

# A coupled isotropic elasto-plastic damage model based on incremental minimization principles

O. Kintzel, J. Mosler

*In the present paper, a variational formulation of an isotropic elasto-plastic damage model is proposed. As prototype model, a coupled formulation originally introduced by LEMAITRE is considered. It is governed by non-linear and non-associative evolution equations. The variational approach advocated within the present paper allows to compute all state variables by means of energy minimization. The performance of the proposed framework is illustrated by a comparison between the novel variational method and a standard return-mapping scheme.*

## 1 Introduction

For the simulation of isotropic ductile damage, two modeling classes are often used. The first of those is based on a sound characterization of the micromechanism of ductile void growth, Gurson (1977), where the damage evolution is mainly driven by the mean stress. However, the consideration of load reversals or void closure effects is difficult and requires further adjustments which are not completely clear, cf. Kintzel (2009). The second class is a rather phenomenological approach which is based on the rules of continuum damage mechanics (CDM). Starting from the early proposition of Kachanov (1958), this class has been applied to a wide range of different damage phenomena. In the present paper, the well-known ductile CDM-model of Lemaitre (1992); Lemaitre and Chaboche (1994); Lemaitre and Desmorat (2005) is used.

The evolution equations of constitutive models are frequently solved by means of a standard return-mapping, Simo and Hughes (1998). Using this approach, the evolution laws are approximated by a suitable time discretization and the constraint resulting from the yield function is enforced directly. However, in this solution strategy the physics behind the equations is neglected. It has been clear beforelong that the underlying physics is reflected mathematically by the presence of certain extremum principles such as the principle of minimum potential energy or the principle of maximum dissipation. This is strongly related to the important physical fact that the final state of a solid is often represented by a certain equilibrium (This state is favored among all neighboring states). Clearly, such a mathematical and physical structure can be recast into a variational framework.

Many topics about variational methods and their implications on mathematical uniqueness and stability (in the case of plasticity, see Drucker (1964); Hill (1958)), their relations to convexity or the symmetry required for the tangent operator, Petryk (2003) have been discussed in literature. However, only relatively recently, Ortiz and Repetto (1999), have elaborated a variational approach for analyzing the development of a local microstructure. Since then, similar methods have been applied to many different rate problems, cf. Ortiz and Stainier (1999); Carstensen et al. (2002). A practical advantage of the variational approach compared to a standard return-mapping scheme is that conventional minimization routines can be exploited.

Today, variational principles based on incremental energy minimization seem to be widely accepted, even though they are seldom applied to more complex rate problems than those showing associative evolution equations (e.g. Mosler and Bruhns (2010)). In case of a relatively simple DRUCKER-PRAGER-type model, the extension to the non-associative case has been recently considered in Mosler (2009).

In the present paper, a variational constitutive update is developed for ductile material damage. Except for relatively simple damage models such as those proposed in Gürses (2007); Gürses et al. (2003), a variationally consistent description of material degradation (damage evolution) has not been elaborated yet. In contrast to the prototype advocated in Gürses (2007); Gürses et al. (2003), the present model is formulated for elasto-plasticity using the well-known effective stress concept. Different to the proposition of LEMAITRE, the plastic part of the free energy is also included in the rate of energy release, cf. Grammenoudis et al. (2009).

The paper is structured as follows: First, a short introduction to the variational method is given considering small strain elasto-plasticity. Afterwards, the constitutive relations for the isotropic ductile damage model are outlined. Then, the coupled model is reformulated within a variationally consistent framework. Finally, the proposed novel algorithmic formulation will be compared to a conventional return-mapping scheme by considering an academic 3D-example.

## 2 Fundamentals of variational constitutive updates

In this section, a variationally consistent framework suitable for constitutive modeling is discussed. Within the framework, all state variables follow jointly from minimizing an incrementally defined energy functional. This method is illustrated by considering one prototype.

