depend on temperature (and maybe on other quantities). The drag stress (cf. Chaboche, 2008) is a positive scalar generally following its own evolution. Finally, the relations (2.34) and (2.35) hold for λ and s_1, s_2.

2.3 Two-mechanism models with two yield criteria

Now we assume for the inelastic part ψ_{in} of the free energy (cf. (2.9) and (2.17))

$$\psi_{in}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q_1, q_2, \theta) := \frac{1}{3\varrho}\left\{c_{11}\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2\,c_{12}\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2\right\} + \\ + \frac{1}{2\varrho}\left\{Q_{11}\,q_1^2 + 2\,Q_{12}\,q_1 q_2 + Q_{22}\,q_2^2\right\} \tag{2.39}$$

with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ as above. q_1 and q_2 are scalar internal variables related to the isotropic hardening of the first and second mechanism, respectively. The coefficient Q_{12} stands for a possible interaction of these two kinds of isotropic hardening (cf. Cailletaud and Saï (1995)). Possible interactions of isotropic and kinematic hardening within ψ_{in} will not be considered here. We refer to Wolff et al. (2010) for an example of such coupling.

Remark 2.5. The inelastic free energy ψ_{in} in (2.39) is convex, if

$$c_{11} \geq 0, \qquad c_{12}^2 \leq c_{11}\, c_{22}, \tag{2.40}$$

$$Q_{11} \geq 0, \qquad Q_{12}^2 \leq Q_{11}\, Q_{22}, \tag{2.41}$$

We restrict ourselves to

$$c_{11} > 0, \qquad c_{22} > 0, \qquad Q_{11} > 0, \qquad Q_{22} > 0. \tag{2.42}$$

The back stresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are defined as in (2.21), the isotropic hardenings R_1 and R_2 are defined by

$$R_1 = \varrho\, \frac{\partial \psi_{in}}{\partial q_1} = Q_{11}\, q_1 + Q_{12}\, q_2, \qquad R_2 = \varrho\, \frac{\partial \psi_{in}}{\partial q_2} = Q_{12}\, q_1 + Q_{22}\, q_2. \tag{2.43}$$

By (2.11), (2.21), (2.39) and (2.43) we infer

$$(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) - R_1\, \dot{q}_1 - R_2\, \dot{q}_2 \geq 0. \tag{2.44}$$

Now, the two yield functions are

$$f_j := J_j - (R_j + R_{0j}) \qquad j = 1, 2, \qquad (J_j \text{ defined by (2.24)}). \tag{2.45}$$

R_{0j} is the initial yield stress of the j^{th} mechanism. Since we are dealing only with plastic behavior, we suppose for all 2M1C models the subsequent constraints

$$f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_{0j}) \leq 0 \qquad j = 1, 2. \tag{2.46}$$

Finally, based on (2.24) and (2.45), for 2M2C models we define

$$\boldsymbol{n}_j := -\frac{\partial f_j}{\partial \boldsymbol{X}_j} = \frac{3}{2} \frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{J_j}. \tag{2.47}$$

Note that the $\boldsymbol{n}_j$ are different for 2M1C and 2M2C models. We assume the subsequent evolution equations:

$$\dot{\boldsymbol{\epsilon}}_j = \lambda_j\, \boldsymbol{n}_j \qquad (\lambda_j \geq 0 \text{ - plastic multipliers, } \boldsymbol{n}_j \text{ defined by (2.47), } j = 1, 2), \tag{2.48}$$

$$\dot{q}_j = r_j\, \lambda_j - \frac{b_j}{Q_{jj}} R_j\, \lambda_j \qquad (j = 1, 2). \tag{2.49}$$

The material parameters b_j, r_j are assumed to fulfil

$$r_j > 0, \qquad b_j > 0, \qquad (j = 1, 2). \tag{2.50}$$

(Again, we neglect the simpler case $b_j = 0$.) (2.14), (2.24), (2.47) and (2.48) yield

$$\lambda_j = \dot{s}_j \qquad (j = 1, 2). \tag{2.51}$$

3 Thermodynamic consistency of some 2M1C models

Now, we discuss two types of 2M1C models differing by their evolution laws for the internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Everything presented in Subsection 2.2 is assumed for both subsequent model variants.

3.1 The model 2M1C-a

The evolution of $\dot{\boldsymbol{\alpha}}_j$ is given by

$$\dot{\boldsymbol{\alpha}}_j = a_j\, \dot{\boldsymbol{\varepsilon}}_j - \frac{3 d_j}{2 c_{jj}} \left\{ (1 - \eta_j)\, \boldsymbol{X}_j + \eta_j (\boldsymbol{X}_j : \boldsymbol{m}_j)\, \boldsymbol{m}_j \right\} \lambda \qquad (j = 1, 2). \tag{3.1}$$

The material parameters a_j, d_j, η_j have to fulfil

$$a_j > 0, \qquad d_j > 0, \qquad 0 \leq \eta_j \leq 1 \qquad (j = 1, 2). \tag{3.2}$$

(d_j = 0 corresponds to a simpler case.) The tensors $\mathbf{m}_j$ are defined as

$$\boldsymbol{m}_j := \boldsymbol{n}_j \, \|\boldsymbol{n}_j\|^{-1} = \frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{\|\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*\|} \qquad (j = 1, 2). \tag{3.3}$$

The isothermal case of this model 2M1C-a (with $a_1 = a_2 = 1$ and $\eta_1 = \eta_2 = 0$) was proposed by Cailletaud and Saï (1995). In Taleb et al. (2006), ratcheting experiments were simulated based on this model. The idea of the projection of $\boldsymbol{X}_j$ onto $\boldsymbol{m}_j$ is due to Burlet and Cailletaud (1987).

Using the evolution equations (2.31), (2.32) as well as (2.27), (2.30), one can re-write the dissipation inequality (2.23) in the form

$$\begin{aligned}(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda \boldsymbol{X}_1 : \boldsymbol{X}_1 + \\ + \frac{3d_1}{2c_{11}}\eta_1 \lambda \, (\boldsymbol{X}_1 : \boldsymbol{m}_1)^2 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda \boldsymbol{X}_2 : \boldsymbol{X}_2 + \frac{3d_2}{2c_{22}}\eta_2 \lambda \, (\boldsymbol{X}_2 : \boldsymbol{m}_2)^2 \geq 0.\end{aligned} \tag{3.4}$$

Clearly, the model 2M1C-a (characterized by (2.17), (2.31), (2.32), (3.1)) is thermodynamically consistent, if the dissipation inequality (3.4) holds. In Wolff and Taleb (2008), the special case $r = 1$ has been considered. The following theorem covers the more general case. To prove it, one has to ensure (3.4) under the assumed conditions.

Theorem 3.1. Assume (2.18) - (2.20), (2.33), (3.2).
(i) In the case of

$$a_1 = a_2 = 1, \tag{3.5}$$

the model 2M1C-a is thermodynamically consistent, if

$$r \leq 1 + 2\sqrt{\frac{b\,R_0}{Q}} \tag{3.6}$$

holds.
(ii) In the general case

$$a_1 \neq 1, \qquad a_2 \neq 1 \tag{3.7}$$

the model 2M1C-a is thermodynamically consistent, if

$$\eta_1 < 1, \quad \eta_2 < 1, \tag{3.8}$$

$$\frac{c_{11}}{d_1(1-\eta_1)}\,|1-a_1|^2 + \frac{c_{22}}{d_2(1-\eta_2)}\,|1-a_2|^2 \leq 4R_0, \tag{3.9}$$

$$r \leq 1 + \sqrt{\frac{b}{Q}\left(4R_0 - \frac{c_{11}}{d_1(1-\eta_1)}\,|1-a_1|^2 - \frac{c_{22}}{d_2(1-\eta_2)}\,|1-a_2|^2\right)} \tag{3.10}$$

Before proving Theorem 3.1, we provide some preliminary results.

