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# The Simplest Model of the Turning Movement of a Car with its Possible Sideslip

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*The simplest model of the turning movement of a car with its possiple sideslip is considered. To this end, a nonholonomic problem with nonretaining constraints is solved. The four possible types of the car motion are studied.*

## 1 Problem Formulation

The complete theory of the motion of a car with deformable wheels has been developed by N. A. Fufaev and detailed in his book (1989). The treatise by V. F. Zhuravlev and N. A. Fufaev (1993) is devoted to mechanics of systems with nonretaining constraints. In this treatise the Boltzmann-Hamel equations are used for studying the motion of nonholonomic systems; and the possibility of restoring nonholonomic constraints is investigated on the basis of behaviour of solution curves in the common space of generalized coordinates and quasivelocities. In this work the Maggi equations, which make it possible to easily determine the generalized reaction forces of nonholonomic constraints, are applied (Zegzhda et al., 2005); the beginning and stop of wheels sideslip is determined by these constraint forces.

In the paper (Sheverdin and Yushkov, 2002) the Maggi equations have been formed for studying the simplest model of the motion of a car while turning (see Figure 1). The car is assumed to consist of the body of mass $M_1$ and the front axle of mass $M_2$. They have moments of inertia $J_1$ and $J_2$, respectively, about the vertical axes that go through their centers of mass. The front axle can rotate about its vertical axis going through the axle's center. Masses of the wheels and back axle as separate parts are neglected. This scheme requires introduction of four generalized coordinates: $q^1 = \varphi$, $q^2 = \theta$, $q^3 = \xi_C$, and $q^4 = \eta_C$. The car is driven by a force $F_1(t)$ acting along its longitudinal axis and moment $L_1(t)$ turning the front axle, $F_1(t)$, $L_1(t)$ being given functions of time. Besides these, a resisting force $F_2(v_C)$ acting in the direction opposite to the direction of the velocity $\mathbf{v}_C$ of the car body centre of mass $C$, resisting moment $L_2(\dot\theta)$ applied to the front axle and opposite to the angular velocity of its rotation, and restoring moment $L_3(\theta)$ are taken into account. A similar scheme is introduced in (Lineikin, 1939) as a simplified mathematical model of the motion of a car while turning. In this formulation, the turning movement of a car is studied, figuratively speaking, under "dynamical control", when the turning moment $L_1(t)$, resisting moment $L_2(\dot\theta)$, and restoring moment $L_3(\theta)$ are applied to the rotating front axle. When introducing two nonholonomic constraints corresponding to the absence of side slipping of the back and front axles of the car (their equations (1) and (2) are given below), two Maggi's equations had to be set up.

Now consider the "kinematic control", under which the turning of the front axle is determined by a driver as a certain time function $\theta = \theta(t)$. In this scheme a turning car has three degrees of freedom: $q^1 = \varphi$, $q^2 = \xi_C$, $q^3 = \eta_C$. The varying function $\theta = \theta(t)$ characterizes only nonstationarity of the problem. As this takes place, nonholonomic constraints

$$\varphi^1 \equiv -\dot\xi_C \sin\varphi + \dot\eta_C \cos\varphi - l_2\dot\varphi = 0 \tag{1}$$

$$\varphi^2 \equiv -\dot\xi_C \sin(\varphi+\theta) + \dot\eta_C \cos(\varphi+\theta) + l_1\dot\varphi\cos\theta = 0 \tag{2}$$

should be satisfied by the car motion (these equations are derived in detail in the treatise (Zegzhda et al., 2005)). We consider these constraints as nonretaining. The active forces $F_1(t)$ and $F_2(t)$ have the same meaning as in the work (Sheverdin and Yushkov, 2002).

## 2 The Turning Movement of a Car with Retaining (bilateral) Constraints

We study the car motion in the horizontal plane with respect to the inertial frame $O\xi\eta\zeta$ (see Figure 1). As was said above, we set its position by generalized coordinates $q^1 = \varphi$, $q^2 = \xi_C$, $q^3 = \eta_C$. The angle $\theta$ is a prescribed time function

$$\theta = \theta(t)\,.$$

Two nonholonomic constraints (1) and (2), expressing the absence of side slipping of the front and rear (back) axles of the car are imposed on the car motion.

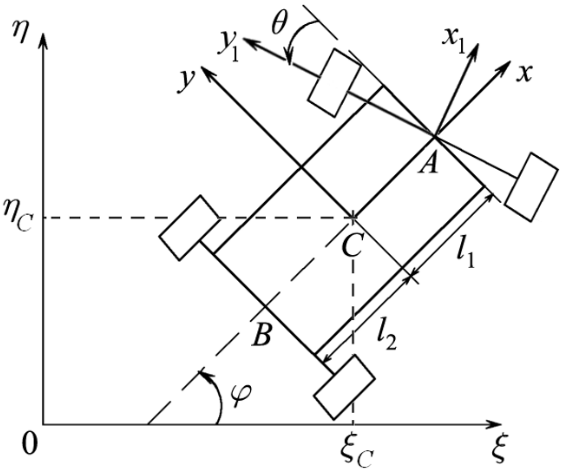


Figure 1. The scheme of a car.