### 2.1 General framework

Starting from the principle of virtual power, where the displacement variations have been replaced by the velocities of the displacements, the balance of power integrated over a finite volume  $\Omega$  can be written as, cf. Bertram (2005),

$$\int_{\Omega} [\mathcal{E}(\dot{\boldsymbol{\epsilon}}) - \dot{W}^{\text{ext}}(\mathbf{u})] dV = 0, \quad (1)$$

where  $\mathbf{u}$  is the displacement field,  $\boldsymbol{\epsilon}$  is the strain tensor,  $\boldsymbol{\sigma}$  is the stress tensor,  $\mathcal{E}(\dot{\boldsymbol{\epsilon}}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}$  is the stress power,  $\dot{W}^{\text{ext}}$  is the power of external forces and the superimposed dot denotes the material time derivative. In Eq. (1), conservative forces have been assumed. Clearly, Eq. (1) is the stationarity condition associated with the variational problem

$$\inf_{\mathbf{u}} I_{tot} = \inf_{\mathbf{u}} \int_{\Omega} [\Psi(\boldsymbol{\epsilon}) - W^{\text{ext}}(\mathbf{u})] dV, \quad \text{with} \quad \Psi = \int \dot{\Psi}(\boldsymbol{\epsilon}) dt = \int \mathcal{E} dt, \quad (2)$$

if hyperelastic continua are considered. In Eq. (2),  $I_{tot}$  is the global potential energy and  $\Psi$  represents the HELMHOLTZ free energy.

Next, a similar variational framework suitable for the analysis of dissipative processes will be discussed. According to Eqs. (1) and (2), the material dependent part of the potential  $I_{tot}$  is defined by the stress power  $\mathcal{E}$ . As a consequence, the functional to be minimized will strongly rely on  $\mathcal{E}$  also for the modeling of dissipative solids. Starting with a HELMHOLTZ free energy of the type  $\Psi = \Psi(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^p, \boldsymbol{\alpha})$  with  $\boldsymbol{\epsilon}^p$  being the plastic strains and  $\boldsymbol{\alpha}$  representing a collection of other internal variables, the stress power is computed as

$$\mathcal{E} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} = \dot{\Psi} - \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^p} : \dot{\boldsymbol{\epsilon}}^p - \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \dot{\boldsymbol{\alpha}} = \dot{\Psi} + \mathcal{D}, \quad (3)$$

where

$$\mathcal{D} = -\frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^p} : \dot{\boldsymbol{\epsilon}}^p - \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \dot{\boldsymbol{\alpha}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \mathbf{Q} \dot{\boldsymbol{\alpha}}, \quad \mathbf{Q} := -\partial_{\boldsymbol{\alpha}} \Psi \quad (4)$$

is the dissipation. Accordingly,  $\mathcal{D} = \mathcal{D}(\boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}}^p, \mathbf{Q}, \dot{\boldsymbol{\alpha}})$  and consequently,  $\mathcal{E} = \mathcal{E}(\dot{\boldsymbol{\epsilon}}, \boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}}^p, \mathbf{Q}, \dot{\boldsymbol{\alpha}})$ . It will be shown in what follows that analogously to hyperelasticity, the minimization of the stress power is also a sound variational principle in case of dissipative processes. Clearly, inadmissible states have to be a priori excluded. For this reason, a space of admissible states is introduced. It is defined by a so-called indicator function  $J$ . In line with plasticity theory, the function  $J$  is assumed to depend on stress-like variables, i.e.,

$$J(\boldsymbol{\sigma}, \mathbf{Q}) := \begin{cases} 0 & \forall (\boldsymbol{\sigma}, \mathbf{Q}) \in \mathbb{E}_{\boldsymbol{\sigma}} \\ \infty & \text{otherwise} \end{cases}. \quad (5)$$

Here,  $\mathbf{Q}$  are the dual variables conjugate to  $\boldsymbol{\alpha}$  and  $\mathbb{E}_{\boldsymbol{\sigma}}$  represents the space of admissible states. In case of elasto-plasticity,  $\mathbb{E}_{\boldsymbol{\sigma}}$  denotes the space of admissible stresses which is usually defined by means of a yield function. With this notation, the stress power is re-written as

$$\mathcal{E}(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}, \mathbf{Q}) = \dot{\Psi}(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) + \mathcal{D}(\boldsymbol{\sigma}, \mathbf{Q}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) + J(\boldsymbol{\sigma}, \mathbf{Q}). \quad (6)$$

Accordingly, inadmissible states result in  $\mathcal{E} \rightarrow \infty$  and hence, if energy minimization is the overriding principle, such states are naturally excluded. For associative evolution equations fulfilling the postulate of maximum dissipation (maximization with respect to the stress-like variables), the unconstrained stress power reads

$$\mathcal{E}(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) = \dot{\Psi}(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) + J^*(\dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}), \quad (7)$$

with  $J^*$  being the LEGENDRE transformation of Eq. (6), i.e.,

$$J^*(\dot{\epsilon}^p, \dot{\alpha}) = \sup_{\sigma, \mathbf{Q}} (\mathcal{D}(\sigma, \mathbf{Q}, \dot{\epsilon}^p, \dot{\alpha}) \mid J(\sigma, \mathbf{Q}) = 0). \quad (8)$$