Lemma 3.2. (i) Let be r, b, Q, $R_0 > 0$. Then there holds the equivalence

$$\Big(\forall\, R \geq 0 \quad : \quad R_0 + (1-r)R + \frac{b}{Q}R^2 \geq 0\Big) \quad \Leftrightarrow \quad r \leq 1 + 2\sqrt{\frac{R_0 b}{Q}} \tag{3.11}$$

(ii) (Young's inequality with a small factor)

$$\forall\, a, b \in \mathbb{R} \quad \forall\, \delta > 0 \quad : \quad |ab| \leq \frac{\delta}{2}\,a^2 + \frac{1}{2\,\delta}\,b^2 \tag{3.12}$$

Proof of Theorem 3.1. The strategy is to estimate the left-hand side of (3.4) from below by simpler expressions, and to show, that, at the end, the last expression is non-negative.

At first, we note that terms containing $(\boldsymbol{X}_j : \mathbf{m}_j)^2$ are non-negative. Hence, they can be omitted in (3.4). Therefore, it is sufficient to prove the validity of

$$(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \\ + \frac{3d_1}{2c_{11}}(1-\eta_1)\,\lambda\boldsymbol{X}_1 : \boldsymbol{X}_1 + \frac{3d_2}{2c_{22}}(1-\eta_2)\,\lambda\boldsymbol{X}_2 : \boldsymbol{X}_2 \geq 0. \quad (3.13)$$

Clearly, in the simple case $a_1 = a_2 = 1$, (3.13) is valid, if

$$R_0 + (1-r)R + \frac{b}{Q}R^2 \geq 0 \qquad \forall\, R \geq 0. \quad (3.14)$$

Due to (3.11), this is the case, because of the assumption (3.6).

In the general case, the terms containing $\boldsymbol{X}_j : \dot{\boldsymbol{\varepsilon}}_j$ are not definite. But, there is a hope to compensate their behavior by the definiteness of the remaining terms. Using (2.25), (2.30), (2.31) as well as Young's inequality (3.12) and Cauchy-Schwarz inequality, one gets the following estimates:

$$\begin{aligned} |(1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1| &= |(1-a_1)\boldsymbol{X}_1 : (\lambda\boldsymbol{n}_1)| = \\ &= \frac{3}{2}|1-a_1| \left\{ \sqrt{\lambda}\frac{J_1^{N-2}}{J^{N-1}}\|\boldsymbol{\sigma}_1^* - \boldsymbol{X}_1^*\| \right\} : \left\{ \sqrt{\lambda}\|\boldsymbol{X}_1\| \right\} \leq \\ &\leq |1-a_1|\lambda\frac{\delta_1}{2}\left(\frac{J_1}{J}\right)^{2(N-1)} + \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2 \leq \\ &\leq |1-a_1|\lambda\frac{\delta_1}{2} + \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2 \end{aligned} \quad (3.15)$$

with $\delta_1 > 0$ which will be chosen later. Analogously, one obtains

$$|(1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2| = |(1-a_2)\boldsymbol{X}_2 : (\lambda\boldsymbol{n}_2)| \leq |1-a_2|\lambda\frac{\delta_2}{2} + \frac{3|1-a_2|}{4\delta_2}\lambda\|\boldsymbol{X}_2\|^2 \quad (3.16)$$

for some $\delta_2 > 0$. From (3.13), (3.15), (3.16), one gets

$$\begin{aligned} &(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \\ &\quad + \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda\boldsymbol{X}_1 : \boldsymbol{X}_1 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda\boldsymbol{X}_2 : \boldsymbol{X}_2 + \\ &\quad \geq (R_0 - |1-a_1|\lambda\frac{\delta_1}{2} - |1-a_2|\lambda\frac{\delta_2}{2} + (1-r)R + \frac{b}{Q}R^2)\lambda + \\ &+ \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda\|\boldsymbol{X}_1\|^2 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda\|\boldsymbol{X}\|^2 - \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2 - \frac{3|1-a_2|}{4\delta_2}\lambda\|\boldsymbol{X}_2\|^2. \end{aligned} \quad (3.17)$$

Since R, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent of each other, it is reasonable to require assumption (3.8). Now, we chose δ_1 and δ_2 such, that the last four terms cancel each other. This can be done by setting

$$\delta_1 := \frac{|1-a_1|c_{11}}{2(1-\eta_1)d_1}, \qquad \delta_2 := \frac{|1-a_2|c_{22}}{2(1-\eta_2)d_2}. \quad (3.18)$$

This implies from (3.17):

$$(R_0 - \frac{|1-a_1|^2c_{11}}{4(1-\eta_1)d_1} - \frac{|1-a_2|^2c_{22}|}{4(1-\eta_2)d_2} + (1-r)R + \frac{b}{Q}R^2)\lambda \geq 0. \quad (3.19)$$

Clearly, it is necessary, that

$$R^* := R_0 - \frac{|1-a_1|^2c_{11}}{4(1-\eta_1)d_1} - \frac{|1-a_2|^2c_{22}}{4(1-\eta_2)d_2} \geq 0. \quad (3.20)$$

This is assumption (3.9)! It remains to ensure that

$$R^* + (1-r)R + \frac{b}{Q}R^2 \geq 0 \qquad \text{for all } R \geq 0. \quad (3.21)$$

Obviously, (3.10) is sufficient for (3.21).

Therefore, in the "trivial case" $a_1 = a_2 = 1$, $r \leq 1$, the model 2M1C-a is thermodynamically consistent. Generally, Theorem 3.1 ensures thermodynamic consistency, if $\eta_j < 1$, and, if the a_j do not differ too much from 1, and, if r is not too much greater than 1.

Remark 3.3. Theorem 3.1 is also valid in the viscoplastic case. The viscoplastic multiplier is only positive, if $J > R_0 + R$, while the plastic multiplier is only positive, if $J = R_0 + R$. Hence, the validity of (3.4) is also sufficient for thermodynamic consistency in the viscoplastic case.

3.2 The model 2M1C-b

Contrary to the 2M1C-a model in subsection 3.1., instead of (3.1), the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are given by

$$\dot{\boldsymbol{\alpha}}_j = a_j\,\dot{\boldsymbol{\varepsilon}}_j - \{(1-\eta_j)\,\boldsymbol{\alpha}_j + \eta_j(\boldsymbol{\alpha}_j : \mathbf{m}_j)\,\mathbf{m}_j\}\,d_j\,\lambda \qquad (j = 1, 2). \tag{3.22}$$

That means, in the right-hand side of (3.22), the back stresses $\boldsymbol{X}_j$ are substituted by the internal variables $\boldsymbol{\alpha}_j$. This approach was proposed in Taleb et al. (2006) in order to get a better description of ratcheting behavior. Analogously, we let the parameters a_j, d_j and η_j fulfil the conditions (3.2). The m_j are defined by (3.3).

