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Kinematics and Balance Relations for Bidimensional Continua

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This work is concerned with the formulation of the kinematics and balance relations for a so-called bidimensional continuum, which can be used in modeling thin layers and interface regions such as phase boundaries. Such a continuum represents a thin, shell-like 3-dimensional region in which the upper and lower surfaces move relative to each other as well as relative to the dividing, non-material interface between them. As such, it is more general than standard interfaces or shells. The standard balance relations of three-dimensional continua are adapted to this dynamic bidimensional geometry using the differential geometric notion of a flow. On this basis, the adapted balance relations are averaged over the dynamic thickness of the bidimensional continuum to obtain reduced 2-dimensional, surface forms of these on the dividing interface. In addition to the usual influence of the surface geometry on their form, the resulting adapted and averaged surface balance relations contain flux terms accounting for the effect of relative motion, i.e., diffusion, on the balances. In the limit that the thickness of the bidimensional continuum goes to zero, the generalized surface balance relations reduce to the classical jump balance relations across an interface.

1 Introduction

The modeling of phase transitions, shock-wave propagation, and other such “abrupt” phenomena as 2-dimensional, moving, non-material continua has a long history (*e.g.*, Scriven, 1960; Slattery, 1967; Moeckel, 1974; Betounes, 1986; Gurtin, Struthers, 1990). An alternative approach, which treats the transition region as a “thin” three-dimensional region has been advocated and developed by Deemer, Slattery (1978); Dumais (1980); Alts, Hutter (1988); Kosinski (1991); dell’Isola, Kosinski (1993). An advantage of this former approach over the latter is that one obtains directly relationships between the standard 3-dimensional thermodynamic fields and their 2-dimensional counterparts on the interface by imposing the kinematic structure of the thin, 3-dimensional transition region on the usual three-dimensional balance relations, and then averaging over the dynamic thickness of the thin region. The resulting thickness-averaged 3-dimensional fields can be identified with surface fields which are introduced formally in the first approach mentioned above. dell’Isolla & Kosiński (1993) have taken a step toward a complete formulation of this type on the basis of classical (*e.g.*, Riemannian) differential geometry. In the realm of solid mechanics, Bövik (1994) used the simple idea of a Taylor expansion of the relevant physical fields in thin regions together with surface differential operators on a curved surface to obtain the representation of a thin interphase by an interface. The idea of a Taylor expansion was also used by Hashin (1991) in deriving the spring-type interface model for soft elastic interphases. All of the above studies have assumed that the interphase is isotropic. Benveniste (2006) generalizes the Bövik model to an arbitrarily curved three-dimensional thin anisotropic layer between two anisotropic media. A comprehensive thermodynamical study of interfaces exists in Gurtin *et al.* (1998).

As it turns out, these previous approaches based on classical methods tacitly neglect effects of the dynamic geometry in the bidimensional context. In particular, these include the fact that both the normal $\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi$ and surface $\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi$ projections of the gradient $D\boldsymbol{n}_\xi$ of the unit normal $\boldsymbol{n}_\xi$ to the dividing surface ξ (see (44)) influence the bidimensional flow geometry and resulting balance relations. This is shown in the current work through the application of modern differential geometric concepts such as flow, adapted differential form, Lie derivative, volume form, relatedness of vector fields generating flows, and so on, as is shown in the current formulation. In doing this, we are following most closely in spirit the work of Betounes (1986). On the other hand, as shown in §5, the fact that the upper, middle and lower surfaces of the bidimensional continuum can move independently of each other leads new and much more complex balance relations than have been derived in the context of moving surfaces such as in Betounes (1986); Bövik (1994); Gurtin *et al.* (1998). Upon introducing the relevant further kinematic restrictions, these reduce to the standard relations.

After introducing certain mathematical concepts and definitions needed for the formulation (§2), we review briefly certain key aspects of the kinematics of moving, non-material surfaces in E (§3). Next, we introduce the kinematics of a bidimensional continuum (§4), whose motion is referenced to that of the interface separating its upper and lower regions, which move relative to one another, just like two phases of a material which is undergoing a phase transition. On the basis of such kinematics, we are in a position to adapt the standard three dimensional balance relations to such a dynamic geometry (§5); since we work with the true 3-dimensional balance relations from the start, and introduce no new physics into the formulation, we insure that the resulting adapted and averaged 2-dimensional balance relations on the interface are physically the same balance relations as those with which we started. Finally, we discuss briefly the form of mass balance on the interface that arises in such a formulation.

2 Mathematical Preliminaries

The bidimensional continuum under consideration in this work is an oriented continuum in the sense of plates and shells. To represent the corresponding geometry of this continuum, we generalize the differential geometric approach of Betounes (1986) to the current bidimensional context in what follows. To this end, let E represent 3-dimensional Euclidean point space, V its oriented linear translation space, $V^* := \mathrm{Lin}(V, \mathbb{R})$ the dual space of V, and $\mathbb{R}$ real number space. Here and in what follows, $\mathrm{Lin}(W, Z)$ stands for the set of all linear mappings between the linear spaces W and Z. In particular, V and V^* are such spaces. For example, the standard metric tensor $\boldsymbol{G}$ on E, defined by $(\boldsymbol{G}\boldsymbol{a})\boldsymbol{b} := \boldsymbol{a} \cdot \boldsymbol{b}$ for all $\boldsymbol{a}, \boldsymbol{b} \in V$, takes values in $\mathrm{Lin}(V, V^*)$. Further, we require the standard volume form $\boldsymbol{\omega}$ of E, defined by $\boldsymbol{\omega}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) := \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$ for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$. This quantity takes values in the linear space $\mathrm{Skw}_3(V^3, \mathbb{R})$ of all completely skew-symmetric multilinear mappings of $V^3 := V \times V \times V$ into $\mathbb{R}$. Next, let $\imath$ represent the interior product operator (*e.g.*, Abraham *et al.*, 1988, Definition 6.4.7), *i.e.*, $\imath_{\boldsymbol{a}}\boldsymbol{\omega}(\boldsymbol{b}, \boldsymbol{c}) := \boldsymbol{\omega}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. Via this operator, $\boldsymbol{\omega}$ maps any unit vector $\boldsymbol{n} \in V$ to a Euclidean two-form $\boldsymbol{\alpha} := \imath_{\boldsymbol{n}}\boldsymbol{\omega}$ perpendicular to $\boldsymbol{n}$. This is meant in the sense that, since $\boldsymbol{\alpha}$ is completely skew-symmetric, $\imath_{\boldsymbol{n}}\boldsymbol{\alpha} = \imath_{\boldsymbol{n}}\imath_{\boldsymbol{n}}\boldsymbol{\omega} = \boldsymbol{0}$ follows. Consequently, $\boldsymbol{\alpha}$ annihilates any element of V parallel to $\boldsymbol{n}$ by linearity. Let $V^{\parallel} \subset V$ represent the corresponding two-dimensional subspace of V consisting of all $\boldsymbol{a} \in V$ for which $\imath_{\boldsymbol{a}}\boldsymbol{\omega} \neq \boldsymbol{0}$ holds. Such vectors are considered parallel to $\boldsymbol{\alpha}$. Further, if $\boldsymbol{\nu} \in V^*$ is any unit one-form parallel to $\boldsymbol{n}$, *i.e.*, $\imath_{\boldsymbol{n}}\boldsymbol{\nu} = 1$, it induces the representation

$$\boldsymbol{\omega} = \boldsymbol{\nu} \wedge \boldsymbol{\alpha} \tag{1}$$

of $\boldsymbol{\omega}$ in terms of the exterior product operator $\wedge$ (*e.g.*, Abraham *et al.*, 1988, Definition 6.1.3). Indeed, we then have $\imath_{\boldsymbol{n}}\boldsymbol{\omega} = (\imath_{\boldsymbol{n}}\boldsymbol{\nu})\boldsymbol{\alpha} - \boldsymbol{\nu} \wedge (\imath_{\boldsymbol{n}}\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ since $\imath_{\boldsymbol{n}}\boldsymbol{\alpha} = \boldsymbol{0}$. Note also that $V^{\parallel} = \ker\,\boldsymbol{\nu}$. The subspace supplementary to $V^{\parallel}$ in V is that $V^{\perp} = \mathrm{ima}\,\boldsymbol{N}$ of all elements of V parallel to $\boldsymbol{n}$, where

$$\boldsymbol{N} := \boldsymbol{n} \otimes \boldsymbol{\nu} \in \mathrm{Lin}(V, V)\,. \tag{2}$$

For example, we can have

$$\boldsymbol{\nu} := \boldsymbol{G}\boldsymbol{n} \tag{3}$$

of $\boldsymbol{\nu}$ induced by $\boldsymbol{G}$. Then

$$\imath_{\boldsymbol{n}}\boldsymbol{\nu} = (\boldsymbol{G}\boldsymbol{n})\boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{n} = 1 \tag{4}$$

holds.

One then has the orthogonal sum $V = V^{\perp} \oplus V^{\parallel}$ and split of V.