Physically speaking,  $J^*$  is the dissipation, if admissible stress states and associative evolution equations are considered. The interesting properties of function (7) become apparent, when the stationarity conditions are analyzed. More precisely, it can be shown that

$$(\dot{\epsilon}^p, \dot{\alpha}) = \arg \inf_{\dot{\epsilon}^p, \dot{\alpha}} \mathcal{E}(\dot{\epsilon}, \dot{\epsilon}^p, \dot{\alpha}) \big|_{\dot{\epsilon}=0} \quad (9)$$

and the stresses are defined by

$$\sigma = \partial_{\dot{\epsilon}} \inf_{\dot{\epsilon}^p, \dot{\alpha}} \mathcal{E}(\dot{\epsilon}, \dot{\epsilon}^p, \dot{\alpha}). \quad (10)$$

Further details are omitted. They can be found elsewhere, cf. Ortiz and Stainier (1999); Carstensen et al. (2002); Mosler and Bruhns (2010); Mosler (2009). Based on Eq. (9) efficient numerical implementations can be developed simply by discretizing the continuous problem (9), i.e.,

$$(\epsilon^p, \alpha) := \arg \inf I_{\epsilon_{loc}}(\epsilon, \epsilon^p, \alpha) \big|_{\epsilon=\text{const}}, \quad I_{\epsilon_{loc}}(\epsilon, \epsilon^p, \alpha) = \int_{t_n}^{t_{n+1}} [\dot{\Psi} + J^*] dt. \quad (11)$$

Clearly, if the considered time discretization is consistent, the resulting update scheme is consistent as well. Since variational updates are, despite their flexibility and efficiency, nowadays still not standard, such schemes will be explained by means of a simple prototype in what follows.

## 2.2 Example: Associative elasto-plasticity at small strains

In this subsection, the aforementioned variational principle is elaborated for the theory of associative plasticity of VON MISES type. For that purpose, the yield function

$$\phi = \phi(\sigma) = \sqrt{\frac{3}{2} \text{dev} \sigma : \text{dev} \sigma} - Q_0^{eq} \leq 0, \quad (12)$$

is considered. Here, the (constant) radius of the yield surface  $Q_0^{eq}$  has been introduced. Applying the classical principle of maximum dissipation, the evolution law reads

$$\dot{\epsilon}^p = \lambda \frac{\partial \phi}{\partial \sigma}, \quad (13)$$

where  $\lambda \geq 0$  is the plastic multiplier. Since the yield function (12) is positively homogeneous of degree one, the dissipation can be calculated explicitly as

$$\mathcal{D} = \sigma : \dot{\epsilon}^p = \lambda Q_0^{eq}. \quad (14)$$

Accordingly, by inserting Eq. (14) into Eq. (11) and considering physically admissible states ( $J = 0 \Rightarrow J^* = \mathcal{D}$ ), the variational constitutive update is given by

$$(\epsilon^p, \alpha) := \arg \inf I_{\epsilon_{loc}}(\epsilon, \epsilon^p, \alpha) \big|_{\epsilon=\text{const}}, \quad I_{\epsilon_{loc}}(\epsilon, \epsilon^p, \alpha) = \Psi_{n+1} - \Psi_n + \Delta \lambda Q_0^{eq}, \quad \Delta \lambda = \int_{t_n}^{t_{n+1}} \lambda dt. \quad (15)$$

As expected, the numerical scheme does depend on the time discretization. However, consistency is fulfilled in any case (Eq. (15) converges to Eq. (9), if  $\Delta t \rightarrow 0$ ). For completing the example, an implicit backward EULER integration is applied here, i.e.,

$$\epsilon_{n+1}^p = \epsilon_n^p + \Delta \lambda \partial_{\sigma} \phi|_{n+1}. \quad (16)$$