Using the evolution equations (2.31), (2.32), (3.22) as well as (2.21), (2.27), (2.30), one can re-write the dissipation inequality (2.23) in the form

$$\begin{aligned}(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + \frac{2}{3}d_1\lambda(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1\} +\\ + \frac{2}{3}(1-a_1)(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : (\lambda\mathbf{n}_1) + \frac{2}{3}(1-a_2)(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : (\lambda\mathbf{n}_2) +\\ + \frac{2}{3}d_2\lambda(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2\} \geq 0.\end{aligned} \tag{3.23}$$

The case $a_1 = a_2 = 1$, $r = 1$ and $\eta_1 = \eta_2$ is dealt with in Wolff and Taleb (2008). In the general case, there arise more complicated conditions to ensure thermodynamic consistency.

Theorem 3.4. Let be given the assumptions (2.18) - (2.20), (2.33), (3.2). The model 2M1C-b is thermodynamically consistent, if

$$r \leq 1 \tag{3.24}$$
$$\eta_1 < 1, \quad \eta_2 < 1, \tag{3.25}$$
$$c_{11}^2(1-a_1)^2 + c_{12}^2(1-a_2)^2 < R_0 d_1 c_{11}(1-\eta_1), \tag{3.26}$$
$$c_{12}^2(1-a_1)^2 + c_{22}^2(1-a_2)^2 < R_0 d_2 c_{22}(1-\eta_2), \tag{3.27}$$

$$\begin{aligned}c_{12}^2(d_1+d_2)^2 \leq 4\,(d_1 c_{11}(1-\eta_1) - \frac{1}{R_0}(c_{11}^2(1-a_1)^2 + c_{12}^2(1-a_2)^2))\,\cdot\\ \cdot\,(d_2 c_{22}(1-\eta_2) - \frac{1}{R_0}(c_{12}^2(1-a_1)^2 + c_{22}^2(1-a_2)^2)).\end{aligned} \tag{3.28}$$

The proof of Theorem 3.4 is similar to the proof of Theorem 3.1, but more complex. Additionally, one needs a result about quadratic forms (cf. Wolff et al. (2010)).

Remark 3.5. (i) In the simpler case $a_1 = a_2 = 1$ and $r = 1$ (cf. Wolff and Taleb (2008)), the above 2M1C-b model is thermodynamically consistent, if (3.25) holds and if

$$(d_1 - d_2)^2 \leq 4\,d_1\,d_2\,\frac{c_{11}c_{22}(1-\eta_1)(1-\eta_2) - c_{12}^2}{c_{12}^2}. \tag{3.29}$$

In contrast to the 2M1C-a model, the condition (3.29) restricts η_1 and η_2 even in the simpler case $a_1 = a_2 = 1$.
(ii) In the case $r > 1$, more complex conditions are sufficient for thermodynamic consistency which involve b and Q.

4 Thermodynamic consistency of some 2M2C models

Now we discuss two types of 2M2C models differing by their evolution laws for the internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Again, all things presented in Subsection 2.3 are assumed for both subsequent model variants.

4.1 The model 2M2C-a

We assume the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\dot{\boldsymbol{\alpha}}_j = a_j\dot{\boldsymbol{\varepsilon}}_j - \frac{3d_j}{2c_{jj}}\left\{(1-\eta_j)\boldsymbol{X}_j + \eta_j(\boldsymbol{X}_j : \boldsymbol{m}_j)\boldsymbol{m}_j\right\}\lambda_j \qquad (j = 1, 2). \tag{4.1}$$

The $\boldsymbol{m}_j$ are defined by (3.3), and the material parameters a_j, d_j, η_j must fulfil (cf. (3.2))

$$a_j > 0, \qquad d_j > 0, \qquad 0 \le \eta_j \le 1 \qquad (j = 1, 2). \tag{4.2}$$

(d_j = 0 corresponds to a simpler case, again.) Repeating arguments as above, the dissipation inequality is

$$\begin{aligned}(R_{01} + (1-r_1)R_1 + \frac{b_1}{Q_{11}}R_1^2)\lambda_1 + (R_{02} + (1-r_2)R_2 + \frac{b_2}{Q_{22}}R_2^2)\lambda_2 + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + \\ + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 : \boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \boldsymbol{m}_1)^2\right\}\lambda_1 + \\ + \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 : \boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \boldsymbol{m}_2)^2\right\}\lambda_2 \ge 0.\end{aligned} \tag{4.3}$$

Thermodynamic consistency can be ensured similarly as in the case of the 2M1C-a model. Since there are *two* multipliers ($\lambda_j = \dot{s}_j$, j = 1,2), there is some "decoupling" (cf. Theorem 3.1).

Theorem 4.1. Assume (2.40) - (2.42), (2.50) and (4.2).
(i) In the case

$$a_1 = a_2 = 1, \tag{4.4}$$

the model 2M2C-a is thermodynamic consistent, if

$$r_j \le 1 + 2\sqrt{\frac{b_j R_{0j}}{Q_{jj}}} \qquad (j = 1, 2) \tag{4.5}$$

(ii) In the general case

$$a_j \ne 1 \qquad \text{for one or both } j, \tag{4.6}$$

the model 2M1C-a is thermodynamic consistent, if

$$\eta_j < 1 \qquad \text{for the same } j \text{ as in (4.6),} \tag{4.7}$$

$$\frac{c_{jj}}{d_j(1-\eta_j)}|1-a_j|^2 \le 4R_{0j} \qquad \text{for the same } j \text{ as in (4.6),} \tag{4.8}$$

$$r_j \le 1 + \sqrt{\frac{b_j}{Q_{jj}}\left(4R_{0j} - \frac{c_{jj}}{d_j(1-\eta_j)}|1-a_j|^2\right)} \qquad \text{for the same } j \text{ as in (4.6).} \tag{4.9}$$

As for the 2M1C-a model, there is a trivial case for the 2M2C-a model: $a_j = 1$, $r_j \le 1$ (cf. Theorem 3.1). Generally, Theorem 4.1 ensures thermodynamic consistency, if $\eta_j < 1$, and, if a_j do not differ too much from 1, and, if r_j is not too much greater than 1. Contrary to Theorem 3.1 for the 2M1C-a model, in Theorem 4.1, the conditions for $j = 1$ and $j = 2$ are separated (cf. (4.6)-(4.9)).

4.2 The model 2M2C-b

Now, we investigate the formal two-criteria analogue to the 2M1C-b model. That means, in (4.1), one could substitute $\boldsymbol{X}_j$ by $\boldsymbol{\alpha}_j$, analogously as in the case of 1C models. Unfortunately, then it becomes very difficult to prove thermodynamic consistency. Hence, instead of (4.1), we assume the following evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$

$$\dot{\boldsymbol{\alpha}}_1 = a_1\,\dot{\boldsymbol{\varepsilon}}_1 - \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1 + d_{12}\,\boldsymbol{\alpha}_2\}\,d_1\,\lambda_1, \tag{4.10}$$

$$\dot{\boldsymbol{\alpha}}_2 = a_2\,\dot{\boldsymbol{\varepsilon}}_2 - \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2 + d_{21}\,\boldsymbol{\alpha}_1\}\,d_2\,\lambda_2. \tag{4.11}$$

a_j, d_j and η_j are supposed to satisfy (4.2); see (3.3) for $\mathbf{m}_j$. For the new material parameters d_{12} and d_{21} we assume

$$d_{12} \neq 0, \quad d_{21} \neq 0. \tag{4.12}$$

Using arguments as above, we obtain from (2.44) the dissipation inequality in the specific form of our 2M1C-b model:

$$\begin{aligned}
&(R_{01} + (1-r_1)R_1 + \frac{b_1}{Q_{11}}R_1^2)\lambda_1 + (R_{02} + (1-r_2)R_2 + \frac{b_2}{Q_{22}}R_2^2)\lambda_2 + \\
&\quad + \frac{2}{3}(1-a_1)(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : (\lambda_1\mathbf{n}_1) + \frac{2}{3}(1-a_2)(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : (\lambda_2\mathbf{n}_2) + \\
&\quad + \frac{2}{3}d_1\lambda_1(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1 + d_{12}\boldsymbol{\alpha}_2\} + \\
&\quad + \frac{2}{3}d_2\lambda_2(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2 + d_{21}\boldsymbol{\alpha}_1\} \geq 0.
\end{aligned} \tag{4.13}$$