The Kinetic energy of the system consists of the kinetic energies of the car body and front axle and is calculated according to

$$2T = M^*(\dot{\xi_C}^2 + \dot{\eta_C}^2) + J^*\dot\varphi^2 + J_2\dot\theta^2 + 2J_2\dot\varphi\dot\theta + 2M_2 l_1\dot\varphi(-\dot\xi_C\sin\varphi + \dot\eta_C\cos\varphi)$$

$$M^* = M_1 + M_2, \quad J^* = J_1 + J_2 + M_2 l_1^2\,. \tag{3}$$

Using the expression for virtual elementary work

$$\delta A = Q_\varphi\delta\varphi + Q_{\xi_C}\delta\xi_C + Q_{\eta_C}\delta\eta_C$$

we find the generalized forces acting on the car. For the rear drive car we obtain the following expressions

$$Q_1 \equiv Q_\varphi = 0$$

$$\begin{aligned} Q_2 \equiv Q_{\xi_C} &= F_1(t)\cos\varphi - F_2(v_C)\dot\xi_C/v_C \\ Q_3 \equiv Q_{\eta_C} &= F_1(t)\sin\varphi - F_2(v_C)\dot\eta_C/v_C \\ v_C &= \sqrt{\dot{\xi_C}^2 + \dot{\eta_C}^2}\,. \end{aligned} \tag{4}$$

In order to form the Maggi equations decribing the vehicle motion we introduce new nonholonomic variables according to (Zegzhda et al., 2005)

$$v_*^1 = \dot\varphi, \quad v_*^2 = -l_2\dot\varphi - \dot\xi_C\sin\varphi + \dot\eta_C\cos\varphi$$

$$v_*^3 = l_1\dot\varphi\cos\theta - \dot\xi_C\sin(\varphi+\theta) + \dot\eta_C\cos(\varphi+\theta)$$

and write the reverse transformation

$$\dot q^1 \equiv \dot\varphi = v_*^1\,, \quad \dot q^2 \equiv \dot\xi_C = \beta_1^2 v_*^1 + \beta_2^2 v_*^2 + \beta_3^2 v_*^3$$

$$\dot q^3 \equiv \dot\eta_C = \beta_1^3 v_*^1 + \beta_2^3 v_*^2 + \beta_3^3 v_*^3 \tag{5}$$

where

$$\beta_1^2 = \frac{l_1\cos\varphi\cos\theta + l_2\cos(\varphi+\theta)}{\sin\theta}$$

$$\beta_2^2 = \frac{\cos(\varphi+\theta)}{\sin\theta}\,, \quad \beta_3^2 = -\frac{\cos\varphi}{\sin\theta} \tag{6}$$

$$\beta_1^3 = \frac{l_1\sin\varphi\cos\theta + l_2\sin(\varphi+\theta)}{\sin\theta}$$

$$\beta_2^3 = \frac{\sin(\varphi+\theta)}{\sin\theta}, \quad \beta_3^3 = -\frac{\sin\varphi}{\sin\theta}\,.$$

The first group of the Maggi equations in this case consists of one equation

$$(MW_1 - Q_1)\frac{\partial \dot q^1}{\partial v_*^1} + (MW_2 - Q_2)\frac{\partial \dot q^2}{\partial v_*^1} + (MW_3 - Q_3)\frac{\partial \dot q^3}{\partial v_*^1} = 0\,. \tag{7}$$

The expressions $MW_\sigma$ can be calculated using kinetic energy by the formulas

$$MW_\sigma = \frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma} - \frac{\partial T}{\partial q^\sigma}\,, \quad \sigma = \overline{1,3}\,.$$

Consequently, using expressions (3), (4), (5), (6), let us represent the motion equation (7) in the following expanded form

$$\begin{gathered} J^*\ddot\varphi + J_2\ddot\theta + M_2 l_1(-\ddot\xi_C\sin\varphi + \ddot\eta_C\cos\varphi) + \\ +\beta_1^2(M^*\ddot\xi_C - M_2 l_1(\ddot\varphi\sin\varphi + \dot\varphi^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot\xi_C/v_C) + \\ +\beta_1^3(M^*\ddot\eta_C + M_2 l_1(\ddot\varphi\cos\varphi - \dot\varphi^2\sin\varphi) - F_1(t)\sin\varphi + F_2(v_C)\dot\eta_C/v_C) = 0\,. \end{gathered} \tag{8}$$

The equations of constraints (1) and (2) should be added to this equation.

If the initial conditions and analytic representation of the functions $F_1(t)$, $F_2(v_C)$ are given, then after numerical integration of the nonlinear system of differential equations (1), (2), (8) we find the car motion

$$\varphi = \varphi(t), \quad \xi_C = \xi_C(t), \quad \eta_C = \eta_C(t). \tag{9}$$

Now we can determine the generalized reaction forces $\Lambda_1, \Lambda_2$. The second group of Maggi's equations will be written as follows

$$\Lambda_1 = (MW_1 - Q_1)\frac{\partial \dot q^1}{\partial v_*^2} + (MW_2 - Q_2)\frac{\partial \dot q^2}{\partial v_*^2} + (MW_3 - Q_3)\frac{\partial \dot q^3}{\partial v_*^2}$$

$$\Lambda_2 = (MW_1 - Q_1)\frac{\partial \dot q^1}{\partial v_*^3} + (MW_2 - Q_2)\frac{\partial \dot q^2}{\partial v_*^3} + (MW_3 - Q_3)\frac{\partial \dot q^3}{\partial v_*^3}$$

or in the extended form for the rear drive vehicle

$$\begin{gathered} \Lambda_1 = \beta_2^2(M^*\ddot\xi_C - M_2 l_1(\ddot\varphi\sin\varphi + \dot\varphi^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot\xi_C/v_C) + \\ +\beta_2^3(M^*\ddot\eta_C + M_2 l_1(\ddot\varphi\cos\varphi - \dot\varphi^2\sin\varphi) - F_1(t)\sin\varphi + \\ +F_2(v_C)\dot\eta_C/v_C) \end{gathered} \tag{10}$$

$$\begin{gathered} \Lambda_2 = \beta_3^2(M^*\ddot\xi_C - M_2 l_1(\ddot\varphi\sin\varphi + \dot\varphi^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot\xi_C/v_C) + \\ +\beta_3^3(M^*\ddot\eta_C + M_2 l_1(\ddot\varphi\cos\varphi - \dot\varphi^2\sin\varphi) - F_1(t)\sin\varphi + \\ +F_2(v_C)\dot\eta_C/v_C)\,. \end{gathered} \tag{11}$$

After inserting the expressions (9) into these formulas we find the generalized reaction forces

$$\Lambda_i = \Lambda_i(t), \quad i = 1, 2\,.$$

These functions allow us to investigate the possibility of realizing the nonholonomic constraints (1), (2). If the reaction forces appear to exceed the forces provided by Coulomb's frictional forces, then these constraints will not be realized and the vehicle will begin to slide along the axles to which the wheels are fastened.