Furthermore and in line with Mosler and Bruhns (2010); Mosler (2009), a parameterization in terms of pseudo stresses  $\tilde{\Sigma} \neq \sigma$  is utilized. Such stresses fulfill the compatibility condition

$$\partial_{\sigma} \phi|_{\sigma_{n+1}} = \partial_{\sigma} \phi|_{\tilde{\Sigma}} \quad (17)$$

and enforce the flow direction explicitly. With Eqs. (16) and (17) the final minimization problem reads

$$(\Delta \lambda, \tilde{\Sigma}) := \arg \inf I_{\epsilon_{loc}}(\epsilon_{n+1}, \Delta \lambda, \tilde{\Sigma}) \big|_{\epsilon_{n+1}=\text{const}}, \quad I_{\epsilon_{loc}}(\epsilon_{n+1}, \Delta \lambda, \tilde{\Sigma}) = \Psi_{n+1} - \Psi_n + \Delta \lambda Q_0^{eq}. \quad (18)$$

The consistency of the scheme can be checked in a straightforward manner by computing the stationarity conditions of Eq. (18). They result in

$$\frac{\partial I_{\epsilon_{loc}}}{\partial \Delta \lambda} = -\boldsymbol{\sigma} : \partial \boldsymbol{\sigma} \phi + Q_0^{eq} = -\phi = 0 \quad (19)$$

and

$$\frac{\partial I_{\epsilon_{loc}}}{\partial \boldsymbol{\Sigma}} = -\Delta \lambda \boldsymbol{\sigma} : \partial \boldsymbol{\sigma}^2 |_{\boldsymbol{\Sigma}} = 0. \quad (20)$$

Accordingly, the yield function is consistently included within the variational scheme (see Eq. (19)), while Eq. (20) can be interpreted as a compatibility condition between the physical stresses and their pseudo counterparts. Eq. (20) enforces the correct flow direction, cf. Mosler and Bruhns (2010); Mosler (2009).

### 3 Modeling of isotropic ductile damage

This section is concerned with a concise review of an isotropic damage model originally introduced by Lemaitre (1992); Lemaitre and Chaboche (1994); Lemaitre and Desmorat (2005). However and in contrast to LEMAITRE's model, the plastic part of the free energy is also included in the rate of the energy release, cf. Grammenoudis et al. (2009).

Introducing a scalar-valued damage variable  $D$  for defining the degree of material degradation, a HELMHOLTZ free energy of the type

$$\Psi = (1 - D) \frac{\boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e}{2} + (1 - D) H_i \frac{\alpha_i^2}{2} + (1 - D) H_k \frac{\boldsymbol{\alpha}_k : \boldsymbol{\alpha}_k}{2} \quad (21)$$

is adopted in what follows. In Eq. (21),  $\mathbb{C}$  is the elastic constitutive tensor,  $H_i$  and  $H_k$  are the hardening moduli for the isotropic as well as for the kinematic part and  $\alpha_i$  and  $\boldsymbol{\alpha}_k$  denote the respective strain-like internal variables. Accordingly, the elastic part and the plastic part of the energy are reduced by the damage variable  $D \in [0, 1]$ . Based on Eq. (21), the stress-like variables are defined in standard fashion, i.e.,

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}^e} = (1 - D) \mathbb{C} : \boldsymbol{\epsilon}^e, \quad \mathbf{Q}_k = -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}_k} = -(1 - D) H_k \boldsymbol{\alpha}_k, \quad Q_i = -\frac{\partial \Psi}{\partial \alpha_i} = -(1 - D) H_i \alpha_i. \quad (22)$$

Here,  $Q_i$  and  $\mathbf{Q}_k$  are the stress-like variables conjugate to  $\alpha_i$  and  $\boldsymbol{\alpha}_k$ . For deriving a physically sound model, the effective stress concept is utilized here. Using this concept, the Helmholtz energy associated with a fictive undamaged state is postulated as

$$\tilde{\Psi} = \frac{\tilde{\boldsymbol{\epsilon}}^e : \mathbb{C} : \tilde{\boldsymbol{\epsilon}}^e}{2} + H_i \frac{\tilde{\alpha}_i^2}{2} + H_k \frac{\tilde{\boldsymbol{\alpha}}_k : \tilde{\boldsymbol{\alpha}}_k}{2}. \quad (23)$$