Consider next the inclusion mapping $\boldsymbol{I} \in \mathrm{Lin}(V^{\parallel}, V)$ of $V^{\parallel}$ into V, and $\boldsymbol{I}^* \in \mathrm{Lin}(V^*, V^{\parallel *})$ its dual mapping. Since the image of $\boldsymbol{I}$ is the kernel of $\boldsymbol{\nu}$ in V, *i.e.*, $\mathrm{ima}\,\boldsymbol{I} = \ker\,\boldsymbol{\nu}$, note that

$$\boldsymbol{I}^*\boldsymbol{\nu} = \boldsymbol{0} \tag{5}$$

holds. For any left inverse $\boldsymbol{P} \in \mathrm{Lin}(V, V^{\parallel})$ of $\boldsymbol{I}$, *i.e.*, $\boldsymbol{P}\boldsymbol{I}$ is equal to the identity $\boldsymbol{I}_{V^{\parallel}}$ on $\mathrm{Lin}(V^{\parallel}, V^{\parallel})$, we have

$$V^{\perp} = \ker\,\boldsymbol{P} \tag{6}$$

as well as the decomposition

$$\boldsymbol{I}_V = \boldsymbol{I}\boldsymbol{P} + \boldsymbol{N} \tag{7}$$

of the identity $\boldsymbol{I}_V$ on $\mathrm{Lin}(V, V)$. In particular, $\boldsymbol{P}$ is compatible with $\boldsymbol{G}$ and the metric

$$\boldsymbol{G}_{\parallel} := \boldsymbol{I}^*\boldsymbol{G}\boldsymbol{I} \tag{8}$$

induced by $\boldsymbol{I}$ on $V^{\parallel}$ when we choose

$$\boldsymbol{P} := \boldsymbol{I}^{\mathrm{T}} , \tag{9}$$

where

$$\boldsymbol{I}^{\mathrm{T}} := \boldsymbol{G}_{\parallel}^{-1} \boldsymbol{I}^{*} \boldsymbol{G} \tag{10}$$

is the transpose of $\boldsymbol{I}$ with respect to $\boldsymbol{G}$ and $\boldsymbol{G}_{\parallel}$. In this case,

$$\boldsymbol{P}\boldsymbol{n} = \boldsymbol{I}^{\mathrm{T}}\boldsymbol{n} = \boldsymbol{G}_{\parallel}^{-1}\boldsymbol{I}^{*}\boldsymbol{\nu} = \boldsymbol{0} \tag{11}$$

follows from (5). Lastly, note that either $\boldsymbol{\alpha}$, or $\boldsymbol{I}$ and $\boldsymbol{P}$, induce the two-dimensional trace

$$\mathrm{tr}_{\parallel}(\boldsymbol{A}) := \frac{\boldsymbol{\alpha}(\boldsymbol{I}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{I}\boldsymbol{v}) + \boldsymbol{\alpha}(\boldsymbol{I}\boldsymbol{u}, \boldsymbol{I}\boldsymbol{A}\boldsymbol{v})}{\boldsymbol{\alpha}(\boldsymbol{I}\boldsymbol{u}, \boldsymbol{I}\boldsymbol{v})} = \mathrm{tr}(\boldsymbol{I}\boldsymbol{A}\boldsymbol{P}) \tag{12}$$

and determinant

$$\det_{\parallel}(\boldsymbol{A}) := \frac{\boldsymbol{\alpha}(\boldsymbol{I}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{I}\boldsymbol{A}\boldsymbol{v})}{\boldsymbol{\alpha}(\boldsymbol{I}\boldsymbol{u}, \boldsymbol{I}\boldsymbol{v})} = \det(\boldsymbol{I}\boldsymbol{A}\boldsymbol{P}) \tag{13}$$

operations, respectively, for all $\boldsymbol{A} \in \mathrm{Lin}(V^{\parallel}, V^{\parallel})$ and $\boldsymbol{u}, \boldsymbol{v} \in V^{\parallel}$.

Let $\boldsymbol{n}$ and $\boldsymbol{m}$ represent the normal and binormal unit vectors to some curve in E at some point along this curve. In this case,

$$\boldsymbol{t} := \boldsymbol{n} \times \boldsymbol{m} \tag{14}$$

is the unit tangent vector to this curve at the point in question, and

$$\boldsymbol{\tau} := \boldsymbol{G}\boldsymbol{t} = \imath_{\boldsymbol{m}}\imath_{\boldsymbol{n}}\boldsymbol{\omega} = \imath_{\boldsymbol{m}}\boldsymbol{\alpha} \in V^{*} \tag{15}$$

the covector associated with $\boldsymbol{t}$, *i.e.*,

$$\imath_{\boldsymbol{t}}\boldsymbol{\tau} = \boldsymbol{t} \cdot \boldsymbol{t} = 1 , \tag{16}$$

analogous to (4), such that $(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{t})$ form an orthogonal, positively-oriented system. Defining the covector

$$\boldsymbol{\mu} := \boldsymbol{G}\boldsymbol{m} = \imath_{\boldsymbol{n}}\imath_{\boldsymbol{t}}\boldsymbol{\omega} = -\imath_{\boldsymbol{t}}\boldsymbol{\alpha} \tag{17}$$

associated with $\boldsymbol{m}$, such that

$$\imath_{\boldsymbol{m}}\boldsymbol{\mu} = \boldsymbol{m} \cdot \boldsymbol{m} = 1 \tag{18}$$

holds, the relations (15) and (16) imply

$$\boldsymbol{\alpha} = \boldsymbol{\mu} \wedge \boldsymbol{\tau} , \tag{19}$$

and so

$$\boldsymbol{\omega} = \boldsymbol{\nu} \wedge \boldsymbol{\mu} \wedge \boldsymbol{\tau} \tag{20}$$

via (1); note that

$$\imath_{\boldsymbol{t}}\boldsymbol{\nu} = 0 , \qquad \imath_{\boldsymbol{t}}\boldsymbol{\mu} = 0 , \qquad \imath_{\boldsymbol{n}}\boldsymbol{\mu} = 0 , \qquad \imath_{\boldsymbol{n}}\boldsymbol{\tau} = 0 , \qquad \imath_{\boldsymbol{m}}\boldsymbol{\tau} = \boldsymbol{0} , \qquad \imath_{\boldsymbol{m}}\boldsymbol{\nu} = \boldsymbol{0} , \tag{21}$$

via (14), (15) and (17). Lastly, we have

$$\boldsymbol{I}\boldsymbol{P} = \boldsymbol{m} \otimes \boldsymbol{\mu} + \boldsymbol{t} \otimes \boldsymbol{\tau} \tag{22}$$

for $\boldsymbol{I}\boldsymbol{P}$ in terms of $\boldsymbol{m}$ and $\boldsymbol{t}$.

Lastly, note that any V-tensor induces a *constant* tensor *field* on any subset of E, and we denote any such tensor and its corresponding constant tensor field by the same symbol in what follows for simplicity. In particular, this will be the case for $\boldsymbol{G}$ and $\boldsymbol{\omega}$.

3 Surface Geometry and Kinematics

Let S represent a regular 2-manifold with boundary bS modeled on E, $I \subset \mathbb{R}$ a time interval. A smooth motion or "flow" of S with respect to E can be represented by a mapping

$$\xi\ :\ I \times S\ \longrightarrow\ E\quad|\quad (t,s) \longmapsto p = \xi(t,s) \tag{23}$$

of $I \times S$ into E such that $\xi_t := \xi(t,\cdot) :\ S \to E$ is an embedding for all $t \in I$, and $\xi_s := \xi(\cdot,s)\ :\ I \to E$ is C^2 for all $s \in S$. By definition, then, ξ_t maps S to a Euclidean 2-submanifold $\xi_t[S] \subset E$ with boundary $b\xi_t[S] \subset E$ at each $t \in I$. The region $R_\xi := \bigcup_{t \in I} \xi_t[S] \subset E$ swept out by S in E via ξ is, on the other hand, 3-dimensional. Basic kinematic quantities associated with ξ include the "material" velocity

$$\delta\xi\ :\ I \times S\ \longrightarrow\ V\quad|\quad (t,s) \longmapsto (\delta_t \xi_s) =: (\delta\xi)(t,s) \tag{24}$$

and "deformation gradient"

$$D\xi\ :\ I \times S\ \longrightarrow\ \mathrm{Lin}(TS,V)\quad|\quad (t,s) \longmapsto (D_s \xi_t) =: (D\xi)(t,s)\ , \tag{25}$$

where δ represents the total time derivative operator, $\delta_t \xi_s$ the total time derivative of $\xi_s \in C^2(I,E)$ at $t \in I$, $(D_s\xi_t) \in \mathrm{Lin}(T_sS,V)$ the differential of $\xi_t\ :\ S \to E$ at $s \in S$, and $\mathrm{Lin}(TS,V) := \bigcup_{s \in S} \mathrm{Lin}(T_sS,V)$. As usual, (24) can also be expressed in the form of the "flow" relation

$$\delta\xi = \boldsymbol{w}_\xi \circ \xi \tag{26}$$

of the spatial velocity field $\boldsymbol{w}_\xi$ of S in E, where $(\boldsymbol{w}_\xi \circ \xi)(t,s) := \boldsymbol{w}_\xi(t,\xi(t,s))$, such that $\boldsymbol{w}_\xi$ represents the velocity of the "flow" or motion ξ of S in E.