In order to write the conditions of the beginning side slipping of the wheels in analytical form, it is necessary to establish the relation between the determined generalized reactions $\Lambda_1$, $\Lambda_2$ and reaction forces $\mathbf{R}_B$, $\mathbf{R}_A$, applied to the wheels from the road.

This is a question of principal importance. So let us consider the relation between the generalized reaction force $\Lambda$ of the nonholonomic constraint and the reaction force $\mathbf{R}$ for the following quite general case. Assume that the equation of the nonholonomic constraint sets the condition that for the plane motion the velocity $\mathbf{v}$ of a point of a mechanical system along the direction of the unit vector $\mathbf{n}$ is equal to zero, i. e. assume that constraint equation written in vectorial form is

$$\varphi^n = \mathbf{v} \cdot \mathbf{n} = 0\,.$$

This equation in a scalar form appears as

$$\varphi^n = \dot{x} n_x + \dot{y} n_y = 0\,.$$

If the constraint is ideal, then the reaction force $\mathbf{R}$ can be represented as

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j} = \Lambda\left(\frac{\partial \varphi^n}{\partial \dot{x}}\mathbf{i} + \frac{\partial \varphi^n}{\partial \dot{y}}\mathbf{j}\right) = \Lambda \mathbf{n}\,,$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in $x$ and $y$ directions. Hence, *the generalized reaction force $\Lambda$ is equal to the projection of the constraint reaction force $\mathbf{R}$ onto the direction of vector $\mathbf{n}$*.

It is apparent that this representation of the vector $\mathbf{R}$ in the form $\Lambda\mathbf{n}$ extends also to the constraints (1), (2). By writing these constraints in the vector form

$$\varphi^1 = \mathbf{v}_B \cdot \mathbf{j} = 0\,, \tag{12}$$

$$\varphi^2 = \mathbf{v}_A \cdot \mathbf{j}_1 = \mathbf{v}_A \cdot (-\underline{\mathbf{i}} \sin\theta + \mathbf{j}\cos\theta) = 0\,, \tag{13}$$

where $\mathbf{j_1}$ is the unit vector of the axis of ordinates of the movable frame $Ax_1y_1$ of the car front axle, we obtain

$$\mathbf{R}_B = \Lambda_1 \mathbf{j}\,, \tag{14}$$

$$\mathbf{R}_A = \Lambda_2(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)\,. \tag{15}$$

Let us remark here that if the constraints (12) and (13) are violated, then nonzero values of $\varphi^1$ and $\varphi^2$ are equal to projections of velocities of the points $B$ and $A$ onto the vectors $\mathbf{j}$ and $\mathbf{j}_1$, correspondingly. The resulting friction forces applied to the wheels can be represented as

$$\mathbf{R}_B^{\text{fr}} = -\Lambda_1^{\text{fr}} \text{sign}(\varphi^1)\mathbf{j}$$

$$\mathbf{R}_A^{\text{fr}} = -\Lambda_2^{\text{fr}} \text{sign}(\varphi^2)(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)\,.$$

Finding the positive values $\Lambda_1^{\text{fr}}$ and $\Lambda_2^{\text{fr}}$ will be reported below.

## 3 The Turning Movement of a Car Nonretaining Constraints

**General remarks.** Let us return to the question considered in the previous paragraphs. Note that the Maggi equation (8) is composed under realization of the constraints (1), (2), i. e. when these nonholonomic constraints are retaining (bilateral).

Let us study the vehicle motion in the case when the constraints (1), (2) may be nonretaining, i. e. when side slipping of the front or rear wheels (or both front and rear wheels simultaneously) begins. The dynamic conditions of realizing the kinematic constraints (1), (2) is the requirement that the forces of interaction between the wheels and the road should not exceed the corresponding Coulomb's friction forces. For the driven front wheels in accordance with formula (15) this is expressed by inequality

$$|\Lambda_2| < F_2^{\text{fr}} = k_2 N_2\,, \tag{16}$$

where $F_2^{\text{fr}}$, $k_2$ are the frictional force and the coefficient of friction between front wheels and the road, respectively, $N_2$ is the normal pressure of the front axle.

When considering the rear driving wheels, it is necessary to take into account that the value of this wheel-road interaction force $\mathbf{F}_B$ is determined by the vector sum of the driving force $\mathbf{F}_1$ and side reaction force $\mathbf{R}_B$ given by formula (14) (see Figure 2). To provide the absence of side slipping of the rear axle, the following condition should be satisfied (the introduced notation is analogous to the notation used for the front axle)

$$F_B = \sqrt{(F_1)^2 + (\Lambda_1)^2} < F_1^{\text{fr}} = k_1 N_1 \,. \tag{17}$$

According to Figure 2 this means that the end of the force vector $\mathbf{F}_B$ should not go beyond the circle of radius $F_1^{\text{fr}}$. Otherwise the road will not be able to develop such reaction value $|\Lambda_1|$ that is required for realization of the nonholonomic constraint (1). Thus, this constraint becomes nonretaining, the side velocity component of driven wheels appears, and Coulomb's friction force $F_1^{\text{fr}}$ starts acting to them from the road. This Coulomb's friction force $F_1^{\text{fr}}$ arises from simultaneous action of the driving force $F_1$ and side friction force $\Lambda_1^{\text{fr}}$, so that

$$(F_B)^2 \equiv (F_1^{\text{fr}})^2 \equiv (k_1 N_1)^2 = (F_1)^2 + (\Lambda_1^{\text{fr}})^2 \,. \tag{18}$$

