

Contact of Multi-Level Roughness with Flat Rigid or Perfectly Plastic Body

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The analysis carried out has shown that the most convenient way of creating the theory of deformation of multi-level roughness is to apply the composite materials methods. The application of a self-consistent method is preferable since it allows defining the effective (average) elastic parameters for all levels. The self-consistent method which has been applied in this investigation consists of the definition of composition of height distributions for all levels. Further, with the help of the composition distribution the elastic coefficients corresponding to the radii of peaks for each level separately have been determined. Finally, the coefficients have been averaged. The pressure applied has been defined as the sum of products of pressures for each level by weight coefficients. The values of the weight coefficient have been defined from self-consistent conditions. This approach allows obtaining the equation of deformation of multi-level roughness which has the same structure as the deformation of the one-level roughness in Demkin-Kragelski theory. The radii of peaks for each level and the reduced elastic modulus are supposed to have scattering. This is a generalisation of Demkin-Kragelski theory. The analytical equations for defining the relative displacement with the help of the average pressure have been obtained.

1 Introduction

The creation of a model of multilevel roughness deformation is one of the most actively developing areas of modern physics (Greenwood J.A. *et al.*, 2001; Persson B.N.J., 2001). Many investigators attract attention to the solution of problems of multilevel models deformation with the help of the finite element method or the creation of alternative theories and equations (Persson B.N.J., 2001).

But insufficient attention has been attracted to the application of the Demkin-Kragelsky theory for the solution of this problem. The theory is based on a more simple hypothesis of the Abbott curve approximation and allows defining the relative level of displacement for one level of roughness in analytical form (Demkin N.B., 1970). There are many experimental investigations confirming this theory.

The worked out analysis has shown that the most convenient way of creating the theory of multi-level roughness deformation is to apply composite material methods. The application of a self-congruent method is preferable since it allows for defining the effective (average) elastic parameters for all levels. The self-congruent method which was applied in this investigation, consists of the definition of the composition distribution of height for all levels. Further, with the help of the composition distribution we have defined the pressure which corresponds to the radii of peaks for each level. The absolute displacement has been assumed to be the same for each level separately. Finally we have defined the applied pressure as the sum of products of pressures for each level by weight coefficients for the same absolute displacement. The values of the weight coefficient were determined from self-congruent conditions (Shermergor T.D., 1977).

This approach allows for obtaining the equation of deformation of the multi-level roughness which has the same structure with deformation of one-level roughness in the Demkin-Kragelski theory. It has been supposed that the radii of peaks for each level and the reduced elastic moduli have scattering. This is a generalisation of the Demkin-Kragelski theory.

2 General Suppositions

There are n separate levels of roughness. The Abbott curve is defined separately for each level of roughness with number k ($1 \leq k \leq n$). A curve with number k ($1 \leq k \leq n$) is defined in frame of references connected with the height of maximum peak $H_{max,k}$ on the level with number k (Figure 1):

$$\eta_{s,k}(\varepsilon_k) = b_k (\varepsilon_k)^{\alpha_k} \quad (1)$$

where ε_k is the relative height of the cross section defined by equation

$$\varepsilon_k = v_k / H_{max,k} \quad (2)$$

v_k is an absolute level of cross-section, $\eta_{s,k}(\varepsilon_k)$ is a relative contact area (Figure 1).

It is assumed that we know the maximum height $H_{max,1..n}$ for the superposition of n levels of roughness. The peaks on different levels are modeled by spherical segments.

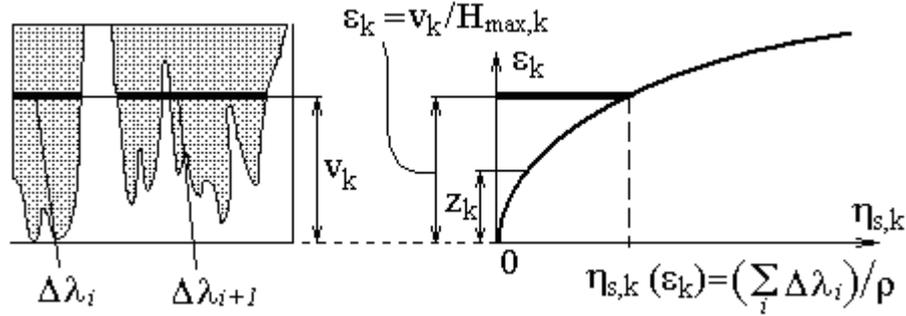


Figure 1. Diagram of a relative profile (Abbott curve)