Among the structures on S induced by, or associated with, ξ, we have an external orientation, represented by a unit normal vector field

$$\boldsymbol{n}_\xi \circ \xi\ :\ I \times S\ \longrightarrow\ V\quad|\quad p \longmapsto \boldsymbol{n}_\xi(\xi(t,s)) =: (\boldsymbol{n}_\xi \circ \xi)(t,s)\ , \tag{27}$$

which is *by definition* perpendicular to ξ, *i.e.*,

$$\xi^* \boldsymbol{\nu}_\xi := \mathbf{0}\ , \tag{28}$$

representing the external orientation of S. Here,

$$\xi^* \boldsymbol{\nu}_\xi = (D\xi)^*(\boldsymbol{\nu}_\xi \circ \xi)\ :\ I \times S\ \longrightarrow\ T^*S \tag{29}$$

represents the pull-back of the one-form

$$\boldsymbol{\nu}_\xi := \boldsymbol{G}\boldsymbol{n}_\xi\ :\ R_\xi\ \longrightarrow\ V^*\quad|\quad p \longmapsto \boldsymbol{G}\boldsymbol{n}_\xi(p) =: \boldsymbol{\nu}_\xi(p) \tag{30}$$

associated with $\boldsymbol{n}_\xi$. In other words, since $\boldsymbol{\nu}_\xi$ on ξ is perpendicular to ths flow, its pull-back has no "component" "parallel" to ξ. The condition (28) implies

$$\mathbf{0} = \delta(\xi^*\boldsymbol{\nu}_\xi) = \xi^*(\mathcal{L}_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi) = \xi^*(\pounds_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\quad\Longrightarrow\quad \pounds_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi = \mathbf{0}\quad\Longrightarrow\quad (D\boldsymbol{\nu}_\xi)\boldsymbol{w}_\xi = -(D\boldsymbol{w}_\xi)^*\boldsymbol{\nu}_\xi \tag{31}$$

via (26) and the fact that $\partial\boldsymbol{\nu}_\xi = \mathbf{0}$ from (27). Here, $\mathcal{L} = \partial + \pounds$ is the dynamic, and $\pounds$ is the autonomous, Lie derivative operator. Further, ∂ represents the partial time-derivative operator. Since $\boldsymbol{G}$ is constant, this last result is equivalent to

$$(D\boldsymbol{n}_\xi)\boldsymbol{w}_\xi = -(D\boldsymbol{w}_\xi)^{\mathrm{T}}\boldsymbol{n}_\xi = -d(\boldsymbol{n}_\xi \cdot \boldsymbol{w}_\xi) + (D\boldsymbol{n}_\xi)^{\mathrm{T}}\boldsymbol{w}_\xi = -(d\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi) + (D\boldsymbol{n}_\xi)^{\mathrm{T}}\boldsymbol{w}_\xi\ , \tag{32}$$

with $(D\boldsymbol{w}_\xi)^{\mathrm{T}} = \boldsymbol{G}^{-1}(D\boldsymbol{w}_\xi)^*\boldsymbol{G}$, the same for $(D\boldsymbol{n}_\xi)^{\mathrm{T}}$, and d represents the exterior derivative operator.

The definition (28) of the external orientation of S implies that the subspace

$$V_p^{\parallel} := (D_s\xi_t)[T_sS] \subset V \tag{33}$$

of V at $p = \xi(t,s) \in R_\xi$ represents all elements of V perpendicular to $\boldsymbol{n}_\xi(p)$, *i.e.*,

$$(D_s\xi_t)[T_sS] = \ker\ \boldsymbol{\nu}_\xi(p)\ . \tag{34}$$

We then have the direct sum

$$V = V_p^{\perp} \oplus V_p^{\parallel} \tag{35}$$

of V at each $p \in R_\xi$, where

$$V_p^\perp := \mathrm{ima}\, \boldsymbol{N}_\xi(p) \tag{36}$$

represents the set of all elements of V at $p \in R_\xi$ parallel to $\boldsymbol{n}_\xi(p)$, with

$$\boldsymbol{N}_\xi := \boldsymbol{n}_\xi \otimes \boldsymbol{\nu}_\xi \;:\; R_\xi \longrightarrow \mathrm{Lin}(V,V) \quad | \quad p \longmapsto \boldsymbol{n}_\xi(p) \otimes \boldsymbol{\nu}_\xi(p) =: \boldsymbol{N}_\xi(p) \tag{37}$$

(see (2)). In what follows, let

$$\boldsymbol{I}_\xi \;:\; R_\xi \longrightarrow \mathrm{Lin}(V_\xi^\parallel, V) \quad | \quad p \longmapsto \boldsymbol{I}_\xi(p) \tag{38}$$

represents the field on ξ induced by the inclusions $\boldsymbol{I}_\xi(p) \in \mathrm{Lin}(V_p^\parallel, V)$, and

$$\boldsymbol{P}_\xi \;:\; R_\xi \longrightarrow \mathrm{Lin}(V, V_\xi^\parallel) \quad | \quad p \longmapsto \boldsymbol{P}_\xi(p) \tag{39}$$

represents that field on ξ induced by the metric-compatible projections $\boldsymbol{P}_\xi(p) \in \mathrm{Lin}(V, V_p^\parallel)$, *i.e.*, (9), with $V_\xi^\parallel := \bigcup_{p \in R_\xi} V_p^\parallel$. Then

$$\boldsymbol{I}_\xi^* \boldsymbol{\nu}_\xi = \boldsymbol{0} \tag{40}$$

and

$$\boldsymbol{P}_\xi\, \boldsymbol{n}_\xi = \boldsymbol{0} \tag{41}$$

hold by extension of (5) and (11), respectively, to ξ via (28).

Since $\boldsymbol{n}_\xi$ is a unit vector field, we have

$$1 = \boldsymbol{n}_\xi \cdot \boldsymbol{n}_\xi = (\boldsymbol{G}\boldsymbol{n}_\xi)\boldsymbol{n}_\xi = \boldsymbol{\nu}_\xi \boldsymbol{n}_\xi \tag{42}$$

via (30), and so

$$\boldsymbol{0} = \boldsymbol{\nu}_\xi(D\boldsymbol{n}_\xi) = (D\boldsymbol{n}_\xi)^* \boldsymbol{\nu}_\xi \;, \tag{43}$$

i.e., $\boldsymbol{\nu}_\xi$ is in the kernel of $(D\boldsymbol{n}_\xi)^*$. On the basis of (43), we obtain the decomposition

$$(D\boldsymbol{n}_\xi) = \boldsymbol{I}_\xi[(\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi + \boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi] \tag{44}$$

of $(D\boldsymbol{n}_\xi)$ via (7), where

$$(\nabla \boldsymbol{n}_\xi) := \boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{I}_\xi \;:\; R_\xi \longrightarrow \mathrm{Sym}(V_\xi^\parallel, V_\xi^\parallel) \tag{45}$$

is the "surface" gradient of $\boldsymbol{n}_\xi$, equal to the negative of the usual curvature tensor of ξ. Unlike $\boldsymbol{N}_\xi(D\boldsymbol{n}_\xi)$, which vanishes on the basis of (28), note that $(D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi$ is in general non-zero. In addition, it is traceless, *i.e.*, $\mathrm{tr}((D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi) = \mathrm{tr}(\boldsymbol{N}_\xi(D\boldsymbol{n}_\xi)) = 0$. Note also that

$$(D\boldsymbol{n}_\xi)^{\mathrm{T}} = [\boldsymbol{I}_\xi(\nabla \boldsymbol{n}_\xi) + \boldsymbol{N}_\xi(D\boldsymbol{n}_\xi)^{\mathrm{T}}\boldsymbol{I}_\xi]\boldsymbol{P}_\xi \tag{46}$$

from (44), where $\boldsymbol{N}_\xi^{\mathrm{T}} = \boldsymbol{N}_\xi$, and (9). From this last result, we also have

$$\mathrm{skw}(D\boldsymbol{n}_\xi) = \mathrm{skw}(\boldsymbol{I}_\xi \boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi) \;. \tag{47}$$

Lastly, the decomposition (7) induces that

$$(D\boldsymbol{n}) = [(D\boldsymbol{n}_\xi)\circ\xi](D\xi) = [(D\boldsymbol{n}_\xi)\circ\xi]\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\xi) = \boldsymbol{I}_\xi[(\nabla\boldsymbol{n}_\xi)\circ\xi]\boldsymbol{P}_\xi(D\xi) \tag{48}$$

of $(D\boldsymbol{n})$ from the chain rule, the fact that $\boldsymbol{N}_\xi(D\xi)$ vanishes via (28), (5), as well as (44).

By point-wise extension of (1) to ξ, $\boldsymbol{\nu}_\xi$ induces a representation

$$\boldsymbol{\omega} = \boldsymbol{\nu}_\xi \wedge \boldsymbol{\alpha}_\xi \tag{49}$$

of $\boldsymbol{\omega}$, where

$$\boldsymbol{\alpha}_\xi := \imath_{\boldsymbol{n}_\xi} \boldsymbol{\omega} \;. \tag{50}$$

Since $\boldsymbol{\nu}_\xi$ is by (28) in addition the normal one-form field to ξ in E, one says that the representation (49) of $\boldsymbol{\omega}$ induced by $\boldsymbol{\nu}_\xi$ is that *adapted to* ξ; in this case, the two-form field $\boldsymbol{\alpha}_\xi$ represents in essence the so-called content form of ξ (*e.g.*, Betounes, 1986). In addition, if $\boldsymbol{m}_\xi$ represents the unit vector field normal to $b\xi$ and parallel to ξ, such that

$$\boldsymbol{t}_\xi := \boldsymbol{n}_\xi \times \boldsymbol{m}_\xi \tag{51}$$

represents the unit tangent vector to $b\xi$, then $\boldsymbol{t}_\xi \in \ker\, \boldsymbol{N}_\xi$, or $\boldsymbol{t}_\xi = \boldsymbol{I}_\xi \boldsymbol{P}_\xi\, \boldsymbol{t}_\xi$, hold. These unit vector fields induce the form

$$\boldsymbol{\alpha}_\xi = \boldsymbol{\mu}_\xi \wedge \boldsymbol{\tau}_\xi \tag{52}$$

of $\boldsymbol{\alpha}_\xi$ adapted to $b\xi$, and so that

$$\boldsymbol{\omega} = \boldsymbol{\nu}_\xi \wedge \boldsymbol{\mu}_\xi \wedge \boldsymbol{\tau}_\xi \tag{53}$$

of $\boldsymbol{\omega}$ adapted to ξ and $b\xi$. Note also that (22) extends to

$$\boldsymbol{I}_\xi \boldsymbol{P}_\xi = \boldsymbol{m}_\xi \otimes \boldsymbol{\mu}_\xi + \boldsymbol{t}_\xi \otimes \boldsymbol{\tau}_\xi \tag{54}$$

for $\boldsymbol{I}_\xi \boldsymbol{P}_\xi$ in terms of $\boldsymbol{m}_\xi$ and $\boldsymbol{t}_\xi$.