Note that at the beginning of side slipping the driver sets

$$F_1 = 0 \,.$$

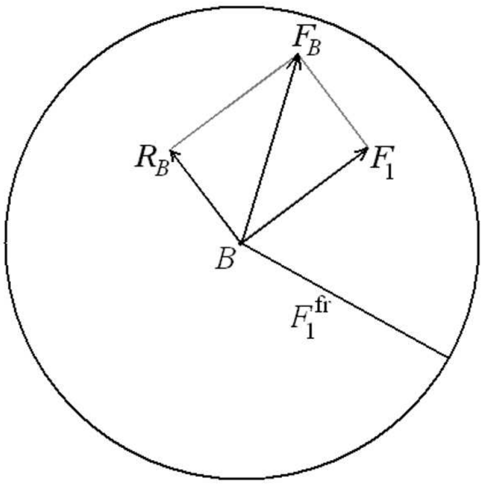


Figure 2. Forces acting on driving wheels.

**Possible types of the car motion.** We shall explain possible different types of motion of the mechanical model of a car. In Figure 3 in the phase space of variables $q^\sigma, \dot{q}^\sigma, \sigma = \overline{1,3}$, we see the representation of two hypersurfaces. The first one corresponds to the constraint given by equation (12), and the second one corresponds to the constraint, given by equation (13). In an explicit form these constraints are presented by formulas (1), (2).

Under simultaneous realization of nonholonomic constraints (1) and (2) the point of the phase space should be located on the line of intersection of these hypersurfaces. It corresponds to the I-st type of the car motion (bold curve I in Figure 3). If the first constraint is violated $(F_B = F_1^{\text{fr}})$ and the second constraint is realized, then the representation point is located at the hypersurface $\varphi^2 = 0$ (II-nd type of motion). If the second constraint is violated, but the first constraint is fulfilled $\varphi^1 = 0$, then the representation point belongs to hypersurface $\varphi^1 = 0$ (III-rd type of motion). In the case if both constraints are violated, the representation point does not belong to hypersurfaces. As this takes place, the vehicle moves in the presence of side friction forces acting on the front and rear axles (IV-th type of motion).

From any type of motion the representation point can change to any other type of motion. For example, in the I-st type of motion, if inequality (17) is not fulfilled , the vehicle becomes released of the constraint (1). If in this case inequality (16) is still fulfilled, then the constraint (2) keeps on working, thus, the representation point can move only over the hypersurface $\varphi^2 = 0$ (the vehicle changes to the II-nd type of motion). Here two cases of the possibility of restoring the I-st type of motion should be distinguished.

In some area $G_1$ the solution curves pierce the curve I, without stopping there (see Figure 3). This instantaneous realization of the constraint (1) corresponds to the stop of side motion of the rear axle in one direction and change of the same axle to the side motion in the reverse direction. In contrast to this the behaviour of solution curves within the area $G_2$ characterizes restoring the constraint $\varphi^1 = 0$ and change from the II-nd type of movement to the I-st one.

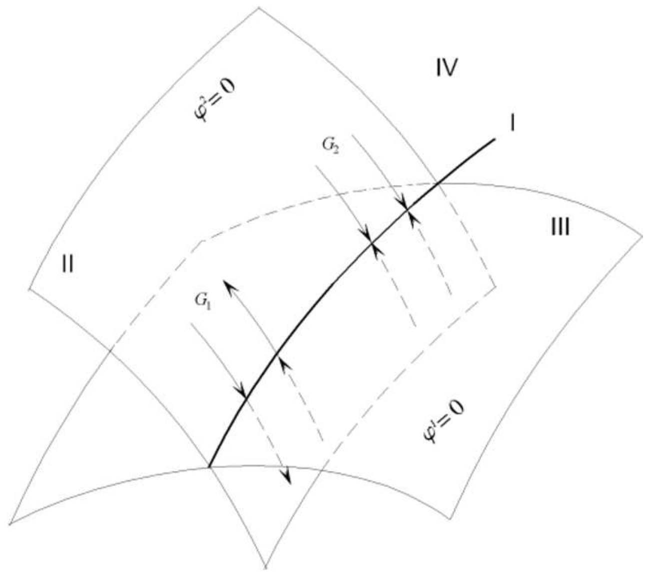


Figure 3. Possible types of the car motion.

Without preliminary studies of the behaviour of the solution curves in the common space of generalized coordinates and quasivelocities [2] it is possible to find out in which area $G_1$ or $G_2$ the equation $\varphi^1 = 0$ turns out to be fulfilled, in the following manner. By the values of phase variables, such that the constraint (1) is fulfilled, let us calculate the reaction $\Lambda_1$ by formula (10). If for the obtained value $\Lambda_1$ the inequality (17) is fulfilled, then the constraint $\varphi^1 = 0$ becomes retaining (bilateral) (the solution curve is within the area $G_2$), otherwise this constraint is not restored (the solution curve is within the area $G_1$).

We note, that when investigating the II-nd type of motion it is necessary also to ensure that inequality (16) is fulfilled, as on its failure the vehicle will change to the IV-th type of motion. If the constraint (1) is restored and the constraint (2) is violated at the same time, then the III-rd type of motion will occur.

Let us write out the car motion equations for these four types of motion.

**I-st type of motion.** For this motion both constraints (1) and (2) are fulfilled

$$\varphi^1 = 0\,, \qquad \varphi^2 = 0\,.$$

Maggi's equation for a rear-wheel drive vehicle takes the form (8), which should be integrated together with the equations of constraints (1) and (2). Having obtained the law of motion

$$\varphi = \varphi(t)\,, \quad \xi_C = \xi_C(t)\,, \quad \eta_C = \eta_C(t)\,,$$

the generalized reactions can be found from (10), (11)

$$\Lambda_1 = \Lambda_1(t)\,, \quad \Lambda_2 = \Lambda_2(t)\,.$$

By these values, fulfillment of inequalities (16) and (17) is checked. When one of them is violated, the vehicle changes to the II-nd or III-rd type of motion, and when they are violated simultaneously it changes to the IV-th type.