Remark 4.2. **(i)** Generally, for 2C models one has $\lambda_1 \neq \lambda_2$. Therefore, if $d_{12} = d_{21} = 0$, some (for the mathematical argument needed) quadratic terms cease to exist in (4.13). Hence, in comparison with (3.23) (and with the exception $c_{12} = 0$), it is more difficult to fulfil the inequality (4.13).
(ii) The coupling in the evolution equations (4.10), (4.11) is a new item in the modelling of 2M models and indicates possible further generalizations.

Theorem 4.3. Assume (2.40) - (2.42), (2.50), (4.2) and (4.12). The model 2M2C-b is thermodynamically consistent, if

$$r_1 \leq 1, \qquad r_2 < 1, \tag{4.14}$$

$$\eta_1 < 1, \qquad \eta_2 < 1, \tag{4.15}$$

$$c_{11}^2(1-a_1)^2 < 2\,R_{01}\,d_1\,c_{11}(1-\eta_1), \qquad c_{12}^2(1-a_1)^2 < 2\,R_{01}\,d_1\,c_{12}\,d_{12}, \tag{4.16}$$

$$c_{22}^2(1-a_2)^2 < 2\,R_{02}\,d_2\,c_{22}(1-\eta_2), \qquad c_{12}^2(1-a_2)^2 < 2\,R_{02}\,d_2\,c_{12}\,d_{21}, \tag{4.17}$$

$$\begin{aligned}
&d_1^2(|c_{12}| + c_{11}|d_{12}|)^2 \leq \\
&\quad \leq 4\left(d_1c_{11}(1-\eta_1) - \frac{1}{2R_{01}}c_{11}^2(1-a_1)^2\right)\left(d_1c_{12}d_{12} - \frac{1}{2R_{01}}c_{12}^2(1-a_1)^2\right),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&d_2^2(|c_{12}| + c_{22}|d_{21}|)^2 \leq \\
&\quad \leq 4\left(d_2c_{22}(1-\eta_2) - \frac{1}{2R_{02}}c_{22}^2(1-a_2)^2\right)\left(d_1c_{12}d_{21} - \frac{1}{2R_{02}}c_{12}^2(1-a_2)^2\right).
\end{aligned} \tag{4.19}$$

Similarly as for the 2M1C-b model, even in the simple case $a_1 = a_2 = 1$, $r_1 \leq 1$, $r_2 \leq 1$, Theorem 4.3 only ensures thermodynamic consistency in the case $\eta_1 < 1$, $\eta_2 < 1$. Besides, (4.18), (4.19) describe smallness conditions with respect to the parameters c_{12}, d_{12}, d_{21} which express the coupling of the two mechanisms.

5 Important relations for the back stresses

It is possible to obtain relations for the isotropic hardenings as well as for the back stresses generalizing the classical Armstrong-Frederick relation. These relations are useful for further mathematical investigations and for numerical simulations. In some cases, the variables q or q_1 and q_2 as well as $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ can be excluded, and *differential* equations can be obtained, even in the case of temperature-dependent parameters. This is very helpful for simulations, when one has to update inelastic quantities in each time step. At first, we consider the isotropic hardening. After this, relations for kinematic hardening are derived.

5.1 Relations concerning isotropic hardening

Since there is an essential difference between 1C and 2C models, we deal separately with them.

5.1.1 Isotropic hardening in the case of 2M1C models

(2.22) and (2.32) imply an integral equation for R

$$R(t) = Q(t) \left\{ \int_0^t r(\tau)\,\lambda(\tau)\,\mathrm{d}\tau - \int_0^t \frac{b(\tau)}{Q(\tau)} R(\tau)\,\lambda(\tau)\,\mathrm{d}\tau \right\}, \tag{5.1}$$

as well as an ordinary differential equation (ODE) (differentiate the relation (2.22) and express q via the same relation)

$$\dot{R}(t) = Q(t)\,r(t)\,\lambda(t) - \left\{ b(t)\,\lambda(t) - \frac{\dot{Q}(t)}{Q(t)} \right\} R. \tag{5.2}$$

For the sake of notational simplicity, we write $Q(t)$ instead of $Q(\theta(t))$ etc. Moreover, the space variable x is suppressed. The unique solution of (5.2) (for the initial value $R(0) = 0$) is given by

$$R(t) = Q(t) \int_0^t r(s)\,\lambda(s) \exp\left(- \int_0^t b(\tau)\,\lambda(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s. \tag{5.3}$$

Moreover, R is non-negative for $t \geq 0$ (cf. (2.20), (2.33), (2.35)). From (5.3) one obtains the estimate

$$0 < R(t) \leq Q(t) \max\{r\} (\min\{b\})^{-1} (1 - \exp(-\min\{b\}\Lambda(t))) \qquad \text{for } t > 0, \tag{5.4}$$

Λ is the primitive of λ (see (2.36)). Maximum and minimum refer to all admissible temperatures (and possibly other quantities). Clearly, if plastic deformation occurs, $R(t)$ is positive. For constant Q, r and b we have

$$R(\Lambda) = \frac{Q\,r}{b}(1 - \exp(-b\,\Lambda)). \tag{5.5}$$

That means, R is a function of Λ alone. The curve $R = R(\Lambda)$ has the initial slope $Q\,r$, and its saturation value is $(Q\,r)/b$. Besides this, R is an increasing function of Λ, as one can expect in the case of isotropic hardening.

5.1.2 Isotropic hardening in the case of 2M2C models

Any attempt to eliminate q_j in order to obtain relations for R_j leads to a substantial difference with respect to the case of 1C models: A system of integral equations comes up. Using (2.43) and (2.49), one obtains the following system of integral equations for R_1 and R_2.

$$\begin{aligned} R_1(t) = Q_{11}(t) \int_0^t \left(r_1(\tau)\lambda_1(\tau) - \frac{b_1(\tau)}{Q_{11}(\tau)} R_1(\tau)\lambda_1(\tau) \right) \mathrm{d}\tau \, + \\ + Q_{12}(t) \int_0^t \left(r_2(\tau)\lambda_2(\tau) - \frac{b_2(\tau)}{Q_{22}(\tau)} R_2(\tau)\lambda_2(\tau) \right) \mathrm{d}\tau, \end{aligned} \tag{5.6}$$

$$\begin{aligned} R_2(t) = Q_{12}(t) \int_0^t \left(r_1(\tau)\lambda_1(\tau) - \frac{b_1(\tau)}{Q_{11}(\tau)} R_1(\tau)\lambda_1(\tau) \right) \mathrm{d}\tau \, + \\ + Q_{22}(t) \int_0^t \left(r_2(\tau)\lambda_2(\tau) - \frac{b_2(\tau)}{Q_{22}(\tau)} R_2(\tau)\lambda_2(\tau) \right) \mathrm{d}\tau. \end{aligned} \tag{5.7}$$