Now, using Cartan's operator relation

$$\pounds = d \circ \imath + \imath \circ d \,, \tag{55}$$

as well as the result $\imath_{\boldsymbol{n}_\xi} \boldsymbol{\alpha}_\xi = \imath_{\boldsymbol{n}_\xi} \imath_{\boldsymbol{n}_\xi} \boldsymbol{\omega} = \boldsymbol{0}$, we have

$$\pounds_{\psi_\xi \boldsymbol{n}_\xi} \boldsymbol{\alpha}_\xi = d\imath_{\psi_\xi \boldsymbol{n}_\xi} \boldsymbol{\alpha}_\xi + \imath_{\psi_\xi \boldsymbol{n}_\xi} d\boldsymbol{\alpha}_\xi = \imath_{\psi_\xi \boldsymbol{n}_\xi} d\imath_{\boldsymbol{n}_\xi} \boldsymbol{\omega} = \psi_\xi \,(\mathrm{div}_{\boldsymbol{\omega}} \boldsymbol{n}_\xi)\, \boldsymbol{\alpha}_\xi \tag{56}$$

for all time-dependent, *linear-space-valued functions* ψ_ξ defined on ξ, where

$$d\boldsymbol{\alpha}_\xi = d\imath_{\boldsymbol{n}_\xi} \boldsymbol{\omega} = \pounds_{\boldsymbol{n}_\xi} \boldsymbol{\omega} = (\mathrm{div}_{\boldsymbol{\omega}} \boldsymbol{n}_\xi)\, \boldsymbol{\omega} \tag{57}$$

follows from $d\boldsymbol{\omega} = \boldsymbol{0}$, (55), and the definition

$$(\mathrm{div}_{\boldsymbol{\omega}} \boldsymbol{u})\boldsymbol{\omega} := \pounds_{\boldsymbol{u}} \boldsymbol{\omega} = d\imath_{\boldsymbol{u}} \boldsymbol{\omega} \tag{58}$$

of the divergence of any smooth V-valued vector field $\boldsymbol{u}$ with respect to $\boldsymbol{\omega}$ on E. Since

$$(\mathrm{div}_{\boldsymbol{\omega}} \boldsymbol{n}_\xi) = \mathrm{tr}(D\boldsymbol{n}_\xi) = \mathrm{tr}(\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi) \boldsymbol{P}_\xi) + \mathrm{tr}(\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi) \boldsymbol{N}_\xi) = \mathrm{tr}_{\parallel}(\nabla \boldsymbol{n}_\xi) \tag{59}$$

follows from the fact that $\boldsymbol{\omega}$ is constant, (44), as well as the fact that $\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi) \boldsymbol{N}$ is traceless, *i.e.*,

$$\mathrm{tr}(\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi) \boldsymbol{N}_\xi) = \mathrm{tr}(\boldsymbol{N}_\xi \boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)) = 0 \,, \tag{60}$$

we then obtain

$$\begin{aligned} \pounds_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi &= \pounds_{\boldsymbol{u}_\xi} \imath_{\boldsymbol{n}_\xi} \boldsymbol{\omega} \\ &= d\imath_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi + \imath_{\boldsymbol{u}_\xi} d\boldsymbol{\alpha}_\xi \\ &= [\mathrm{div}_{\boldsymbol{\alpha}} \boldsymbol{u}_\xi + (\imath_{\boldsymbol{u}_\xi} \boldsymbol{\nu}_\xi)\, \mathrm{tr}_{\parallel}(\nabla \boldsymbol{n}_\xi)]\, \boldsymbol{\alpha}_\xi - \mathrm{tr}_{\parallel}(\nabla \boldsymbol{n}_\xi)\, \boldsymbol{\nu}_\xi \wedge \imath_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi \end{aligned} \tag{61}$$

from (56) and (59), where

$$(\mathrm{div}_{\boldsymbol{\alpha}} \boldsymbol{u}_\xi)\, \boldsymbol{\alpha}_\xi := d\imath_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi \tag{62}$$

defines the divergence of a vector field $\boldsymbol{u}_\xi$ on ξ with respect to $\boldsymbol{\alpha}_\xi$. Since $\imath_{\boldsymbol{N}_\xi \boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi = \boldsymbol{0}$, and so $\imath_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi = \imath_{\boldsymbol{I}_\xi (\boldsymbol{P}_\xi \boldsymbol{u}_\xi)} \boldsymbol{\alpha}_\xi$, holds, note that only the tangential part $\boldsymbol{u}_\xi$ actually contributes to $\mathrm{div}_{\boldsymbol{\alpha}} \boldsymbol{u}_\xi$, *i.e.*,

$$\mathrm{div}_{\boldsymbol{\alpha}} \boldsymbol{u}_\xi = \mathrm{div}_{\boldsymbol{\alpha}} (\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{u}_\xi) \tag{63}$$

holds in general.

On the basis of (62), we also have the form

$$\int_{b\xi} \imath_{\boldsymbol{u}_\xi} \boldsymbol{\alpha}_\xi = \int_\xi (\mathrm{div}_{\boldsymbol{\alpha}} \boldsymbol{u}_\xi)\, \boldsymbol{\alpha}_\xi \tag{64}$$

of Stokes' theorem on ξ, where the notation

$$\int_\xi f \boldsymbol{\alpha}_\xi \,:\, I \longrightarrow \mathbb{R} \quad | \quad t \longmapsto \int_{\xi_t[P]} f_t \boldsymbol{\alpha}_\xi =: \left\{ \int_\xi f \boldsymbol{\alpha}_\xi \right\} (t) \tag{65}$$

for any time-dependent function f defined on ξ, and any $P \subset S$, has been introduced. Using (64), we also obtain the transport relation

$$
\begin{aligned}
\delta \int_\xi \psi_\xi \, \boldsymbol{\alpha}_\xi &= \int_\xi \mathcal{L}_{\boldsymbol{w}_\xi}(\psi_\xi \, \boldsymbol{\alpha}_\xi) \\
&= \int_\xi (\partial\psi_\xi)\boldsymbol{\alpha}_\xi + d\imath_{\psi_\xi \boldsymbol{w}_\xi}\boldsymbol{\alpha}_\xi + \imath_{\boldsymbol{w}_\xi} d(\psi_\xi \, \boldsymbol{\alpha}_\xi) \\
&= \int_\xi [(\partial\psi_\xi) + (\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\,\mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\psi_\xi + \mathrm{div}_{\boldsymbol{\alpha}}(\psi_\xi \boldsymbol{w}_\xi)]\,\boldsymbol{\alpha}_\xi
\end{aligned}
\tag{66}
$$

(*e.g.*, see Abraham *et al.*, 1988, §7.1) for any differentiable, time-dependent, linear space valued function ψ_ξ on ξ via the result $\imath_{\boldsymbol{w}_\xi} d(\psi_\xi \boldsymbol{\alpha}_\xi)|_\xi = \psi_\xi \, \imath_{\boldsymbol{w}_\xi}(d\boldsymbol{\alpha}_\xi)|_\xi$, and (61)–(63). This last form of a time derivative associated with ξ avoids the ambiguity associated with the related concept of "displacement derivative" (*e.g.*, Bowen, Wang, 1975; Kosinski, 1991; dell'Isola, Kosinski, 1993).

4 Bidimensional Geometry and Kinematics

A bidimensional continuum represents one in which one of the three dimensions is thin in comparison to the other two, *e.g.*, a shell. The motion of such a continuum relative to S can be represented by a pair of time-dependent, scalar-valued fields

$$
h_\xi^\pm \diamond \xi \,:\, I \times S \longrightarrow \mathbb{R} \quad | \quad (t,s) \longmapsto h_\pm(t, \xi(t,s)) =: (h_\xi^\pm \diamond \xi)(t,s) \tag{67}
$$

on S, which induce the flows

$$
\zeta_\pm \,:\, I \times S \longrightarrow E \quad | \quad (t,s) \longmapsto \xi(t,s) + h_\xi^\pm(t,\xi(t,s))\,\boldsymbol{n}_\xi(\xi(t,s)) =: \zeta_\pm(t,s) \tag{68}
$$

of the bidimensional continuum on the $+$ and $-$ sides of ξ, *i.e.*,

$$
\zeta_\pm = \lambda_\pm \diamond \xi \,, \tag{69}
$$

with

$$
\lambda_\pm := \boldsymbol{I}_{\xi\subset E} + h_\xi^\pm \, \boldsymbol{n}_\xi \,, \tag{70}
$$

where $\boldsymbol{I}_{\xi\subset E}$ is the inclusion mapping of ξ into E.

Since the entire remaining development is the same for the $+$ and $-$ sides of the bidimensional flow, we drop the $\pm$ sub- and superscripts on all relations involving $\zeta_\pm$ for notational simplicity in what follows. Noting that ζ represents, like ξ, a motion of S in E, we can apply the discussion and results of the previous section for ξ to ζ by analogy. In particular, in terms of the flow relations (26) and

$$
\delta\zeta = \boldsymbol{w}_\zeta \diamond \zeta \tag{71}
$$

the total time derivative of (69) takes the form

$$
\boldsymbol{w}_\zeta = \boldsymbol{w}_\lambda + \lambda_* \boldsymbol{w}_\xi \,, \tag{72}
$$

with

$$
\partial\lambda = \boldsymbol{w}_\lambda \diamond \lambda = (\partial h_\xi)\boldsymbol{n}_\xi \tag{73}
$$

and

$$
\lambda_* \boldsymbol{w}_\xi = (D\lambda)(\boldsymbol{w}_\xi \diamond \lambda^{-1}) \,, \tag{74}
$$

where

$$
(D\lambda) = \boldsymbol{I}_V + \boldsymbol{n}_\xi \otimes (dh_\xi) + h_\xi(D\boldsymbol{n}_\xi) = \boldsymbol{H}_\xi + h_\xi(D\boldsymbol{n}_\xi) \tag{75}
$$

is obtained from (70), with

$$
\boldsymbol{H}_\xi := \boldsymbol{I}_V + \boldsymbol{n}_\xi \otimes (dh_\xi) \,. \tag{76}
$$