**II-nd type of motion.** For this type of motion only the second constraint is fulfilled

$$\varphi^1 \neq 0\,, \qquad \varphi^2 = 0\,.$$

The rear axle of the vehicle executes lateral motion, therefore the lateral frictional force $\Lambda_1^{\text{fr}}$ calculated by formula (18) is applied to it. As this takes place, if $\varphi^1 > 0$, then according to formula (12) the rear wheels sideslip in the positive direction of the $y$-axis. Therefore, the lateral frictional force is opposed to the $y$-axis, and if $\varphi^1 < 0$, it is aligned with the $y$-axis (see Figure 1).

Let us compose Maggi's equations in the presence of one constraint (2). Let us proceed to quasivelocities by formulas

$$v_*^1 = \dot{\varphi}\,, \qquad v_*^2 = \dot{\xi}_C\,,$$

$$v_*^3 = -\dot{\xi}_C \sin(\varphi + \theta) + \dot{\eta}_C \cos(\varphi + \theta) + l_1 \dot{\varphi} \cos\theta \,.$$

Let us find the inverse transformation

$$\dot{\varphi} = v_*^1 \,, \quad \dot{\xi}_C = v_*^2 \,, \quad \dot{\eta}_C = \beta_1^3 v_*^1 + \beta_2^3 v_*^2 + \beta_3^3 v_*^3$$

where

$$\beta_1^3 = -l_1 \cos\theta / \cos(\varphi + \theta) \,, \quad \beta_2^3 = \mathtt{tg}(\varphi + \theta) \,, \quad \beta_3^3 = 1/\cos(\varphi + \theta) \,.$$

Now we may compose two Maggi's equations for the rear-wheel drive vehicle

$$\begin{gathered} J^* \ddot{\varphi} + J_2 \ddot{\theta} + M_2 l_1 (-\ddot{\xi}_C \sin\varphi + \ddot{\eta}_C \cos\varphi) - \Lambda_1^{\text{fr}} \text{sign}(\varphi^1) l_2 + \\ + \beta_1^3 (M^* \ddot{\eta}_C + M_2 l_1 (\ddot{\varphi} \cos\varphi - \dot{\varphi}^2 \sin\varphi) - \\ - F_1(t) \sin\varphi + F_2(v_C) \dot{\eta}_C / v_C + \Lambda_1^{\text{fr}} \text{sign}(\varphi^1) \cos\varphi) = 0 \\ M^* \ddot{\xi}_C - M_2 l_1 (\ddot{\varphi} \sin\varphi + \dot{\varphi}^2 \cos\varphi) - F_1(t) \cos\varphi + F_2(v_C) \dot{\xi}_C / v_C - \\ - \Lambda_1^{\text{fr}} \text{sign}(\varphi^1) \sin\varphi + \beta_2^3 (M^* \ddot{\eta}_C + M_2 l_1 (\ddot{\varphi} \cos\varphi - \dot{\varphi}^2 \sin\varphi) - F_1(t) \sin\varphi + \\ + F_2(v_C) \dot{\eta}_C / v_C + \Lambda_1^{\text{fr}} \text{sign}(\varphi^1) \cos\varphi) = 0 \,. \end{gathered} \tag{19}$$

From the second group of Maggi's equations there remains one equation for determination of the generalized reaction $\Lambda_2$. For the vehicle with driving rear wheels it is as follows

$$\begin{gathered} \Lambda_2 = \beta_3^3 (M^* \ddot{\eta}_C + M_2 l_1 (\ddot{\varphi} \cos\varphi - \dot{\varphi}^2 \sin\varphi) - F_1(t) \sin\varphi + \\ + F_2(v_C) \dot{\eta}_C / v_C + \Lambda_1^{\text{fr}} \text{sign}(\varphi^1) \cos\varphi) \,. \end{gathered} \tag{20}$$

The equations of motion (19) are integrated together with the constraint equation (2). If the dynamic condition (16) for the constraint (2) to be realized holds for the obtained value of $\Lambda_2$, then the II-nd type of motion continues. If the condition (16) is violated, then the vehicle will change to IV-th type of motion.

In the course of checking inequality (16) it is necessary to keep watching if the constraint $\varphi^1 = 0$ begins to hold. If this constraint is realized under certain obtained values of $t, q^\sigma, \dot{q}^\sigma, \sigma = \overline{1,3}$, and if inequality (17) holds for the value $\Lambda_1$ calculated by formula (10), then the constraint $\varphi^1 = 0$ is restored, the rear axle ceases to execute lateral motion and the car changes to the I-st type of motion. If inequality (17) is not fulfilled for the value $\Lambda_1$ calculated by formula (10), then the car keeps the II-nd type of motion (rear axle begins lateral motion in the opposite direction).

Theoretically the car may change from the II-nd type of motion to the III-rd one. For this purpose, at a certain time instant inequality (16) must cease to hold and simultaneously the constraint $\varphi^1 = 0$ must be restored.

**III-rd type of motion.** This motion is studied in a similar way to the II-nd type. Now the following should be fulfilled

$$\varphi^1 = 0 \,, \qquad \varphi^2 \neq 0 \,.$$

Due to side slipping of the front axle of the car this front axle is acted upon by the side friction force

$$\Lambda_2^{\text{fr}} = k_2 N_2 \,. \tag{21}$$

In order to compose Maggi's equations for this nonholonomic problem with one constraint (1) let us change to quasi-velocities by using the formulas

$$v_*^1 = \dot{\varphi} \,, \qquad v_*^2 = \dot{\xi}_C \,,$$
$$v_*^3 = -\dot{\xi}_C \sin\varphi + \dot{\eta}_C \cos\varphi - l_2 \dot{\varphi} \,.$$