Again, the dependence on the space variable x is suppressed, and $Q_{11}(t)$ stands for $Q_{11}(\theta(t))$. In the subsequent cases, one can obtain from (5.6), (5.7) differential equations:

1) For *constant* Q_{ij}, differentiation in (5.6), (5.7) leads to a coupled system of differential equations:

$$\dot{R}_1(t) = Q_{11} r_1(t)\lambda_1(t) + Q_{12} r_2(t)\lambda_2(t) - b_1(t)R_1(t)\lambda_1(t) - Q_{12}\frac{b_2(t)}{Q_{22}} R_2(t)\lambda_2(t), \tag{5.8}$$

$$\dot{R}_2(t) = Q_{12} r_1(t)\lambda_1(t) + Q_{22} r_2(t)\lambda_2(t) - Q_{12}\frac{b_1(t)}{Q_{11}} R_1(t)\lambda_1(t) - b_2(t)R_2(t)\lambda_2(t). \tag{5.9}$$

Note that the two systems (5.6), (5.7) and (5.8), (5.9) are equivalent, if one assumes the usual initial condition $R_1(0) = R_2(0) = 0$. In comparison to the case of 1C models, a simple solution of (5.8), (5.9) like (5.3) does not exist. Thus, there is a mathematical challenge to formulate appropriate conditions such that $R_j + R_{0j} > 0$. Furthermore, due to the interaction in the isotropic hardening (if $Q_{12} < 0$), there can be a softening in one mechanism caused by the hardening in the other one.

2) In the regular case

$$\Delta_Q := Q_{11}Q_{22} - Q_{12}^2 > 0 \qquad \text{for all admissable arguments,} \tag{5.10}$$

one gets a coupled system of ordinary differential equations for R_1 and R_2 as well for non-constant parameters Q_{ij}. The argumentation is similar as in the regular case for kinematic hardening in Subsection 5.2. We refer to Wolff et al. (2010) for more details.

3) The singular case

$$\Delta_Q := Q_{11}Q_{22} - Q_{12}^2 = 0 \qquad \text{for all admissible arguments,} \tag{5.11}$$

is dealt with in Wolff et al. (2010).

5.2 Generalized Armstrong-Frederick relations for the 2MnC-a model

We distinguish between the models 2MnC-a and 2MnC-b (with $n = 1$ or $n = 2$). Concerning the models 2MnC-a, the only difference is that one has *one common* multiplier λ in the case of 1C models, and *two* multipliers λ_1 and λ_2 otherwise. We formulate the subsequent formulas for the 2M2C-a model. Setting $\lambda = \lambda_1 = \lambda_2$, one obtains the case for the 2M1C-a model. (2.21) and (3.1) imply integral equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_1$:

$$\begin{aligned} \boldsymbol{X}_1(t) = \frac{2}{3}c_{11}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \boldsymbol{m}_1)\boldsymbol{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{12}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \boldsymbol{m}_2\boldsymbol{m}_2\right\}\lambda_1\,\mathrm{d}\tau\right\}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \boldsymbol{X}_2(t) = \frac{2}{3}c_{12}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \boldsymbol{m}_1)\boldsymbol{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{22}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \boldsymbol{m}_2)\boldsymbol{m}_2\right\}\lambda_1\,\mathrm{d}\tau\right\}. \end{aligned} \tag{5.13}$$

Note: (5.12) and (5.13) do *not* involve $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Analogously as in the case of two isotropic hardenings, R_1 and R_2, in Subsection 5.1.2, one can derive differential equations. This follows from (5.12), (5.13) under some additional conditions:

1) For *constant* c_{11}, c_{12}, c_{22} one can differentiate (5.12), (5.13) with respect to time t. This yields

$$\begin{aligned} \dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda_1 + \\ - \frac{c_{12}d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda_2, \end{aligned} \tag{5.14}$$

$$\begin{aligned} \dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda_1 + \\ - d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda_2. \end{aligned} \tag{5.15}$$

These last two equations generalize the Armstrong-Frederick equation (cf. Armstrong and Frederick (1966), Lemaitre and Chaboche (1990), Haupt (2002) e.g.) as well as the approach by Burlet and Cailletaud (1987). Indeed, in the case of only one inelastic strain (i.e. $\boldsymbol{\varepsilon}_{in} = \boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2 = 0$, $\boldsymbol{\alpha}_2 = 0$, $\boldsymbol{X}_1 = \boldsymbol{X}$, $\boldsymbol{X}_2 = 0$, $\lambda = \dot{s}_{in}$), (5.14) reduces to

$$\dot{\boldsymbol{X}} = \frac{2}{3}\,c\,a\,\dot{\boldsymbol{\varepsilon}}_{in} - d\{(1-\eta)\boldsymbol{X} + \eta(\boldsymbol{X} : \boldsymbol{m})\boldsymbol{m}\}\dot{s}_{in}. \tag{5.16}$$

Finally, for $\eta = 0$, (5.16) turns into the classical Armstrong-Frederick relation; for $\eta = 1$, one gets the proposal by Burlet and Cailletaud (1987).

2) In the regular case

$$\Delta_c := c_{11}c_{22} - c_{12}^2 > 0 \qquad \text{for all admissible arguments,} \tag{5.17}$$

the brackets {} in (5.12), (5.13) can be expressed by $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

$$\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1:\boldsymbol{m}_1)\boldsymbol{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} = \frac{3}{2\Delta_c}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2), \quad (5.18)$$

$$\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2:\boldsymbol{m}_2)\boldsymbol{m}_2\right\}\lambda_2\,\mathrm{d}\tau\right\} = \frac{3}{2\Delta_c}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \quad (5.19)$$

Differentiating (5.12), (5.13) and using (5.18),(5.19), one gets differential equations not containing $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\begin{aligned}\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1:\boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda_1 +& \\ - \frac{c_{12}}{d_2}c_{22}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2:\boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda_2 +& \\ + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{11}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1),& \quad (5.20)\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}}{d_1}c_{11}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1:\boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda_1 +& \\ - d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2:\boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda_2 +& \\ + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{22}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1).& \quad (5.21)\end{aligned}$$

3) For the singular case

$$\Delta_c := c_{11}c_{22} - c_{12}^2 = 0 \qquad \text{for all admissible arguments} \qquad (5.22)$$

we refer to Wolff et al. (2010).

5.3 Generalized Armstrong-Frederick relations for the 2MnC-b model

Since the 2M2C-b model is more complex than the 2M1C-b model (cf. (4.10), (4.11)), we write down only the expressions for the 2M1C-b model. Analogously to Subsection 5.2, from (2.21) and (3.22) we obtain integral equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

$$\begin{aligned}\boldsymbol{X}_1(t) = \frac{2}{3}c_{11}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1:\boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda\,\mathrm{d}\tau\right\} +& \\ + \frac{2}{3}c_{12}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2:\boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda\,\mathrm{d}\tau\right\},& \quad (5.23)\end{aligned}$$

$$\begin{aligned}\boldsymbol{X}_2(t) = \frac{2}{3}c_{12}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1:\boldsymbol{m}_1)\boldsymbol{m}_1\}\lambda\,\mathrm{d}\tau\right\} +& \\ + \frac{2}{3}c_{22}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2:\boldsymbol{m}_2)\boldsymbol{m}_2\}\lambda\,\mathrm{d}\tau\right\}.& \quad (5.24)\end{aligned}$$