Here and henceforth, the superimposed tilde symbol highlights variables associated with the fictive undamaged state. With Eq. (23), the respective effective stress-like variables result in

$$\tilde{\boldsymbol{\sigma}} = \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{\epsilon}}^e} = \mathbb{C} : \tilde{\boldsymbol{\epsilon}}^e, \quad \tilde{\mathbf{Q}}_k = -\frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{\alpha}}_k} = -H_k \tilde{\boldsymbol{\alpha}}_k, \quad \tilde{Q}_i = -\frac{\partial \tilde{\Psi}}{\partial \tilde{\alpha}_i} = -H_i \tilde{\alpha}_i. \quad (24)$$

According to the principle of strain equivalence which states that  $\tilde{\boldsymbol{\epsilon}}^e = \boldsymbol{\epsilon}^e$  and  $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$ , the relationships

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{(1 - D)} = \mathbb{C} : \boldsymbol{\epsilon}^e, \quad \tilde{\mathbf{Q}}_k = \frac{\mathbf{Q}_k}{(1 - D)} = -H_k \boldsymbol{\alpha}_k \quad \text{and} \quad \tilde{Q}_i = \frac{Q_i}{(1 - D)} = -H_i \alpha_i \quad (25)$$

between the effective (fictive undamaged) and the real stresses hold, cf. Lemaitre (1992). For defining reversible as well as irreversible processes a yield function is defined. It is described by means of effective stress-like variables. In the following, the VON MISES-type function

$$\phi = \sqrt{\frac{3}{2} \text{dev}(\tilde{\boldsymbol{\sigma}} - \tilde{\mathbf{Q}}_k) : \text{dev}(\tilde{\boldsymbol{\sigma}} - \tilde{\mathbf{Q}}_k) - (\tilde{Q}_i + Q_0^{eq})} \leq 0 \quad (26)$$

is adopted. For providing enough flexibility for the evolution equations, a plastic potential  $\bar{\phi}$  is utilized. It is assumed to be

$$\bar{\phi} = \phi + \frac{B_k}{H_k} \frac{\tilde{\mathbf{Q}}_k : \tilde{\mathbf{Q}}_k}{2} + \frac{B_i}{H_i} \frac{\tilde{Q}_i^2}{2} + \frac{Y^M}{M S_1 (1 - D)}. \quad (27)$$

The additional quadratic terms depending on the stress-like internal variables  $\tilde{\mathbf{Q}}_k$  and  $\tilde{Q}_i$  are associated with nonlinear kinematic and isotropic hardening of ARMSTRONG-FREDERICK-type, while the last term corresponds to the evolution law of the damage related variable  $D$ . In Eq. (27),  $B_k$ ,  $H_k$ ,  $B_i$ ,  $H_i$ ,  $M$  and  $S_1$  are material parameters and  $Y$  denotes the rate of energy release defined by

$$Y = -\frac{\partial \Psi}{\partial D} = \frac{\boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e}{2} + H_i \frac{\alpha_i^2}{2} + H_k \frac{\boldsymbol{\alpha}_k : \boldsymbol{\alpha}_k}{2}. \quad (28)$$

Based on the convex function (27) the evolution equations are postulated to be

$$\dot{\boldsymbol{\epsilon}}^p = p \mathbf{n}, \quad \dot{\boldsymbol{\alpha}}_k = -p (\mathbf{n} + B_k \boldsymbol{\alpha}_k), \quad \dot{\alpha}_i = -p(1 + B_i \alpha_i) \quad \text{and} \quad \dot{D} = p \frac{Y^{M-1}}{S_1}, \quad (29)$$

where  $p := \frac{\lambda}{(1-D)}$  and  $\mathbf{n} := \partial_{\boldsymbol{\sigma}} \bar{\phi}$ . Considering Eq. (29), the damage evolution and the elasto-plastic rate problem are uncoupled, since the plastic variables  $\Delta p$  and  $\mathbf{n}$  are defined completely by the elasto-plastic equations. Based on this uncoupling efficient numerical implementations can be derived. One efficient, variationally consistent method is discussed in the next section.

#### 4 A variational constitutive update for ductile damage

In the present section, a variational constitutive update for the isotropic ductile material damage model as described in the previous subsection is elaborated.