This corresponds to the reverse transformation

$$\dot{\varphi} = v_*^1 \,, \quad \dot{\xi}_C = v_*^2 \,, \quad \dot{\eta}_C = \beta_1^3 v_*^1 + \beta_2^3 v_*^2 + \beta_3^3 v_*^3$$

where

$$\beta_1^3 = l_2/\cos\varphi\,, \quad \beta_2^3 = \mathtt{tg}\varphi\,, \quad \beta_3^3 = 1/\cos\varphi\,.$$

Two Maggi's equations for the car with driving rear wheels have the form

$$\begin{gathered} J^*\ddot{\varphi} + J_2\ddot{\theta} + M_2 l_1(-\ddot{\xi}_C\sin\varphi + \ddot{\eta}_C\cos\varphi) + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2) l_1\cos\theta + \\ +\beta_1^3(M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - \\ -F_1(t)\sin\varphi + F_2(v_C)\dot{\eta}_C/v_C + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\cos(\varphi+\theta)) = 0 \\ M^*\ddot{\xi}_C - M_2 l_1(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot{\xi}_C/v_C - \\ -\Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\sin(\varphi+\theta) + \beta_2^3(M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - \\ -F_1(t)\sin\varphi + F_2(v_C)\dot{\eta}_C/v_C + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\cos(\varphi+\theta)) = 0\,. \end{gathered} \tag{22}$$

The generalized reaction $\Lambda_1$ is expressed as

$$\begin{gathered} \Lambda_1 = \beta_3^3(M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - F_1(t)\sin\varphi + \\ +F_2(v_C)\dot{\eta}_C/v_C + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\cos(\varphi+\theta))\,. \end{gathered} \tag{23}$$

The equations of motion (22) are integrated together with the constraint equation (1). If the dynamic condition (17) for realizing the constraint (1) is satisfied for the value of $\Lambda_1$ obtained by formula (23), then the III-rd type of motion continues. If the condition (17) is violated, then the car changes to the IV-th type of motion.

In the course of checking inequality (17) it is necessary to keep watching if the constraint $\varphi^2 = 0$ begins to be realized. If this constraint is realized under certain calculated values $t, q^\sigma, \dot{q}^\sigma, \sigma = \overline{1,3}$, then these values of the variables should be substituted in formula (11). If for the obtained $\Lambda_2$ the inequality (16) is satisfied, then the constraint $\varphi^2 = 0$ is restored, the front axle ceases to execute lateral motion, and the car changes to the I-st type of motion. If for the calculated value of $\Lambda_2$ inequality (16) is not satisfied, then the car continues the III-rd type of motion (the front axle begins lateral motion in the opposite direction).

Theoretically the III-rd type of motion can change to the II-nd one: for this purpose, at a certain instant inequality (17) must cease to hold, and at the same time the constraint $\varphi^2 = 0$ must be restored.

**IV-th type of motion.** For such motion the following must take place

$$\varphi^1 \neq 0\,, \qquad \varphi^2 \neq 0\,.$$

This means that the car moves as a holonomic system when its wheels are acted upon by side frictional forces $\Lambda_1^{\text{fr}}$ and $\Lambda_2^{\text{fr}}$ set by formulas (18) and (21). The motion of the rear wheel-drive car is determined by the following Lagrange equations of the second kind

$$\begin{gathered} J^*\ddot{\varphi} + J_2\ddot{\theta} + M_2 l_1(-\ddot{\xi}_C\sin\varphi + \ddot{\eta}_C\cos\varphi) - \\ -\Lambda_1^{\text{fr}}\text{sign}(\varphi^1) l_2 + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2) l_1\cos\theta = 0 \\ M^*\ddot{\xi}_C - M_2 l_1(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot{\xi}_C/v_C - \\ -\Lambda_1^{\text{fr}}\text{sign}(\varphi^1)\sin\varphi - \Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\sin(\varphi+\theta) = 0 \\ M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - F_1(t)\sin\varphi + F_2(v_C)\dot{\eta}_C/v_C + \\ +\Lambda_1^{\text{fr}}\text{sign}(\varphi^1)\cos\varphi + \Lambda_2^{\text{fr}}\text{sign}(\varphi^2)\cos(\varphi+\theta) = 0\,. \end{gathered} \tag{24}$$

In course of calculation of motion by equations (24) it is necessary to keep watching if either function $\varphi^1$ or $\varphi^2$ vanishes, or both functions $\varphi^1$ and $\varphi^2$ do so for the current values of

$$t,\ q^\sigma\,,\ \dot{q}^\sigma\,, \qquad \sigma = \overline{1,3}\,. \tag{25}$$

If $\varphi^1 = 0$ holds for the values (25), then $\Lambda_1$ should be calculated for these values of variables by formula (23). If for this value of $\Lambda_1$ inequality (17) is satisfied, then the car changes to the III-rd type of motion, otherwise it keeps the motion of the IV-th type.

If it turns out that $\varphi^2 = 0$ for the values (25), then for these values of variables $\Lambda_2$ should be calculated by formula (20). If the inequality (16) is satisfied for this value of $\Lambda_2$, then the car changes to the II-nd type of motion, otherwise it keeps the motion of the IV-th type.

If it turns out that for the values (25) the both functions $\varphi^1$ and $\varphi^2$ vanish simultaneously, then $\Lambda_1$ and $\Lambda_2$ should be found from formulas (10), (11). If both inequalities (16) and (17) are fulfilled for these values, then the car changes to the I-st type of motion. If only inequality (16) is satisfied, then the II-nd type of motion begins. If only inequality (17) is fulfilled, then from this point on the car will execute the III-rd type of motion.