Further, with $\boldsymbol{n}_\zeta$ the unit normal to ζ,

$$
\zeta^* \boldsymbol{\nu}_\zeta := \boldsymbol{0} \tag{77}
$$

holds by definition, analogous to (28). Like $\boldsymbol{\nu}_\xi$, $\boldsymbol{\nu}_\zeta$ induces a representation

$$
\boldsymbol{\omega} = \boldsymbol{\nu}_\zeta \wedge \boldsymbol{\alpha}_\zeta \tag{78}
$$

of $\boldsymbol{\omega}$ (*c.f.* (49)), where

$$\boldsymbol{\alpha}_\zeta := \imath_{\boldsymbol{n}_\zeta}\boldsymbol{\omega} \tag{79}$$

(*c.f.* (50)). And with $\boldsymbol{m}_\zeta$ the unit vector field normal to $b\zeta$ and parallel to ζ,

$$\boldsymbol{t}_\zeta := \boldsymbol{n}_\zeta \times \boldsymbol{m}_\zeta \tag{80}$$

represents the unit tangent vector to $b\zeta$. These vectors induce the representation

$$\boldsymbol{\alpha}_\zeta = \boldsymbol{\mu}_\zeta \wedge \boldsymbol{\tau}_\zeta \tag{81}$$

of $\boldsymbol{\alpha}_\zeta$ adapted to $b\zeta$, and so that

$$\boldsymbol{\omega} = \boldsymbol{\nu}_\zeta \wedge \boldsymbol{\mu}_\zeta \wedge \boldsymbol{\tau}_\zeta \tag{82}$$

of $\boldsymbol{\omega}$ adapted to ζ and $b\zeta$.

The formulation of bidimensional balance relations in the next section relies on certain relations between the dynamic geometries of ζ and ξ, which are connected by the relative motion λ as given in (69). For example, (69) and (77) imply that

$$\xi^*(\lambda^*\boldsymbol{\nu}_\zeta) = \boldsymbol{0}\,, \tag{83}$$

and so in turn that

$$\lambda^*\boldsymbol{\nu}_\zeta = (D\lambda)^*(\boldsymbol{\nu}_\zeta\circ\lambda) = (\boldsymbol{\nu}_\zeta\circ\lambda) + h_\xi(D\boldsymbol{n}_\xi)^*(\boldsymbol{\nu}_\zeta\circ\lambda) + \imath_{\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)\,(dh_\xi) \tag{84}$$

is proportional to $\boldsymbol{\nu}_\xi$, *i.e.*,

$$\lambda^*\boldsymbol{\nu}_\zeta = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)\,\boldsymbol{\nu}_\xi\,, \tag{85}$$

with

$$\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta) = \imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda) \tag{86}$$

from (44) and (84). On the other hand, we have

$$\lambda^*\boldsymbol{\mu}_\zeta = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\boldsymbol{\nu}_\xi + \imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\boldsymbol{\mu}_\xi + \imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\boldsymbol{\tau}_\xi \tag{87}$$

and

$$\lambda^*\boldsymbol{\tau}_\zeta = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\boldsymbol{\nu}_\xi + \imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\boldsymbol{\mu}_\xi + \imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\boldsymbol{\tau}_\xi \tag{88}$$

in general, where the coefficients

$$\begin{array}{rcl}
\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)\\
\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)\\
\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)\\
\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\\
\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\\
\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta) &=& \imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda) + h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)
\end{array} \tag{89}$$

follow from (44) and (84).

¿From (78) and (85), we obtain

$$\lambda^*\boldsymbol{\omega} = \det(D\lambda)\,\boldsymbol{\omega} = \lambda^*\boldsymbol{\nu}_\zeta \wedge \lambda^*\boldsymbol{\alpha}_\zeta = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)\,\boldsymbol{\nu}_\xi \wedge \lambda^*\boldsymbol{\alpha}_\zeta\,, \tag{90}$$

and so the expression

$$\det(D\lambda) = \imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\omega}) = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)\,\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\alpha}_\zeta) \tag{91}$$

for $\det(D\lambda)$ from (90) with $\imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}\boldsymbol{\omega} = 1$. An alternative expression for $\det(D\lambda)$ can be obtained as follows. Let $\boldsymbol{a},\boldsymbol{b}\in V$ be two arbitrary, linearly-independent elements of V. Then

$$\begin{array}{rcl}
\imath_{\boldsymbol{b}}\imath_{\boldsymbol{a}}(\lambda^*\boldsymbol{\omega}) &=& \imath_{(D\lambda)\boldsymbol{b}}\imath_{(D\lambda)\boldsymbol{a}}\boldsymbol{\omega}\\
&=& \imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\lambda)\boldsymbol{b}}\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\lambda)\boldsymbol{a}}\boldsymbol{\omega} + \imath_{\boldsymbol{N}_\xi(D\lambda)\boldsymbol{b}}\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\lambda)\boldsymbol{a}}\boldsymbol{\omega} + \imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\lambda)\boldsymbol{b}}\imath_{\boldsymbol{N}_\xi(D\lambda)\boldsymbol{a}}\boldsymbol{\omega}\\
&=& c_{\boldsymbol{\nu}}(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{\nu}_\xi + c_{\boldsymbol{\mu}}(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{\mu}_\xi + c_{\boldsymbol{\tau}}(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{\tau}_\xi
\end{array} \tag{92}$$

follows from the fact that $\imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{n}_\xi}\boldsymbol{\omega} = \boldsymbol{0}$, where

$$
\begin{aligned}
c_{\boldsymbol{\nu}}(\boldsymbol{a},\boldsymbol{b}) \;&:=\; \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\lambda)\boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\lambda)\boldsymbol{a}}\boldsymbol{\alpha}_\xi \\
&=\; \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi \\
&+\; h_\xi \left\{ \mathrm{tr}_{\|}(\nabla \boldsymbol{n}_\xi)\,(\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi) + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi \right\} \\
&+\; h_\xi^2 \left\{ \mathrm{det}_{\|}(\nabla \boldsymbol{n}_\xi)\,(\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi) + \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi \right. \\
&\qquad \left. + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{b}}\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{a}}\boldsymbol{\alpha}_\xi \right\} ,
\end{aligned}
\tag{93}
$$

is obtained from (12), (13) and (75), and

$$
\begin{aligned}
c_{\boldsymbol{\mu}}(\boldsymbol{a},\boldsymbol{b}) \;&:=\; \imath_{\boldsymbol{H}_\xi \boldsymbol{a}}\boldsymbol{\nu}_\xi \left\{ (\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\boldsymbol{\tau}_\xi) + h_\xi \left[\imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{b}}\boldsymbol{\tau}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{b}}\boldsymbol{\tau}_\xi \right] \right\} \\
&-\; \imath_{\boldsymbol{H}_\xi \boldsymbol{b}}\boldsymbol{\nu}_\xi \left\{ (\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\tau}_\xi) + h_\xi \left[\imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\tau}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{a}}\boldsymbol{\tau}_\xi \right] \right\} , \\
c_{\boldsymbol{\tau}}(\boldsymbol{a},\boldsymbol{b}) \;&:=\; \imath_{\boldsymbol{H}_\xi \boldsymbol{a}}\boldsymbol{\nu}_\xi \left\{ (\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{b}}\boldsymbol{\mu}_\xi) + h_\xi \left[\imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{b}}\boldsymbol{\mu}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{b}}\boldsymbol{\mu}_\xi \right] \right\} \\
&-\; \imath_{\boldsymbol{H}_\xi \boldsymbol{b}}\boldsymbol{\nu}_\xi \left\{ (\imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\mu}_\xi) + h_\xi \left[\imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{a}}\boldsymbol{\mu}_\xi + \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{N}_\xi \boldsymbol{a}}\boldsymbol{\mu}_\xi \right] \right\} ,
\end{aligned}
\tag{94}
$$

from (75) and (76). In particular, setting $\boldsymbol{a} = \boldsymbol{m}_\xi$ and $\boldsymbol{b} = \boldsymbol{t}_\xi$, (92) yields

$$
\begin{aligned}
\det(D\lambda) \;&=\; \imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\omega}) \\
&=\; \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)[\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta) - \imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)] \\
&=\; c_{\boldsymbol{\nu}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi) \\
&=\; \jmath_\xi
\end{aligned}
\tag{95}
$$

for $\det(D\lambda)$ from (91), (93) and $\imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}\boldsymbol{\omega} = 1$, where

$$
\jmath_\xi := \mathrm{det}_{\|}(\boldsymbol{K}_\xi) = 1 + h_\xi\,\mathrm{tr}_{\|}(\nabla \boldsymbol{n}_\xi) + h_\xi^2\,\mathrm{det}_{\|}(\nabla \boldsymbol{n}_\xi)
\tag{96}
$$

and

$$
\boldsymbol{K}_\xi := \boldsymbol{P}_\xi(D\lambda)\boldsymbol{I}_\xi = \boldsymbol{I}_{V_\xi^\|} + h_\xi(\nabla \boldsymbol{n}_\xi) : \; R_\xi \longrightarrow \mathrm{Sym}(V_\xi^\|, V_\xi^\|)
\tag{97}
$$

have been introduced.