An elimination of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ is only possible under the additional condition (5.17). Then the equations in (2.21) are uniquely solvable with respect to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\boldsymbol{\alpha}_1 = \frac{3}{2\Delta_c}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2), \qquad \boldsymbol{\alpha}_2 = \frac{3}{2\Delta_c}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \qquad (5.25)$$

Inserting (5.25) into (5.23), (5.24), one obtains integral equations *not* containing $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.
Again, for *constant* c_{ij} one can take the derivatives with respect to t and one obtains the following generalizations of Armstrong-Frederick relations:

$$\begin{aligned}\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 +& \\ - c_{11}\frac{d_1}{\Delta_c}\left\{(1-\eta_1)(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \eta_1(c_{22}(\boldsymbol{X}_1:\boldsymbol{m}_1) - c_{12}(\boldsymbol{X}_2:\boldsymbol{m}_1))\boldsymbol{m}_1\right\}\lambda +& \\ - c_{12}\frac{d_2}{\Delta_c}\left\{(1-\eta_2)(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1) + \eta_2(c_{11}(\boldsymbol{X}_2:\boldsymbol{m}_2) - c_{12}(\boldsymbol{X}_1:\boldsymbol{m}_2))\boldsymbol{m}_2\right\}\lambda,& \quad (5.26)\end{aligned}$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 +$$
$$- c_{12}\frac{d_1}{\Delta_c}\left\{(1-\eta_1)(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \eta_1(c_{22}(\boldsymbol{X}_1 : \boldsymbol{m}_1) - c_{12}(\boldsymbol{X}_2 : \boldsymbol{m}_1))\boldsymbol{m}_1\right\}\lambda +$$
$$- c_{22}\frac{d_2}{\Delta_c}\left\{(1-\eta_2)(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1) + \eta_2(c_{11}(\boldsymbol{X}_2 : \mathrm{m}_2) - c_{12}(\boldsymbol{X}_1 : \boldsymbol{m}_2))\boldsymbol{m}_2\right\}\lambda. \quad (5.27)$$

Remark 5.1. **(i)** In the case of constant c_{ij}, the Armstrong-Frederick relations (5.14), (5.15) and (5.26), (5.27) have a similar structure. But, in the case of 2M1C-b model, in (5.26), (5.27), there are the additional coupling terms $(\boldsymbol{X}_2 : \boldsymbol{m}_1)\boldsymbol{m}_1$, $(\boldsymbol{X}_1 : \boldsymbol{m}_2)\boldsymbol{m}_2$.
(ii) In the case of the 2M2C-b model, one gets similar integral equations as in (5.23), (5.24). In the regular case (5.17), $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ can be excluded.
(iii) In the regular case (5.17), one gets elaborated differential equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, if some of the c_{ij} depend on the temperature.

Remark 5.2. Consider the regular case (5.17): As in the case of the classical Armstrong-Frederick relation for 1M models, the back stresses, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and the internal variables, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, are traceless. This is a mathematical consequence of the Volterra equations (5.12), (5.13) and (5.23), (5.24), resp. For details we refer to Wolff et al. (2010).

6 An extension concerning kinematic hardening

The 2M models described above have been applied (besides the new proposal for the 2M2C-b model in (4.10), (4.11)), or they are simple extensions of such models. Here, we want to present a possible extension concerning the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ which lead to non-symmetric Armstrong-Frederick relations.

Besides the proposal made in (4.10), (4.11), the evolution equations for the internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ involve only quantities with the same index. Hence, instead of the simple approach

$$\dot{\boldsymbol{\alpha}}_j = \dot{\boldsymbol{\varepsilon}}_j - \frac{3}{2}\,d_j\boldsymbol{X}_j\lambda_j \qquad j = 1, 2, \quad (6.1)$$

we may propose

$$\dot{\boldsymbol{\alpha}}_j = \dot{\boldsymbol{\varepsilon}}_j - \frac{3}{2}\sum_{i=1}^{2} d_{ji}\boldsymbol{X}_i\sqrt{\lambda_j\lambda_i} \qquad j = 1, 2, \quad (6.2)$$

In the case of 1C models, one sets $\lambda = \lambda_1 = \lambda_2$. Generally, one has to suppose sufficient conditions for the matrix d, that the dissipation inequality is fulfilled. Assuming (2.31), (2.32) with $r \leq 1$ or (2.48), (2.49) with $r_1 \leq 1$, $r_2 \leq 1$, the remaining interesting part of the dissipation inequality becomes

$$\frac{3}{2}\sum_{i,j=1}^{2} d_{ji}\sqrt{\lambda_j\lambda_i}\;\boldsymbol{X}_i : \boldsymbol{X}_j. \quad (6.3)$$

This part is non-negative, if d is positive semi-definite, i.e., if d fulfils

$$\sum_{i,j=1}^{2} d_{ji}\,\xi_j\,\xi_i \geq 0 \qquad \text{for all real vectors } \xi = (\xi_1, \xi_2). \quad (6.4)$$

Note that d is generally not symmetric (contrary to the matrix c). Thus, thermodynamic consistence of 2M models extended in the above way can be ensured. Clearly, in more complex cases, additional assumptions on d may be needed (cf. Theorem 3.1), e.g. In order to show the possibilities of modelling via the approach in (6.2), we present a simple example.

Example 6.1. For a 2M1C model we suppose (6.2) (with $\lambda = \lambda_1 = \lambda_2$). Assuming

$$d_{11} = d_{22} = d_{21} = 1, \quad d_{12} = 0, \qquad c_{11} > 0, \ \ c_{22} > 0 \ \ c_{12} = 0, \quad (6.5)$$

from (2.21) and (6.2), one obtains a *non-symmetric* Armstrong-Frederick relations:

$$\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}\dot{\boldsymbol{\varepsilon}}_1 - c_{11}\,\boldsymbol{X}_1\,\lambda \quad (6.6)$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{22}\dot{\boldsymbol{\varepsilon}}_2 - c_{22}\,\boldsymbol{X}_2 - c_{22}\,\boldsymbol{X}_1\,\lambda. \quad (6.7)$$

Thus, the back stress $\boldsymbol{X}_1$ influences the evolution of $\boldsymbol{X}_2$, but not vice versa.

Finally, the underlying idea of (6.2) can be applied to more complex approaches like in (3.1), (3.22), (4.1), (4.10), (4.11). Clearly, in every case one has to ensure thermodynamic consistency, assuming suitable conditions. Moreover, there arise more complex Armstrong-Frederick relations.

7 An application of 2M models to modelling and simulation of ratcheting

As already mentioned above, 2M models have been used for modelling and simulation of ratcheting (see Section 1 for some comments). But, up to now, the majority of contributions to ratcheting concerns extensions of the Chaboche model (= 1M model). We refer to Abdel-Karim (2009), Abdel-Karim (2010) and Hassan et al. (2008) for detailed explanations and references. One might say that there is no model which sufficiently well describes ratcheting also in complex situations (biaxial ratcheting under stress control, e.g.). Thus, there is a wide field of current research. Here, our aim is to compare exemplarily a 1M model (an extended Chaboche model) and a 2M1C-b model (as in Subsection 3.3). A short description of these models will be given below. The subsequent results stem from Hassan et al. (2008).

At first, uniaxial ratcheting is considered. The experimental response of the steel SS304L and simulations are compared. The mean stress is 50 MPa, the equivalent stress amplitude is 200 MPa.