##### 4.1 Time-continuous rate problem

First, the time-continuous case is considered here. For deriving a variational constitutive update, the stress power represents again the starting point. Considering the HELMHOLTZ energy (21), together with Eqs. (22) and (28), the dissipation is computed as

$$\mathcal{D}(\boldsymbol{\sigma}, \mathbf{Q}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \mathbf{Q}_k : \dot{\boldsymbol{\alpha}}_k + Q_i \dot{\alpha}_i + Y \dot{D}. \quad (30)$$

Combining Eq. (30) with the evolution equations (29), results finally in the reduced stress power (the evolution equations and the yield function are already included)

$$\mathcal{E}(\dot{\boldsymbol{\epsilon}}, \dot{\boldsymbol{\epsilon}}^p, \dot{\boldsymbol{\alpha}}) = \dot{\Psi} + p(1-D) Q_0^{eq} + p(1-D) (B_i H_i \alpha_i^2 + B_k H_k \boldsymbol{\alpha}_k : \boldsymbol{\alpha}_k) + p \frac{Y^M}{S_1}. \quad (31)$$

Alternatively, by using Eqs. (26), (21) and (29), the stress power can be reformulated as

$$\mathcal{E}(\dot{\boldsymbol{\epsilon}}, p, \tilde{\boldsymbol{\Sigma}}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - p(1-D) \underbrace{\{\partial_{\tilde{\boldsymbol{\sigma}}} \phi |_{\tilde{\boldsymbol{\Sigma}}} : [\tilde{\boldsymbol{\sigma}} - \tilde{\mathbf{Q}}_k] - (\tilde{Q}_i + Q_0^{eq})\}}_{= \phi}. \quad (32)$$

Here, a parameterization in terms of pseudo stresses  $\tilde{\boldsymbol{\Sigma}}$  has been adopted, compared to Eq. (17). With Eq. (32), consistency of the algorithm, i.e.,

$$(p, \tilde{\boldsymbol{\Sigma}}) = \arg \inf_{p, \tilde{\boldsymbol{\Sigma}}} \mathcal{E}(\dot{\boldsymbol{\epsilon}}, p, \tilde{\boldsymbol{\Sigma}}) |_{\boldsymbol{\epsilon} = \text{const}} \quad (33)$$

can be proved in a straightforward manner. More precisely, the respective stationarity conditions read

$$\delta_p \mathcal{E}(\dot{\boldsymbol{\epsilon}}, p, \tilde{\boldsymbol{\Sigma}}) = 0 \quad \Leftrightarrow \quad \phi = 0 \quad (34)$$

and

$$\delta_{\tilde{\boldsymbol{\Sigma}}} \mathcal{E}(\dot{\boldsymbol{\epsilon}}, p, \tilde{\boldsymbol{\Sigma}}) = \mathbf{0} \quad \Leftrightarrow \quad \partial_{\tilde{\boldsymbol{\sigma}}}^2 \phi |_{\tilde{\boldsymbol{\Sigma}}} : [\tilde{\boldsymbol{\sigma}} - \tilde{\mathbf{Q}}_k] = \mathbf{0}. \quad (35)$$

Consequently, the yield function is naturally included within the variational update (see Eq. (34)) and the compatibility condition (35) between the pseudo stresses and their physical counterparts enforces naturally the flow direction, cf. Mosler and Bruhns (2010); Mosler (2009).

## 4.2 The incremental variational functional

Next, a discrete approximation of the continuous variational update as explained in the previous section is briefly discussed here. For that purpose, the continuous Eq. (31) or (32) is discretized by using a suitable time integration. Clearly, if this integration is consistent, consistency of the resulting numerical scheme is a priori guaranteed, i.e., in the limiting case  $\Delta t \rightarrow 0$ , the algorithmic formulation converges to its underlying continuous variational update. In line with the standard return-mapping scheme, a backward EULER integration scheme is utilized here. Thus, the incremental energy functional

$$I_{\boldsymbol{\epsilon}_{loc}} = \Psi|_n^{n+1} + \Delta p (1 - D_{n+1}) Q_0^{eq} + \Delta p (1 - D_{n+1}) (B_i H_i \alpha_{i n+1}^2 + B_k H_k \boldsymbol{\alpha}_{k n+1} : \boldsymbol{\alpha}_{k n+1}) + \Delta p \frac{Y_{n+1}^M}{S_1} \quad (36)$$

and the associated variational principle

$$(\tilde{\boldsymbol{\Sigma}}_{n+1}, \Delta p) = \arg \inf_{\tilde{\boldsymbol{\Sigma}}_{n+1}, \Delta p} I_{\boldsymbol{\epsilon}_{loc}} |_{\boldsymbol{\epsilon} = \text{const}} \quad (37)$$