## 4 Calculation of Motion of a Certain Car

As an example, let us consider the motion of the hypothetical compact motor car with $M_1 = 1000\ kg$; $M_2 = 110\ kg$; $J_1 = 1500\ kg{\cdot}m^2$; $J_2 = 30\ kg{\cdot}m^2$; $l_1 = 0.75\ m$; $l_2 = 1.65\ m$; $k_{1\text{fr}} = 0.4$; $k_{2\text{fr}} = 0.4$ with the power characteristics

$$F_2(v_C) = k_2 v_C\ N; \qquad k_2 = 100\ N{\cdot}s{\cdot}m^{-1}.$$

The following car motion is studied. In the beginning the vehicle moves rectilinearly (the planes of the front and rear wheels are parallel) during eight seconds, in this case $\varphi = \pi/6$. During this time the function $F_1(t)$ is changed by the law $F_1(t) = 200t$ ($F_1$ is measured in Newtons, $t$ is measured in seconds), i. e. at the initial time $F_1(0) = 0$, and at the end of rectilinear motion $F_1(8) = 1600$. Graphs of dependences of coordinates on time are presented in Figure 4.

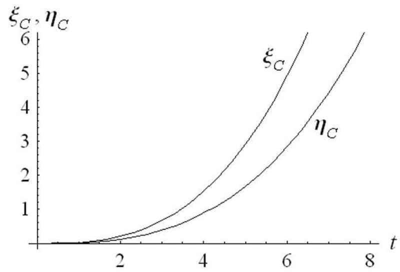


Figure 4. Rectilinear motion.

After eight seconds of rectilinear motion the driver starts to turn the steering wheel at a smooth manner at the angle $\theta = \pi(t-8)/8$, that is, in two seconds the angle $\theta$ is equal to $\pi/4$. For this motion $F_1(t) = 1600$. By the computed values of constraint reactions $\Lambda_1$ and $\Lambda_2$ we get graphs shown in Figure 5. It follows from the graphs that inequality (17) is satisfied, but condition (16) is violated when $t_1 = 9.5147$, $\theta(t_1) = 0.5948$. Thus, when $8 < t < 9.5147$, the car moves by the I-st type, but after $t_1 = 9.5147$ it changes to the III-rd one.

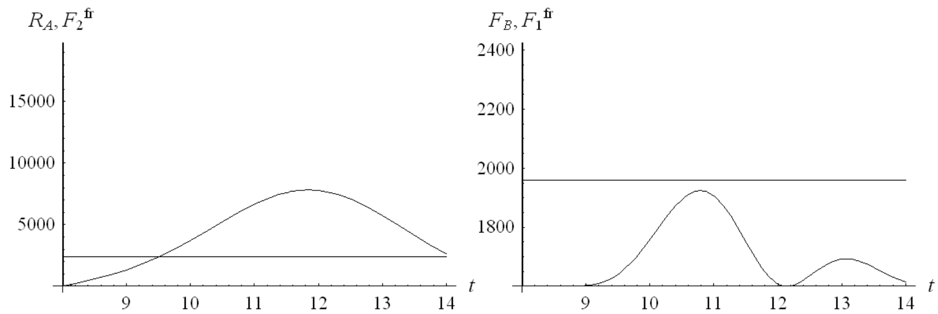


Figure 5. Change to the III-rd type of motion.

After occurrence of the III-rd type of motion a driver tries to eliminate the side slipping of front wheels of the car, by the way of setting $F_1 = 0$ and changing a turning angle of the front axle according to the law $\theta = -10(t-t_1)+\theta_1$. Let us calculate the constraint reaction $\Lambda_1$ and check if dynamic condition (17) is satisfied. As we can see from Figure 6 the force $F_B$ does not exceed the friction force at least during the interval of time $9.5147 < t < 13$, that is the dynamic condition (17) of realizing the constraint (1) is satisfied during this interval. At the same time we check if the condition $\varphi^2 = 0$ is fulfilled. As follows from Figure 6, it starts to be satisfied at the moment $t_2 = 9.8415$, in this case the constraint reaction force $|\Lambda_2|$ becomes close in value to the friction force between wheels and the road, and the front axle stops moving in side direction.

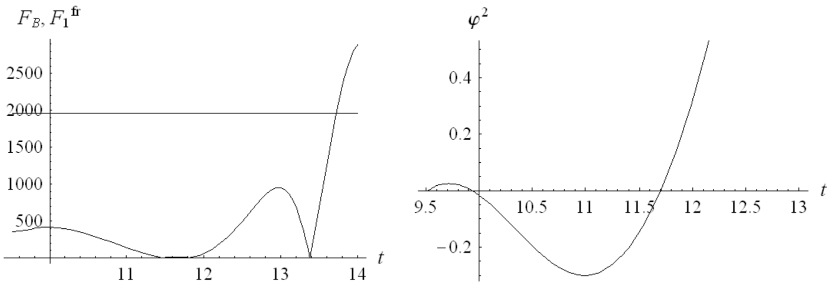


Figure 6. Change to the I-st type of motion.

Thus, for $t_1 < t < t_2$ the car moves according to the III-rd type, but after $t_2 = 9.8415$ it returns to the I-st type of motion.

Now suppose that for $t_1 < t < 14$ the driving force is varied by the law $F_1 = 200(t - t_2)/(2 - t_2)$ . In this case, according to Figure 7 dynamic conditions (16) and (17) are satisfied, that is, the restoring force $\varphi^2 = 0$ will be realized further. So, the car is in the I-st type of motion.

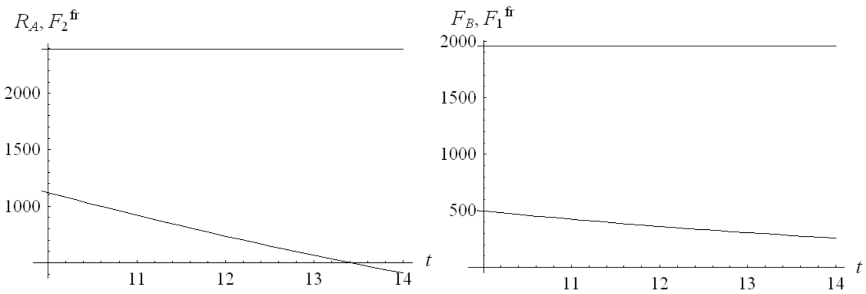


Figure 7. Check of the I-th type of motion.