¿From the definition of the adjunct form of a linear mapping, we also have the result

$$
\imath_{\boldsymbol{b}}\imath_{\boldsymbol{a}}(\lambda^*\boldsymbol{\omega}) = \det(D\lambda)\,(D\lambda)^{-*}(\imath_{\boldsymbol{b}}\imath_{\boldsymbol{a}}\boldsymbol{\omega})
\tag{98}
$$

which we can use to determine the pull-back $\lambda^*\boldsymbol{z}_\zeta = (D\lambda)^{-1}(\boldsymbol{z}_\zeta \diamond \lambda)$ of some time-dependent vector field $\boldsymbol{z}$ defined on ζ as follows. First, note that

$$
\begin{aligned}
\imath_{\boldsymbol{m}_\xi}\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\omega}) \;&=\; \jmath_\xi\,(D\lambda)^{-*}\boldsymbol{\tau}_\xi \\
\imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\omega}) \;&=\; \jmath_\xi\,(D\lambda)^{-*}\boldsymbol{\mu}_\xi \\
\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\omega}) \;&=\; \jmath_\xi\,(D\lambda)^{-*}\boldsymbol{\nu}_\xi
\end{aligned}
\tag{99}
$$

(compare the last two with dell'Isola and Kosinski, 1994, 2.16 and 2.29, respectively) follow from (53), (95) and (98). Expressing $\lambda^*\boldsymbol{z}_\zeta$ then in the form

$$
\lambda^*\boldsymbol{z}_\zeta = (\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\nu}_\xi)\,\boldsymbol{n}_\xi + (\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\mu}_\xi)\,\boldsymbol{m}_\xi + (\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\tau}_\xi)\,\boldsymbol{t}_\xi
\tag{100}
$$

relative to ξ, we have

$$
\begin{aligned}
\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\nu}_\xi \;&=\; \jmath_\xi^{-1}\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\imath_{\boldsymbol{m}_\xi}\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\omega}) \\
&=\; \jmath_\xi^{-1}[c_{\boldsymbol{\nu}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\nu}_\xi + c_{\boldsymbol{\mu}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\mu}_\xi + c_{\boldsymbol{\tau}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\tau}_\xi] \\
\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\mu}_\xi \;&=\; \jmath_\xi^{-1}\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\imath_{\boldsymbol{n}_\xi}\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\omega}) \\
&=\; \jmath_\xi^{-1}[c_{\boldsymbol{\nu}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\nu}_\xi + c_{\boldsymbol{\mu}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\mu}_\xi + c_{\boldsymbol{\tau}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\tau}_\xi] \\
\imath_{\lambda^*\boldsymbol{z}_\zeta}\boldsymbol{\tau}_\xi \;&=\; \jmath_\xi^{-1}\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\imath_{\boldsymbol{t}_\xi}\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\omega}) \\
&=\; \imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\nu}_\xi + \jmath_\xi^{-1}[c_{\boldsymbol{\mu}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\mu}_\xi + c_{\boldsymbol{\tau}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi)\,\imath_{(\boldsymbol{z}_\zeta \diamond \lambda)}\boldsymbol{\tau}_\xi]
\end{aligned}
\tag{101}
$$

from (92), (95) and (99), where

$$
\begin{aligned}
c_{\boldsymbol{\nu}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi) &= -h_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\tau}_\xi + h_\xi^2\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{m}_\xi} \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\alpha}_\xi \ , \\
c_{\boldsymbol{\mu}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi) &= h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{m}_\xi} \boldsymbol{\tau}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{m}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\tau}_\xi \right\} \ , \\
c_{\boldsymbol{\tau}}(\boldsymbol{n}_\xi,\boldsymbol{m}_\xi) &= \imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi + h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{m}_\xi} \boldsymbol{\mu}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{m}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\mu}_\xi \right\} \ ,
\end{aligned}
\tag{102}
$$

as well as

$$
\begin{aligned}
c_{\boldsymbol{\nu}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi) &= -h_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\mu}_\xi + h_\xi^2\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{t}_\xi} \boldsymbol{\alpha}_\xi \ , \\
c_{\boldsymbol{\mu}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi) &= -\imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi + h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{t}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\tau}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{t}_\xi} \boldsymbol{\tau}_\xi \right\} \ , \\
c_{\boldsymbol{\tau}}(\boldsymbol{t}_\xi,\boldsymbol{n}_\xi) &= h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{t}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi \boldsymbol{P}_\xi (D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi} \boldsymbol{\mu}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{n}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{t}_\xi} \boldsymbol{\mu}_\xi \right\} \ ,
\end{aligned}
\tag{103}
$$

and

$$
\begin{aligned}
c_{\boldsymbol{\nu}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi) &= \jmath_\xi \ , \\
c_{\boldsymbol{\mu}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi) &= \imath_{\boldsymbol{H}_\xi \boldsymbol{m}_\xi} \boldsymbol{\nu}_\xi + h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{m}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{t}_\xi} \boldsymbol{\tau}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{t}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{m}_\xi} \boldsymbol{\tau}_\xi \right\} \ , \\
c_{\boldsymbol{\tau}}(\boldsymbol{m}_\xi,\boldsymbol{t}_\xi) &= -\imath_{\boldsymbol{H}_\xi \boldsymbol{t}_\xi} \boldsymbol{\nu}_\xi + h_\xi \left\{ \imath_{\boldsymbol{H}_\xi \boldsymbol{m}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{t}_\xi} \boldsymbol{\mu}_\xi - \imath_{\boldsymbol{H}_\xi \boldsymbol{t}_\xi} \boldsymbol{\nu}_\xi\, \imath_{\boldsymbol{I}_\xi (\nabla \boldsymbol{n}_\xi)\boldsymbol{P}_\xi \boldsymbol{m}_\xi} \boldsymbol{\mu}_\xi \right\} \ ,
\end{aligned}
\tag{104}
$$

are obtained from (93) and (94).

5 Bidimensional Balance Relations

Let $\chi : I \times B \to E$ represent the motion, and $\boldsymbol{v}$ the corresponding spatial velocity, of a material body B in E, such that $\delta\chi = \boldsymbol{v} \diamond \chi$ holds, where δ represents a variation in time, *i.e.*, the total time derivative operator. The general balance relation for some additive thermodynamic quantity with spatial density ψ, production rate density π, flux density $\boldsymbol{\phi}$ (into χ), and supply rate density σ is given by

$$
\delta \int_\chi \psi \boldsymbol{\omega} = \int_\chi \pi \boldsymbol{\omega} + \int_{b\chi} \imath_{\boldsymbol{\phi}} \boldsymbol{\omega} + \int_\chi \sigma \boldsymbol{\omega} \ , \tag{105}
$$

where we have introduced the notation

$$
\int_\chi \psi \boldsymbol{\omega} : \ I \longrightarrow Z \quad | \quad t \longmapsto \int_{\chi_t[P]} \psi_t \, \boldsymbol{\omega} =: \left(\int_\chi \psi \boldsymbol{\omega} \right)(t) \tag{106}
$$

for the integral on the motion or flow χ of any subbody $P \subset B$, analogous to (65). Note that ψ, π and σ are time-dependent fields on χ taking values in some normed linear space Z, while $\boldsymbol{\phi}$ is such a field taking values in $\mathrm{Lin}(V^*, Z)$. With the help of the transport relation[1]

$$
\delta \int_\chi \psi \boldsymbol{\omega} = \int_\chi \mathcal{L}_{\boldsymbol{v}}(\psi \boldsymbol{\omega}) \ , \tag{107}
$$

as well as Stokes' theorem

$$
\int_{b\chi} \imath_{\boldsymbol{\phi}} \boldsymbol{\omega} = \int_\chi d\, \imath_{\boldsymbol{\phi}} \boldsymbol{\omega} = \int_\chi \pounds_{\boldsymbol{\phi}} \boldsymbol{\omega} \ , \tag{108}
$$

(105) can be expressed in the alternative form

$$
\int_\chi \mathcal{L}_{\boldsymbol{v}}(\psi \boldsymbol{\omega}) = \int_\chi [\pi \boldsymbol{\omega} + \pounds_{\boldsymbol{\phi}} \boldsymbol{\omega} + \sigma \boldsymbol{\omega}] \ . \tag{109}
$$

Via continuity of the integrands, then, (109) takes the local form

$$
\mathcal{L}_{\boldsymbol{v}}(\psi \boldsymbol{\omega}) = \pi \boldsymbol{\omega} + \pounds_{\boldsymbol{\phi}} \boldsymbol{\omega} + \sigma \boldsymbol{\omega} \ , \tag{110}
$$

[1] The appearance of the dynamic Lie derivative of ψ in (107) is based on the *fixed* linear space structure of $\mathcal{Z}$, with respect to which it is equal to the material time derivative of ψ.

with which we work in what follows. To show that (110) corresponds to the usual general local spatial balance, note that[2]

$$\mathcal{L}_{\boldsymbol{v}}(\psi\boldsymbol{\omega}) = (\partial\psi)\boldsymbol{\omega} + \pounds_{\boldsymbol{v}}(\psi\boldsymbol{\omega}) = [(\partial\psi) + \mathrm{div}_{\boldsymbol{\omega}}(\psi\boldsymbol{v})]\,\boldsymbol{\omega} \tag{111}$$

via (58) and the relation $\pounds_{\boldsymbol{u}}(\psi\boldsymbol{\omega}) = \pounds_{\psi\boldsymbol{u}}\boldsymbol{\omega}$, which follows from the identity $\pounds_{\boldsymbol{u}}(\psi\boldsymbol{\gamma}) = \pounds_{\psi\boldsymbol{u}}\boldsymbol{\gamma} + \imath_{\boldsymbol{u}}[(d\psi)\wedge\boldsymbol{\gamma}]$, valid for any differentiable r-form $\boldsymbol{\gamma}$ ($r \leq 3$ here), differentiable vector field $\boldsymbol{u}$, and volume form $\boldsymbol{\omega}$. From (58), we also have

$$\pounds_{\boldsymbol{\phi}}\boldsymbol{\omega} = d\imath_{\boldsymbol{\phi}}\boldsymbol{\omega} = (\mathrm{div}_{\boldsymbol{\omega}}\boldsymbol{\phi})\,\boldsymbol{\omega}\ . \tag{112}$$

Substituting $(111)_2$ and (112) into (110), we obtain

$$\dot{\psi} + \psi\,(\mathrm{div}\,\boldsymbol{v}) = \pi + \mathrm{div}\,\boldsymbol{\phi} + \sigma\ , \tag{113}$$

i.e., the usual form of the general local balance, with $\dot{\psi} = \partial\psi + (D\psi)\boldsymbol{v}$, and $\mathrm{div} = \mathrm{div}_{\boldsymbol{\omega}}$ the usual divergence operator in E.