3 Definition of Real Average Contact Pressure for a Spherical Peak of Roughness

Let us define the peak height H_i by the equation (Figure 2, 3)

$$H_i = H_{max,k} \cdot (1 - z_k), \quad (0 \leq z_k \leq \varepsilon_k). \quad (3)$$

In this case we can define the pressure which is applied in the area of the peak contact with the smooth rigid plate. We know the Hertz formulas for a spherical segment with radius $R_{i,k}$ and height H_i (Demkin N.B., 1970; Johnson K.L., 1985; Ponomarev S.D. *et. al.*, 1958) (Figure 2, 3)

$$\bar{p}_{real,i,j}(\varepsilon_k, z_k) = \frac{4}{3\pi} E_j^* \sqrt{\frac{\delta_k(\varepsilon_k, z_k)}{R_{i,k}}} = \frac{4}{3\pi} E_j^* \sqrt{\frac{H_{max,k} \cdot (1 - z_k) \cdot (\varepsilon_k - z_k)}{R_{i,k}}}, \quad (4)$$

where

$$E_j^* = \frac{E_j}{1 - \nu_j^2},$$

E_j^* is the reduced modulus of elasticity of the peak, ν_j is the Poisson coefficient, E_j is the modulus of elasticity.

4 Definition of Pressure Applied to the Base of a Peak

A peak of roughness is in equilibrium. It means that the value of an integral of pressure which is applied to the contact area is equal to the value of an integral of pressure which is applied to the base of a peak. But the peak base is larger than the contact area. Let us define the pressure applied to the peak base by equation (Demkin N.B., 1970)

$$\gamma(\varepsilon_k - z_k) \cdot \bar{p}_{real,i,j}(\varepsilon_k, z_k), \quad (5)$$

where $\gamma(\varepsilon_k - z_k)$ is a shape function which defines the ratio of real contact area $S_{real,i}(\varepsilon_k, z_k)$ and the area of base $S_{base,i}(z_k)$ for a single peak with height H_i (3) (Figure 2, 3)

$$\gamma(\varepsilon_k - z_k) = \frac{S_{real,i}(\varepsilon_k, z_k)}{S_{base,i}(z_k)} \quad (6)$$

5 Definition of a Shape Function $\gamma(\varepsilon_k - z_k)$ for Spherical Segments

It is necessary to determine the ratio (shape function) of the real contact area and the area of the peak base. It allows for defining the pressure on the peak base with the help of the pressure applied to the real contact area. Obviously, the real contact area is not congruent to the section area in the case of a spherical segment (Figure 2, 3). The definition of a shape function is defined by the geometry of a segment and Hertz solutions for spherical bodies (Demkin N.B., 1970; Kragelski I.V., 1968; Ponomarev S.D. *et.al.*, 1958).

Let ε_k be a relative displacement of the level roughness with number k . It is necessary to define $\gamma(\varepsilon_k - z_k)$ for all peaks which have a contact, i.e. for peaks with heights H_i (3) (Figure 2). We can define the area of a peak base $S_{base,i}(z_k)$ on relative level ε_k (Figure 2) by equation

$$S_{base,i}(z_k) = \pi \cdot R_{i,k}^2 \left(1 - \left(1 - \frac{H_i}{R_{i,k}} \right)^2 \right) \approx 2\pi \cdot R_{i,k} \cdot H_i = 2\pi \cdot R_{i,k} \cdot H_{max,k} \cdot (1 - z_k) \quad (7)$$

The Hertz formulas are valid for any elastic peak with the radius $R_{i,k}$ (Demkin N.B., 1970; Johnson, 1985). The real area of contact is defined by a contact radius $a_{real,i}(\varepsilon_k, z_k)$ (Demkin N.B., 1970)

$$a_{real,i}(\varepsilon_k, z_k) = \sqrt{R_{i,k} \cdot \delta_k(\varepsilon_k, z_k)}, \quad (8)$$

where $\delta_k(\varepsilon_k, z_k)$ is an elastic displacement of a single smooth peak with height $H_{max,k} \cdot (1 - z_k)$ (Demkin N.B., 1970)

$$\delta_k(\varepsilon_k, z_k) = H_{max,k} \cdot (1 - z_k) \cdot (\varepsilon_k - z_k) \quad (9)$$

Therefore taking into account (8), (9) we get

$$S_{real,i}(\varepsilon_k, z_k) = \pi \cdot R_{i,k} \cdot H_{max,k} \cdot (1 - z_k) \cdot (\varepsilon_k - z_k) \quad (10)$$