are considered. For computing the updated internal variables, a standard backward EULER integration is applied as well. Though consistency of the scheme is guaranteed, the respective stationarity conditions are given for the sake of completeness. Considering the limiting case  $\Delta t \rightarrow 0$ , they result, as expected, in

$$\left. \frac{\partial I_{\boldsymbol{\epsilon}_{loc}}}{\partial \Delta p} \right|_{\Delta t \rightarrow 0} = 0 \quad \Leftrightarrow \quad \phi = 0 \quad (38)$$

and

$$\left. \frac{\partial I_{\boldsymbol{\epsilon}_{loc}}}{\partial \tilde{\boldsymbol{\Sigma}}_{n+1}} \right|_{\Delta t \rightarrow 0} = \mathbf{0} \quad \Leftrightarrow \quad \partial_{\tilde{\boldsymbol{\sigma}}}^2 \phi |_{\tilde{\boldsymbol{\Sigma}}} : [\tilde{\boldsymbol{\sigma}} - \tilde{\mathbf{Q}}_k] = \mathbf{0}. \quad (39)$$

This confirms the consistency of the scheme: The yield function as well as the flow direction are naturally included within the variational update. Note that the damage variable has to be updated during the minimization procedure which makes the algorithm a little bit elaborate. This will be explained in more detail in a forthcoming contribution, Kintzel and Mosler (2010). The tangent is constructed by linearizing the constraints considering the yield function and the definition of the normal as residua, as is conventionally done, cf. Kintzel (2006).

Remark: Although the proposed variational constitutive update is, similarly to the return-mapping scheme, based on a standard backward EULER integration, both algorithmic formulations are not identical. Within the return-mapping scheme, the discretized evolution equations, together with the yield function, define usually the residuals. Contrariwise, the natural residuals associated with the variational scheme are the gradients of the function to be minimized. However, even in the limiting case  $\Delta t \rightarrow 0$ , they are not identical to the residuals defining the return-mapping scheme. Though constraint (38) is identical, the flow direction is enforced by a different equation, cf. Eq. (39).

## 5 Numerical example

In the present section, the accuracy of the variational constitutive update for ductile material damage is analyzed. The variational method is based on the coupled elasto-plastic damage model as described in section 4.2. Additionally, the same problem is solved by a standard return-mapping scheme. The selected material parameters are summarized in Tab. 1. The predicted CAUCHY-stresses for uniaxial tension (monotonic loading) are plotted in

Young's modulus $E$ :	200000 Mpa,	Poisson's ratio $\nu$ :	0.30,	Yield stress $Q_0^{eq}$ :	300 Mpa
Hardening modulus $H_i$ :	2850 Mpa,	Saturation parameter $B_i$ :	20.0,	$S_1$ -parameter:	0.003
Hardening modulus $H_k$ :	3000 Mpa,	Saturation parameter $B_k$ :	30.0,	$M$ -exponent:	2.0

Table 1: Material parameters employed for the ductile damage law.

Fig. 1. Accordingly, the novel variational update, as considered for only 20 load steps, leads to almost the same mechanical response as the return-mapping scheme based on 600 load steps. Hence, the method is robust and accurate and convergence is obtained even for a relatively coarse time discretization.

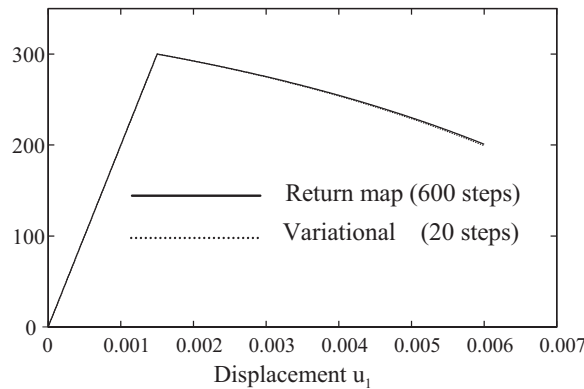


Figure 1: Uniaxial tension test: Stress-strain response as predicted by the novel variational constitutive update and by a standard return mapping scheme (3D-computation).

## 6 Conclusions

In this paper, a novel variational approach suitable for solving the non-linear and non-associative evolution equations for isotropic ductile material damage has been proposed. Considering the advocated approach, the governing equations have been formulated within a variationally consistent framework. Every aspect is consistently driven by incremental energy minimization. The complete ductile damage model has been examined numerically for a uniaxial one-element tension test. Particularly, the accuracy of the aforementioned variational approach has been compared to that of a conventional return-mapping scheme. This comparison revealed the robustness and reliability of the proposed variational constitutive update.