The usual form (110) of the local spatial balance relative to the material motion χ with time-dependent velocity field $\boldsymbol{v}$ can be expressed relative to an arbitrary (not necessarily material) motion with time-dependent velocity field $\boldsymbol{u}$ as follows. First, note that the linearity of the dynamic Lie derivative operator

$$\mathcal{L}_{\boldsymbol{v}} := \partial + \pounds_{\boldsymbol{v}} \tag{114}$$

yields the relation

$$\mathcal{L}_{\boldsymbol{u}} = \mathcal{L}_{\boldsymbol{v}} + \pounds_{\boldsymbol{u}-\boldsymbol{v}} \tag{115}$$

between such operators with respect to $\boldsymbol{v}$ and $\boldsymbol{u}$. Combining (110) and (115), we obtain the alternative form

$$\begin{aligned} \mathcal{L}_{\boldsymbol{u}}(\psi\boldsymbol{\omega}) &= \mathcal{L}_{\boldsymbol{v}}(\psi\boldsymbol{\omega}) + \pounds_{\boldsymbol{u}-\boldsymbol{v}}(\psi\boldsymbol{\omega}) \\ &= \pi\boldsymbol{\omega} + \pounds_{\boldsymbol{\varphi}}\boldsymbol{\omega} + \sigma\boldsymbol{\omega} \end{aligned} \tag{116}$$

of the general local balance (110) relative to the motion associated with $\boldsymbol{u}$, where

$$\boldsymbol{\varphi} := \psi(\boldsymbol{u}-\boldsymbol{v}) + \boldsymbol{\phi}\ , \tag{117}$$

follows from $\pounds_{\boldsymbol{u}}(\psi\boldsymbol{\omega}) = \pounds_{\psi\boldsymbol{u}}\boldsymbol{\omega} = d\imath_{\psi\boldsymbol{u}}\boldsymbol{\omega}$ via (55), and $d\boldsymbol{\omega} = \boldsymbol{0}$. It is worth emphasizing that (110) and (116) are simply two different *mathematical* forms of the same *physical* balance relation. In the latter case, however, the "extra" flux $\psi(\boldsymbol{u}-\boldsymbol{v})$ appearing in (117) can be used to represent the process of diffusion, where $\boldsymbol{u}$ represents the velocity of the diffusion "front," as we will see in what follows.

The general balance relation relative to the bidimensional flow to be developed next is based on the corresponding flow geometry

$$\begin{aligned} \zeta &:= \zeta_+ \cup \xi \cup \zeta_-\ , \\ b\zeta &= b^o\zeta_+ \cup b\xi \cup b^o\zeta_-\ , \\ b^o\zeta_\pm &= b^c\zeta_\pm \cup b^s\zeta_\pm\ , \\ b^s\zeta_\pm &= b^l\zeta_\pm \cup b^r\zeta_\pm\ , \end{aligned} \tag{118}$$

relative to S, where $b^o\zeta_\pm$ represents the "outer" boundary, $b\zeta_\pm$ the complete boundary, and $b^s\zeta_\pm$ the two sides, of $\zeta_\pm$, while $b^c\zeta_+$ represents the "top" of ζ_+, and $b^c\zeta_-$ the "bottom" of ζ_-. On the basis of (116) and (118), then, the general balance relation relative to the bidimensional flow takes the form

$$\delta\int_\zeta \psi_\zeta\,\boldsymbol{\omega} = \int_\zeta \pi_\zeta\,\boldsymbol{\omega} + \int_{b\zeta} \imath\boldsymbol{\varphi}_\zeta\,\boldsymbol{\omega} + \int_\zeta \sigma_\zeta\,\boldsymbol{\omega}\ . \tag{119}$$

¿From the structure $(118)_1$ of the bidimensional flow ζ, we have

$$\int_\zeta \psi_\zeta\,\boldsymbol{\omega} = \int_{\zeta_+} \psi_\zeta\,\boldsymbol{\omega} + \int_{\zeta_-} \psi_\zeta\,\boldsymbol{\omega}\ , \tag{120}$$

[2]As in (107), the *fixed* linear space structure of $\mathcal{Z}$ is used here.

and likewise for π_ζ and σ_ζ via the fact that ξ has volume measure zero. Now, with the help of Fubini's theorem, (120) can be rewritten in the form

$$\begin{aligned}\int_\zeta \psi_\zeta\,\boldsymbol{\omega} &= \int_\xi \int_{h_\xi^-}^{h_\xi^+} \lambda^*(\psi_\zeta\,\boldsymbol{\omega})\\ &= \int_\xi \int_{h_\xi^-}^{h_\xi^+} (\lambda^*\psi_\zeta)\,(\lambda^*\boldsymbol{\omega})\\ &= \int_\xi \int_{h_\xi^-}^{h_\xi^+} \jmath_\xi\,(\lambda^*\psi_\zeta)\,\boldsymbol{\nu}_\xi\wedge\boldsymbol{\alpha}_\xi\\ &= \int_\xi \psi_{\mathrm{s}}\,\boldsymbol{\alpha}_\xi\end{aligned}\tag{121}$$

via (52), (90) and (95), where

$$\psi_{\mathrm{s}} := \int_{h_\xi^-}^{h_\xi^+} \jmath_\xi\,(\lambda^*\psi_\zeta)\,\boldsymbol{\nu}_\xi\tag{122}$$

represents the "normal average" of ψ_ζ relative to ξ. Note that the integrand in (122) is given by $\jmath_\xi^\pm\,(\lambda_\pm^*\psi_\zeta)$ in the $\pm$ regions of ζ, respectively. Similar expressions hold for π_ζ and σ_ζ. The appearance of the pull back in (121) accounts for the relative motion between ζ and ξ.

Consider next the result

$$\int_{b\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} = \int_{b^c\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} + \int_{b^s\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega}\,,\tag{123}$$

which follows from $(118)_{2,3}$ and the fact that $b\xi$ has area measure zero, where

$$\int_{b^c\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} = \int_{b^c\zeta_+} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} + \int_{b^c\zeta_-} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega}\tag{124}$$

and

$$\int_{b^s\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} = \int_{b^s\zeta_+} \imath\boldsymbol{\varphi}_\zeta^+\boldsymbol{\omega} + \int_{b^s\zeta_-} \imath\boldsymbol{\varphi}_\zeta^-\boldsymbol{\omega}\,,\tag{125}$$

with

$$\boldsymbol{\varphi}_\zeta := \psi_\zeta(\boldsymbol{w}_\zeta - \boldsymbol{v}_\zeta) + \boldsymbol{\phi}_\zeta\tag{126}$$

the general flux $\boldsymbol{\varphi}_\zeta$ relative to ζ, in analogy with (117). With the result

$$\begin{aligned}\int_{b^c\zeta_\pm} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} &= \int_{b^c\zeta_\pm} (\imath\boldsymbol{\varphi}_\zeta\boldsymbol{\nu}_\zeta)\,\boldsymbol{\alpha}_\zeta\\ &= \int_\xi \lambda_\pm^*[(\imath\boldsymbol{\varphi}_\zeta\boldsymbol{\nu}_\zeta)\,\boldsymbol{\alpha}_\zeta]\\ &= \int_\xi \jmath_\xi^\pm\,(\imath_{\lambda_\pm^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi)\,\boldsymbol{\alpha}_\xi\end{aligned}\tag{127}$$

via $\boldsymbol{\nu}_\zeta|_{b^c\zeta_\pm} = \mathbf{0}$, (90) and (95), where

$$\lambda_\pm^*\boldsymbol{\varphi}_\zeta := (\lambda_\pm^*\psi_\zeta)[\boldsymbol{w}_\xi + \lambda_\pm^*(\boldsymbol{w}_\lambda - \boldsymbol{v}_\zeta)] + \lambda_\pm^*\boldsymbol{\phi}_\zeta\tag{128}$$

follows from (72) and (126), we have

$$\int_{b^c\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} = \int_\xi [\jmath_\xi^+\,(\imath_{\lambda_+^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi) + \jmath_\xi^-\,(\imath_{\lambda_-^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi)]\,\boldsymbol{\alpha}_\xi\tag{129}$$

for the flux integral along the "top" and "bottom" of the bidimensional flow. Similarly,

$$\begin{aligned}\int_{b^s\zeta} \imath\boldsymbol{\varphi}_\zeta\boldsymbol{\omega} &= \int_{b^s\zeta} (\imath\boldsymbol{\varphi}_\zeta\boldsymbol{\mu}_\zeta)\,\boldsymbol{\tau}_\zeta\wedge\boldsymbol{\nu}_\zeta\\ &= \int_{b\xi}\int_{h_\xi^-}^{h_\xi^+} \lambda^*[(\imath\boldsymbol{\varphi}_\zeta\boldsymbol{\mu}_\zeta)\,\boldsymbol{\tau}_\zeta\wedge\boldsymbol{\nu}_\zeta]\\ &= \int_{b\xi}\int_{h_\xi^-}^{h_\xi^+} [\imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)\,\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}(\lambda^*\boldsymbol{\mu}_\zeta)]\,\boldsymbol{\tau}_\xi\wedge\boldsymbol{\nu}_\xi\,,\end{aligned}\tag{130}$$

using $\boldsymbol{\mu}_\zeta|_{b^s\zeta} = \mathbf{0}$, $\boldsymbol{\mu}_\xi|_{b\xi} = \mathbf{0}$, Fubini's theorem, (85) and (87). Using the representation

$$\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}(\lambda^*\boldsymbol{\mu}_\zeta) = \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi + \imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\mu}_\xi + \imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta)\,\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\tau}_\xi \tag{131}$$

of $\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}(\lambda^*\boldsymbol{\mu}_\zeta)$ on ξ, the relation (95), and the fact that 2-forms vanish on $b\xi$, (130) simplifies to