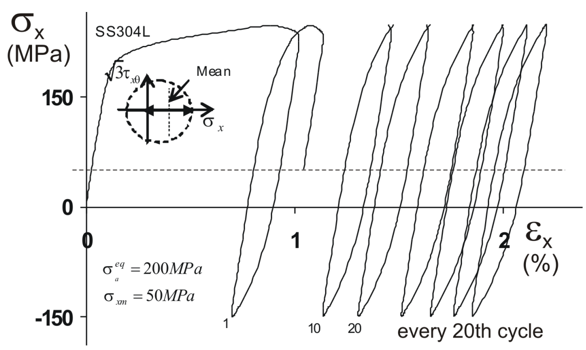


Figure 3: Uniaxial ratcheting: Hysteresis loops from experiment

Figure 3 shows the hysteresis loops from experiment, while Figure 4 presents the results of simulations.

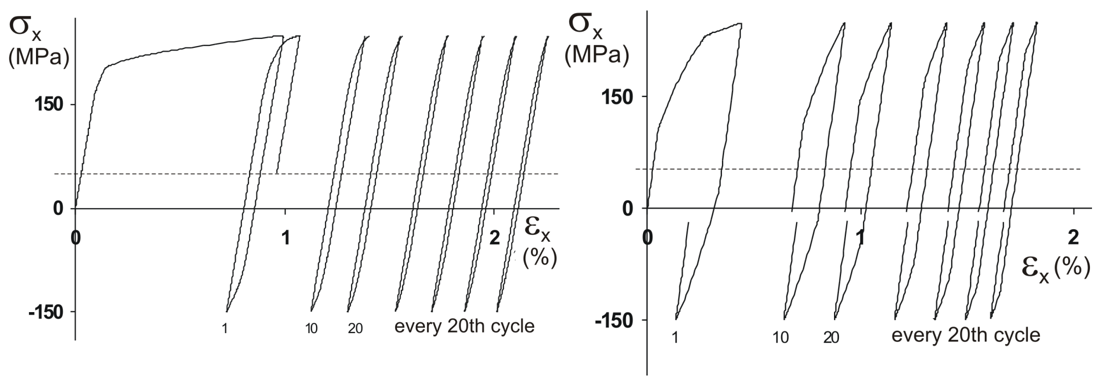


Figure 4: Uniaxial ratcheting: Simulation by modified Chaboche model (left), and by a 2M1C-b model (right)

Secondly, a biaxial experiment is considered. The mean stress and the equivalent stress amplitude remain the same. Again, experimental results (Fig. 5) are confronted with simulations (Fig. 6). At first view, one notes that, contrary to the first loop, the subsequent loops are better represented by the 2M model in both cases. However, in general, the capabilities of these models are similar in these simulations despite the significant difference related to the number of material parameters of the two models: 23 for the modified Chaboche model and only 12 parameters for the 2M model (see Hassan et al. (2008) and belove for a short overview). Besides, both models represent the smaller ratcheting strain in the biaxial experiment.

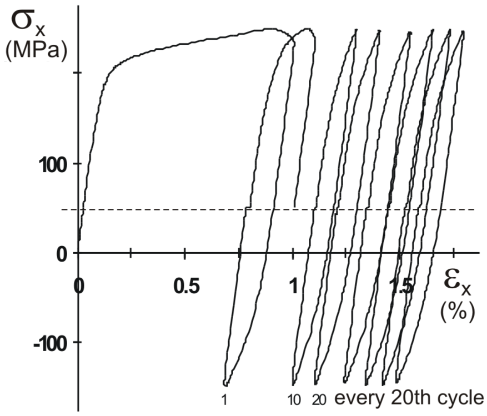
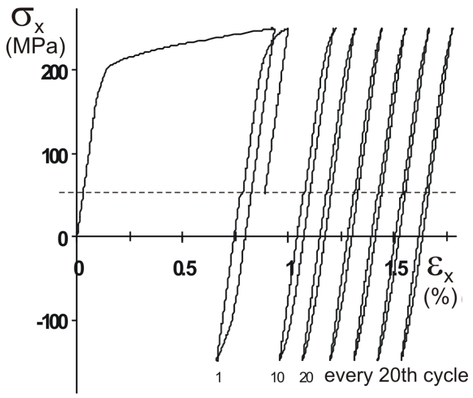


Figure 5: Biaxial ratcheting: Hysteresis loops from experiment

Thus, the 2M models (or, more generally nM models) have a potential for modelling and simulation of ratcheting, even in the light of limitation of material parameters. One can note that more recent versions of the modified Chaboche model (Krishna et al. (2009)) and the 2M model (Taleb and Cailletaud (2010)) have been proposed.



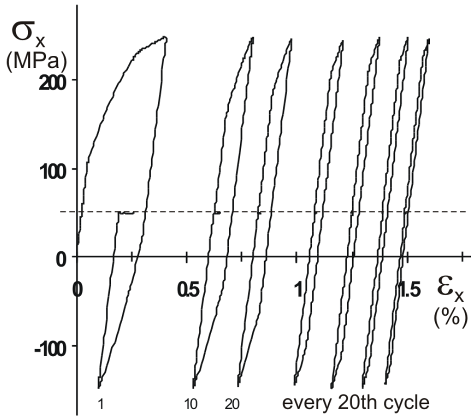


Figure 6: Biaxial ratcheting: Simulation by a modified Chaboche model (left), and by a 2M1C-b model (right)

For a better readability we give short descriptions of the above both models. For detailed explanations, discussions, references and parameter identification we refer to Hassan et al. (2008).

Description of the modified Chaboche model used for simulations

A temperature-independent version of the model is used. Flow function, additive strain decomposition, Hooke's law and flow rule are standard. Young's modulus E, Poisson's ratio ν and the initial yield stress σ_0 are the elastic parameters. The back stress is the sum of four partial back stresses.

$$\boldsymbol{X} = \sum_{i=1}^{4} \boldsymbol{X}_i. \tag{7.1}$$

Note that our notation differs from the one in Hassan et al. (2008). The evolution of the back stresses is given by

$$\dot{\boldsymbol{X}}_i = \frac{2}{3}\, c_i\, \boldsymbol{\varepsilon}_p - \gamma_i\, (\delta\, \boldsymbol{X}_i + (1-\delta)(\boldsymbol{X}_i : \boldsymbol{n})\boldsymbol{n})\, \dot{s}_p \qquad \text{for } i = 1, 2, 3, \tag{7.2}$$

$$\dot{\boldsymbol{X}}_4 = \frac{2}{3}\, c_4\, \boldsymbol{\varepsilon}_p - \gamma_4\, (\delta\, \boldsymbol{X}_4 + (1-\delta)(\boldsymbol{X}_4 : \boldsymbol{n})\boldsymbol{n}) \left\langle 1 - \frac{a_4}{\sigma_{eq}(\boldsymbol{X}_4)} \right\rangle \dot{s}_p. \tag{7.3}$$

$\boldsymbol{n}$ is the normal to the yield surface, c_i, γ_i $(i = 1, \ldots, 4)$, δ and a_4 are material parameters, σ_{eq} is the equivalent von Mises stress (cf. (2.24)), $\langle \cdot \rangle$ are the McCauley brackets (cf. (2.38)). Moreover, a non-proportionality parameter A is defined by

$$A = 1 - \cos^2(\alpha) \qquad \text{with} \quad \cos(\alpha) = \frac{\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\sigma}}^*}{\varepsilon_{eq}(\dot{\boldsymbol{\varepsilon}}_p)\, \sigma_{eq}(\dot{\boldsymbol{\sigma}}^*)}. \tag{7.4}$$