## References

- Bertram, A.: *Elasticity and Plasticity of Large Deformations*. Springer, Berlin (2005).
- Carstensen, C.; Hackl, K.; Mielke, A.: Non-convex potentials and microstructures in finite-strain plasticity. *Proceedings of the Royal Society London A*, 458, (2002), 299–317.
- Drucker, D.: On the postulate of stability of materials in the mechanics of continua. *Journal Mecanique*, 3, (1964), 235 – 249.
- Grammenoudis, P.; Reckwerth, D.; Tsakmakis, C.: Continuum Damage Models based on Energy Equivalence: Part I - Isotropic Material Response. *International Journal of Damage Mechanics*, 18 (No.1), (2009), 31–64.
- Gürses, E.: *Aspects of Energy Minimization in Solid Mechanics: Evolution of Inelastic Microstructures and Crack Propagation*. Ph.D. thesis, Institute for Mechanics, Stuttgart, Germany (2007).
- Gürses, E.; Lambrecht, M.; Miehe, C.: Application of relaxation techniques to a nonconvex isotropic damage model. *PAMM - Proceedings of applied Mathematics and Mechanics*, 3, (2003), 222 – 223.
- Gurson, A.: Continuum theory of ductile rupture by void nucleation and growth: Part I - Yield criteria and flow rules for porous ductile media. *Journal of Engineering Materials and Technology*, January, (1977), 2 – 15.
- Hill, R.: A general theory of uniqueness and stability in elasto-plastic solids. *Journal of the Mechanics and Physics of Solids*, 6, (1958), 236 – 249.
- Kachanov, L.: Time of the Rupture Process under Creep Conditions. *Isv. Akad. Nauk. SSR. Otd Tekh. Nauk.*, 8, (1958), 26–31.
- Kintzel, O.: *Modeling of elasto-plastic material behavior and ductile micropore damage of metallic materials at large deformations*. Ph.D. thesis, Ruhr-University Bochum, Germany. Download at: <http://www-brs.ub.ruhr-uni-bochum.de/netahtml/HSS/Diss/KintzelOlafE> (2006).
- Kintzel, O.: A comparison of ductile damage models with application to LCF. *PAMM- Proceedings of applied Mathematics and Mechanics*, in press, (2009), –.

- Kintzel, O.; Mosler, J.: A novel isotropic ductile-brittle damage model solved via variational constitutive updates. *prepared for International Journal for numerical Methods in Engineering*, -, (2010), –.
- Lemaitre, J.: *A course on damage mechanics*. Springer, Berlin (1992).
- Lemaitre, J.; Chaboche, J.-L.: *Mechanics of Solid Mechanics*. Cambridge University Press (1994).
- Lemaitre, J.; Desmorat, R.: *Engineering damage mechanics*. Springer, Berlin (2005).
- Mosler, J.: Variational consistent modeling of finite strain plasticity theory with non-linear kinematic hardening. *Computer Methods in Applied Mechanics and Engineering*, in press, (2009), –.
- Mosler, J.; Bruhns, O.: On the implementation of rate-independent standard dissipative solids at finite strain - variational constitutive updates. *Computer Methods in Applied Mechanics and Engineering*, 199, Issues 9-12, (2010), 417–429.
- Ortiz, M.; Repetto, E.: Nonconvex energy minimization and dislocation structures in ductile single crystals. *Journal of the Mechanics and Physics of Solids*, 47, (1999), 397 – 462.
- Ortiz, M.; Stainier, L.: The variational formulation of viscoplastic constitutive updates. *Computer Methods in applied Mechanics and Engineering*, 171, (1999), 419 – 444.
- Petryk, H.: Incremental energy minimization in dissipative solids. *Comptes Rendus Mecanique*, 331, (2003), 469 – 474.
- Simo, J.; Hughes, T.: *Computational Inelasticity*. Springer, New York (1998).

---

*Address:* Dr.-Ing. Olaf Kintzel and Prof. Dr.-Ing. Jörn Mosler, Materials Mechanics, Institute of Materials Research, GKSS-Research Centre Geesthacht, D-21502 Geesthacht, Germany.  
correspondence to: olaf.kintzel@gkss.de, joern.mosler@gkss.de