$$\begin{aligned} \int_{b^s\zeta} \imath_{\boldsymbol{\varphi}_\zeta}\boldsymbol{\omega} &= \int_{b\xi} \imath_{\boldsymbol{\varphi}_{\mathrm{S}}}\boldsymbol{\alpha}_\xi \\ &= \int_\xi (\mathrm{div}_{\boldsymbol{\alpha}}\boldsymbol{\varphi}_{\mathrm{S}})\,\boldsymbol{\alpha}_\xi \ , \end{aligned} \tag{132}$$

via (64), where

$$\boldsymbol{\varphi}_{\mathrm{S}} := \int_{h_\xi^-}^{h_\xi^+} (\jmath_\xi + k_\xi)\,(\lambda^*\boldsymbol{\varphi}_\zeta)\,\boldsymbol{\nu}_\xi \tag{133}$$

is the "normal average" of $\lambda^*\boldsymbol{\varphi}_\zeta$ on ξ, and the weighting

$$\begin{aligned} k_\xi &:= \imath_{\boldsymbol{n}_\xi}(\lambda^*\boldsymbol{\nu}_\zeta)\,\imath_{\boldsymbol{m}_\xi}(\lambda^*\boldsymbol{\tau}_\zeta)\,\imath_{\boldsymbol{t}_\xi}(\lambda^*\boldsymbol{\mu}_\zeta) \\ &= \imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\,\imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) \\ &+ h_\xi\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)[\imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda) + \imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)] \\ &+ h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)[\imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) + \imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)] \\ &+ h_\xi\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)[\imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda) + \imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)] \\ &+ h_\xi^2\,\imath_{\boldsymbol{H}_\xi\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) \\ &+ h_\xi^2\,\imath_{\boldsymbol{H}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda) \\ &+ h_\xi^2\,\imath_{\boldsymbol{H}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda) \\ &+ h_\xi^3\,\imath_{\boldsymbol{I}_\xi\boldsymbol{P}_\xi(D\boldsymbol{n}_\xi)\boldsymbol{n}_\xi}(\boldsymbol{\nu}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{m}_\xi}(\boldsymbol{\tau}_\zeta\circ\lambda)\,\imath_{\boldsymbol{I}_\xi(\nabla\boldsymbol{n}_\xi)\boldsymbol{P}_\xi\boldsymbol{t}_\xi}(\boldsymbol{\mu}_\zeta\circ\lambda) \end{aligned} \tag{134}$$

represents the effect of the dynamic side geometry of the bidimensional flow on the averaged flux density via (86) and (89). Substituting the expression (128) for $\lambda^*\boldsymbol{\varphi}_\zeta$ into (133), $\boldsymbol{\varphi}_{\mathrm{S}}$ takes the form

$$\boldsymbol{\varphi}_{\mathrm{S}} = \boldsymbol{\gamma}_{\mathrm{S}} + \psi_{\mathrm{S}}(\boldsymbol{w}_\xi - \boldsymbol{v}_{\mathrm{S}}) + \boldsymbol{\phi}_{\mathrm{S}} \tag{135}$$

via (122), where

$$\boldsymbol{\gamma}_{\mathrm{S}} := \left\{\int_{h_\xi^-}^{h_\xi^+} k_\xi\,(\lambda^*\psi_\zeta)\,\boldsymbol{\nu}_\xi\right\}\boldsymbol{w}_\xi + \int_{h_\xi^-}^{h_\xi^+} (\jmath_\xi + k_\xi)\,\lambda^*(\psi_\zeta\boldsymbol{w}_\lambda)\,\boldsymbol{\nu}_\xi \tag{136}$$

is a surface flux of the averaged thermodynamic quantity in question due to the bidimensional motion, while the fluxes $\boldsymbol{\phi}_{\mathrm{S}}$ and $\psi_{\mathrm{S}}\boldsymbol{v}_{\mathrm{S}}$ are defined analogous to $\boldsymbol{\varphi}_{\mathrm{S}}$ in (133).

Lastly, the form (121) yields the transport relation

$$\delta\int_\zeta \psi_\zeta\boldsymbol{\omega} = \int_\xi [(\partial\psi_{\mathrm{S}}) + (\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\,\mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\psi_{\mathrm{S}} + \mathrm{div}_{\boldsymbol{\alpha}}(\psi_{\mathrm{S}}\boldsymbol{w}_\xi)]\,\boldsymbol{\alpha}_\xi \tag{137}$$

via (66). Substituting (123) with (129) and (132) into (119), then, we obtain the local form

$$(\partial\psi_{\mathrm{S}}) + (\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\,\mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\psi_{\mathrm{S}} + \mathrm{div}_{\boldsymbol{\alpha}}(\psi_{\mathrm{S}}\boldsymbol{v}_{\mathrm{S}}) = \pi_{\mathrm{S}} + \mathrm{div}_{\boldsymbol{\alpha}}(\boldsymbol{\phi}_{\mathrm{S}} + \boldsymbol{\gamma}_{\mathrm{S}}) + [\![\jmath_\xi(\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi)]\!] + \sigma_{\mathrm{S}} \tag{138}$$

of the general integral balance relation (119) adapted to the flow of the bidimensional continuum, where

$$[\![\jmath_\xi(\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi)]\!] := \jmath_\xi^+(\imath_{\lambda_+^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi) + \jmath_\xi^-(\imath_{\lambda_-^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi) = \jmath_\xi^+(\lambda_+^*\boldsymbol{\varphi}_\zeta)_\perp - \jmath_\xi^-(\lambda_-^*\boldsymbol{\varphi}_\zeta)_\perp \tag{139}$$

is the usual "jump" bracket, and

$$\boldsymbol{\varphi}_{\mathrm{S}} - \psi_{\mathrm{S}}\boldsymbol{w}_\xi = \boldsymbol{\gamma}_{\mathrm{S}} + \boldsymbol{\phi}_{\mathrm{S}} - \psi_{\mathrm{S}}\boldsymbol{v}_{\mathrm{S}} \tag{140}$$

holds via (135). On the basis of (63), only the surface divergences of the *tangential* parts of $\psi_{\mathrm{S}}\boldsymbol{v}_{\mathrm{S}}$, $\boldsymbol{\phi}_{\mathrm{S}}$ and $\boldsymbol{\gamma}_{\mathrm{S}}$ contribute to (138), *e.g.*,

$$\boldsymbol{I}_\xi\boldsymbol{P}_\xi\,\boldsymbol{\phi}_{\mathrm{S}} = (\imath_{\boldsymbol{\phi}_{\mathrm{S}}}\boldsymbol{\tau}_\xi)\,\boldsymbol{t}_\xi + (\imath_{\boldsymbol{\phi}_{\mathrm{S}}}\boldsymbol{\mu}_\xi)\,\boldsymbol{m}_\xi \ . \tag{141}$$

In addition, we have

$$
\begin{aligned}
[\![\jmath_\xi(\imath_{\lambda^*\boldsymbol{\varphi}_\zeta}\boldsymbol{\nu}_\xi)]\!] - (\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\,\mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\psi_{\mathrm{S}} &= \Big\{[\![\jmath_\xi(\lambda^*\psi_\zeta)]\!] - \mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\psi_{\mathrm{S}}\Big\}(\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi) \\
&+ [\![\jmath_\xi\,(\lambda^*\psi_\zeta)\,(\imath_{\lambda^*(\boldsymbol{w}_\lambda-\boldsymbol{v}_\zeta)}\boldsymbol{\nu}_\xi)]\!] \\
&+ [\![\jmath_\xi(\imath_{\lambda^*\boldsymbol{\phi}_\zeta}\boldsymbol{\nu}_\xi)]\!]
\end{aligned}
\tag{142}
$$

from (128).

As an example of (138), consider the simplest case, *i.e.*, mass balance. In this case, we have $\psi_\zeta = \varrho_\zeta$, $\pi_\zeta = 0$, $\boldsymbol{\phi}_\zeta = \mathbf{0}$ and $\sigma_\zeta = 0$. Substituting these choices into (138) yields its reduced form

$$
(\partial\varrho_{\mathrm{S}}) + (\imath_{\boldsymbol{w}_\xi}\boldsymbol{\nu}_\xi)\,\mathrm{tr}_\parallel(\nabla\boldsymbol{n}_\xi)\,\varrho_{\mathrm{S}} + \mathrm{div}_{\boldsymbol{\alpha}}(\varrho_{\mathrm{S}}\boldsymbol{v}_{\mathrm{S}}) = \mathrm{div}_{\boldsymbol{\alpha}}\gamma_{\mathrm{S}} + [\![\jmath_\xi(\lambda^*\varrho_\zeta)(\imath_{\boldsymbol{w}_\xi+\lambda^*(\boldsymbol{w}_\lambda-\boldsymbol{v}_\zeta)}\boldsymbol{\nu}_\xi)]\!] \tag{143}
$$

via (128), where the surface flux

$$
\gamma_{\mathrm{S}} := \left\{\int_{h_\xi^-}^{h_\xi^+} k_\xi\,(\lambda^*\varrho_\zeta)\,\boldsymbol{\nu}_\xi\right\}\boldsymbol{w}_\xi + \int_{h_\xi^-}^{h_\xi^+} (\jmath_\xi + k_\xi)\,\lambda^*(\varrho_\zeta\boldsymbol{w}_\lambda)\,\boldsymbol{\nu}_\xi \tag{144}
$$

is, even in the case of mass balance, in general non-zero, and represents a contribution to mass balance on the surface due to mass diffusion in the system that arises because each distinct part of the boundary of the bidimensional region identified above moves relative to the other parts as well as to ξ. Note that, if we let the thickness of the bidimensional region go to zero, (143) reduces to

$$
[\![\varrho_\xi(\imath_{\boldsymbol{w}_\xi-\boldsymbol{v}_\xi}\boldsymbol{\nu}_\xi)]\!] = 0\,, \tag{145}
$$

representing the classical mass jump balance relation across ξ. Clearly, the other balance relations can be reduced to their classical jump relation counterparts in a similar fashion.

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