ε_{eq} is the equivalent strain (cf. (2.7)). The evolution of the isotropic hardening variable R is given by

$$\dot{R} = D_g(A)\,(R^{AS}(A) - R)\,\dot{s}_p, \quad R(0) = 0. \tag{7.5}$$

With given parameters g, R^0, R^∞, d_R and f_R, the functions D_g and R^{AS} are assumed as

$$D_g(A) = (d_R - f_R)A + f_R, \qquad R^{AS}(A) = \frac{gR^\infty A + (1-A)R^0}{gA + (1-A)}. \tag{7.6}$$

To take the influence of non-proportionality on kinematic hardening into account, the parameters γ_i ($i = 1, \ldots, 4$) are supposed to be functions fulfilling

$$\dot{\gamma}_i = D_{\gamma i}(A)\,(\gamma_i^{AS}(A) - \gamma_i)\dot{s}_p, \quad \gamma_i(0) = \gamma_{0i} \quad \text{for } i = 1, \ldots, 4. \tag{7.7}$$

The initial values γ_{0i} of γ_i are parameters which must be defined. With given parameters γ_i^∞, γ_i^0, $d_{\gamma i}$, $f_{\gamma i}$ and g, the functions $D_{\gamma i}$ and γ_i^{AS} are defined by

$$D_{\gamma i}(A) = (d_{\gamma i} - f_{\gamma i})A + f_{\gamma i}, \qquad \gamma_i^{AS}(A) = \frac{g\gamma_i^\infty A + (1-A)\gamma_i^0}{gA + (1-A)}. \tag{7.8}$$

(i) In the case of **proportional loading** (as in the case of uniaxial ratcheting), one has $A = 0$, and (7.6) and (7.8) yield

$$D_R(A) = f_R, \quad R^{AS}(A) = R^0, \quad D_{\gamma i}(A) = f_{\gamma i}, \quad \gamma_i^{AS}(A) = \gamma_i^0. \tag{7.9}$$

Therefore, the equations (7.5) and (7.7) get the special form

$$\dot{R} = f_R\,(R^0 - R)\,\dot{s}_p, \quad \dot{\gamma}_i = f_{\gamma i}\,(\gamma_i^0 - \gamma_i)\dot{s}_p, \quad \text{for } i = 1, \ldots, 4. \tag{7.10}$$

Summarizing, in the case of uniaxial ratcheting, the simulation by the modified Chaboche model (see Fig. 4, left) has been performed with the following 23 parameters (cf. Hassan et al. (2008))

$$\begin{array}{llll} E = 180\,\text{GPa} & \nu = 0.30 & \sigma_0 = 153.2\,\text{MPa} & \\ c_1 = 540.2\,\text{MPa} & c_2 = 1937\,\text{MPa} & c_3 = 625\,\text{MPa} & c_4 = 73.25\,\text{MPa} \\ \gamma_{01} = 28.285 & \gamma_{02} = 740 & \gamma_{03} = 12.8 & \gamma_{04} = 6084 \\ a_4 = 14.1\,\text{MPa} & \delta = 0.13 & f_R = 0.8 & R^0 = 10\,\text{MPa} \\ f_{\gamma 1} = 16 & f_{\gamma 2} = 7.7 & f_{\gamma 3} = 3.45 & f_{\gamma 4} = 9 \\ \gamma_1^0 = 12.524 & \gamma_2^0 = 340 & \gamma_3^0 = 8.78 & \gamma_4^0 = 1952 \end{array} \tag{7.11}$$

(ii) In the case of **non-proportional loading** (as in the case of biaxial ratcheting as above), one has $A = 1$, and from (7.6) and (7.8) it follows

$$D_R(A) = d_R, \quad R^{AS}(A) = R^\infty, \quad D_{\gamma i}(A) = d_{\gamma i}, \quad \gamma_i^{AS}(A) = \gamma_i^\infty. \tag{7.12}$$

Thus, (7.5) and (7.7) are reduced to

$$\dot{R} = d_R\,(R^\infty - R)\,\dot{s}_p, \quad \dot{\gamma}_i = d_{\gamma i}\,(\gamma_i^\infty - \gamma_i)\dot{s}_p, \quad \text{for } i = 1, \ldots, 4. \tag{7.13}$$

Finally, in the case of biaxial ratcheting as above, the simulation by the modified Chaboche model (see Fig. 6, left) has been performed with the following 23 parameters (cf. Hassan et al. (2008))

$$\begin{array}{llll} E = 180\,\text{GPa} & \nu = 0.30 & \sigma_0 = 153.2\,\text{MPa} & \\ c_1 = 540.2\,\text{MPa} & c_2 = 1937\,\text{MPa} & c_3 = 625\,\text{MPa} & c_4 = 73.25\,\text{MPa} \\ \gamma_{01} = 28.285 & \gamma_{02} = 740 & \gamma_{03} = 12.8 & \gamma_{04} = 6084 \\ a_4 = 14.1\,\text{MPa} & \delta = 0.13 & d_R = 5.0 & R^\infty = 30\,\text{MPa} \\ d_{\gamma 1} = 66 & d_{\gamma 2} = 84 & d_{\gamma 3} = 3.45 & d_{\gamma 4} = 82.5 \\ \gamma_1^\infty = 9549 & \gamma_2^\infty = 291 & \gamma_3^\infty = 8.78 & \gamma_4^\infty = 1688 \end{array} \tag{7.14}$$

The strategy for determining the material parameters from experimental data is explained in Hassan et al. (2008). For completeness we note that in the case of general non-proportional ratcheting, the modified Chaboche model above needs 34 parameters. The parameter g is only needed in this general case (see Hassan et al. (2008)).

Description of the two-mechanism model used for simulations
This 2M model is a temperature-independent 2M1C-b model (see Subsection 3.2). The parameters A_1 and A_2 in (2.12) are taken equal to one. The back stresses $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and the isotropic hardening R are defined in (2.21) and (2.22). The evolution of $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ and of q is governed by (2.31) and (2.32) (with $r = 1$). The evolution of the internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ is given by (3.22) with $a_1 = a_2 = 1$ and with $\eta_1 = \eta_2 = \eta$. Thus, the simulation by this 2M model (see Fig. 4 and 6, right) requires the following 12 material parameters which are given as (cf. Hassan et al. (2008))

$$
\begin{array}{lll}
E = 180\,\mathrm{GPa} & \nu = 0.30 & R_0 = 200\,\mathrm{MPa} \\
c_{11} = 481.1\,\mathrm{MPa} & c_{12} = 5458\,\mathrm{MPa} & c_{22} = 13.53\,\mathrm{MPa} \\
D_1 = 15.13 & D_2 = 21 & \eta = 0.71 \\
N = 1 & Q = 4000\,\mathrm{MPa} & b = 20
\end{array}
\tag{7.15}
$$

Finally, we note that the parameter identifications for both models have been performed separately. Therefore, the values of the initial yield stress differ. This effect can be compensated by the different structure of the models. Moreover, a non-proportionality effect has not yet been included into the two-mechanism model.

8 Conclusions

Two-mechanism models are the subject of this study. Our new results are:

- The proof of thermodynamic consistency for some types of 2M models,
- The derivation of useful relations for back stresses generalizing the Armstrong-Frederick relations known for 1M models.
- A reasonable extension within the evolution equations of the internal variables which allows non-symmetric Armstrong-Frederick relations for the back stresses.

Finally, we have presented a comparison of special 1M and 2M models in the simulation of ratcheting behavior in order to illustrate the possibilities and problems. This presentation is taken from Hassan et al. (2008).

We are well aware that there remains a lot of further work to do.

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