In Figure 8 graphs of functions during all the time interval of the car motion are given.

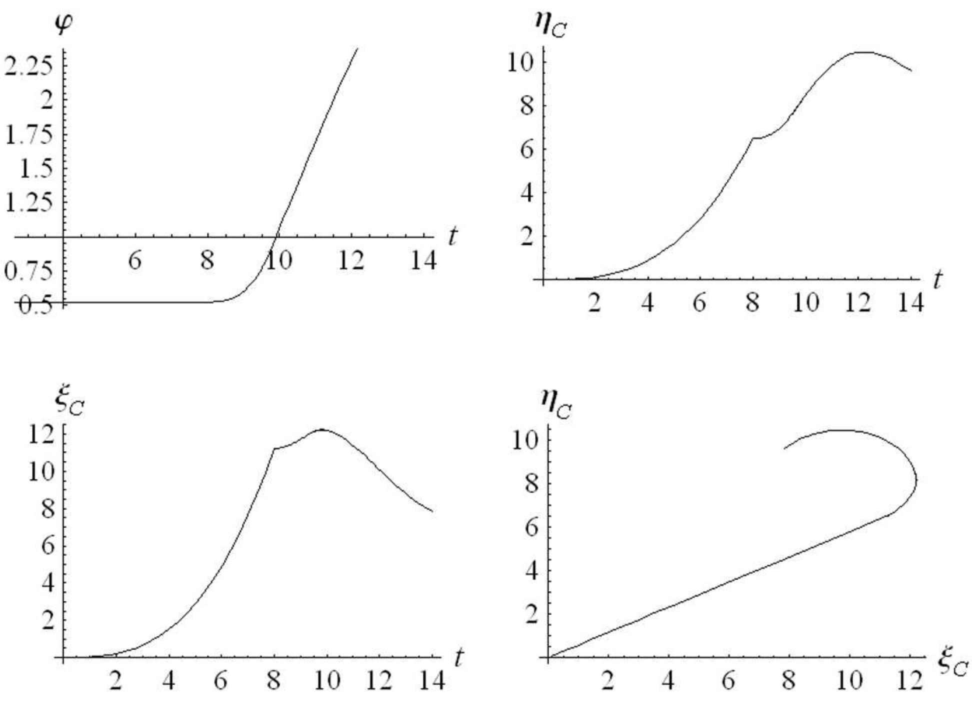


Figure 8. Graphs of functions during the car motion.

## 5 Rational Choice of Quasi-Velocities

Previously, when studying possible types of the car motion, we had to use different forms of the equations of motion (8), (19), (22), (24). This makes certain difficulties, especially when numerically integrating the given systems of differential equations with the help of computer. For similar problems with nonholonomic nonretaining constraints N. A. Fufayev (Zhuravlev and Fufaev, 1993) suggests to use a single form of Boltzmann-Hamel's equations. Let us see, how this idea may be applied in the case of using Maggi's equations in analogous problems. (We notice that for solving similar problems the equations of motion of nonholonomic systems with variable kinematic structure (Byachkov and Suslonov, 2002) can be effective).

Quite different forms of the equations of motion (8), (19), (22), (24) are obtained due to the fact that for different types of the car motion new transition formulas for quasi-velocities are chosen every time or the generalized coordinates are used directly to get the Lagrange equations of the second kind. Now we shall use the form of Maggi's equations for all the four types of motion, the quasi-velocities being always introduced by the same formulas (5). In these formulas the quasi-velocities have a certain physical meaning: $v_*^1$ is the angular velocity of rotation of the car body, $v_*^2$ and $v_*^3$ are, according to formulas (1) and (2), the side velocities of the rear and front axles, correspondingly, aligned with the vectors $\mathbf{j}$ and $\mathbf{j}_1$. If the nonholonomic constraints (1), (2) are realized, quasivelocities $v_*^2$ and $v_*^3$ vanish, and if these constraints turn out to be nonretaining, then these quasi-velocities have real nonzero values (except for the instant stops of axles in their side motion).

For the motion of the I-st type we still use the equation of motion (8) and the formulas for determination of the generalized reactions (10), (11).

For the II-nd type of motion, if the constraint $\varphi^2 = 0$ holds, then the generalized reaction $\Lambda_2$ calculated by the formula (11) arises. In this case the equation of motion (8) should be completed with the differential equation (10), where $\Lambda_1$ is changed by the projection of the side friction force $(-\Lambda_1^{\text{fr}}\text{sign}(\varphi^1))$ acting on the rear axle during its side slipping. It is necessary to add the constraint equation (2) to these differential equations.

For the III-rd type of motion $\Lambda_1$ is calculated in the same way by formula (10), and equation (11), in which the reaction $\Lambda_2$ is replaced with the projection of the side friction force $(-\Lambda_2^{\text{fr}}\text{sign}(\varphi^2))$, is added to equation (8). The constraint equation (1) is added to these differential equations.

Maggi's equations are linear combinations of the Lagrange equations of the second kind. Therefore in order to keep the uniformity of differential equations and for the IV-th type of motion corresponding to the holonomic problem, it is convenient to use the form of Maggi's equations. Eventually the equations of motion will take the form (8), (10), (11), where $\Lambda_1$ and $\Lambda_2$ are replaced with $(-\Lambda_1^{\text{fr}}\text{sign}(\varphi^1))$ and $(-\Lambda_2^{\text{fr}}\text{sign}(\varphi^2))$.

The logic of change from one type of motion to another is the same as in item 3.

Note that the obtained equations of motion have a singularity at $\theta = 0$. Therefore, the difficulties may occur in calculations, when turning begins with the rectilinear motion. In this case, instead of some possible modifications of the system of differential equations, which we used in the calculations given above, we can advise to change initially to the special system of curvelinear coordinates suggested in the work (Kalenova et al., 2004).

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