From (6), (7), (10) it follows that

$$\gamma(\varepsilon_k - z_k) = \frac{S_{real,i}(\varepsilon_k, z_k)}{S_{base,i}(z_k)} = \frac{(\varepsilon_k - z_k)}{2} \quad (11)$$

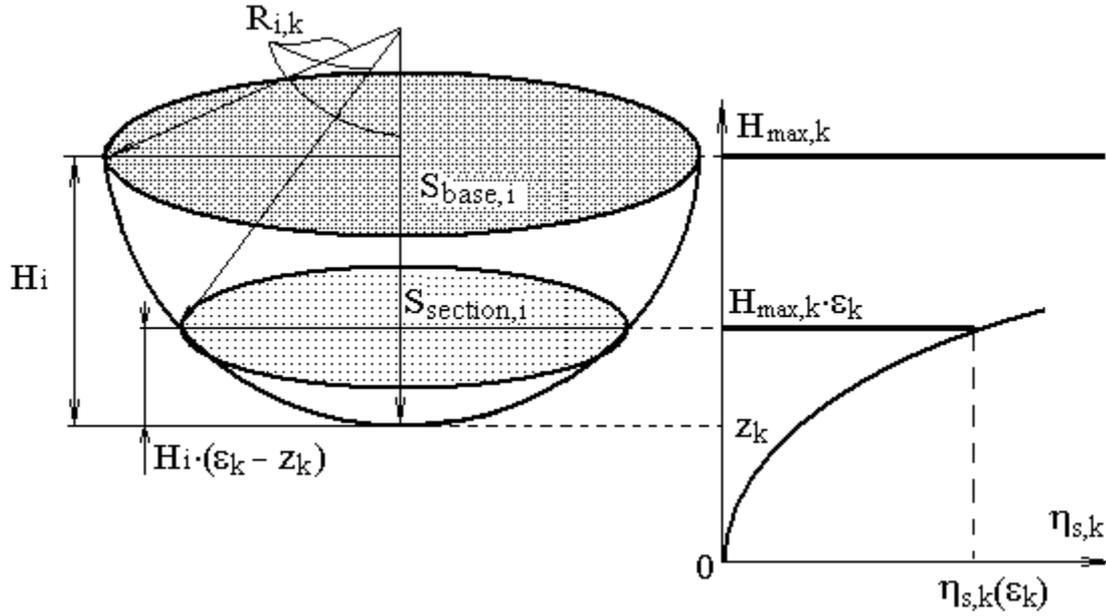


Figure 2. Cross-sections of the peak

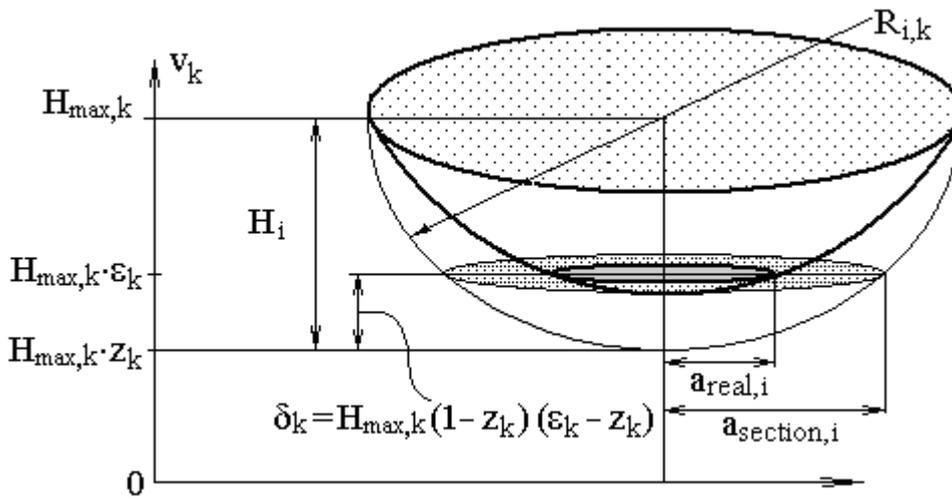


Figure 3. Real area of contact of the peak

6 Effect of Scatterings of Radii and the Reduced Moduli of Elasticity on the Average Pressure

Let us suppose that the rate of radii $R_{i,k}$ of curvature is $\omega_{i,k}^{radius}$ ($\sum_i \omega_{i,k}^{radius} = 1$) for the level with number k .

The rate of elasticity coefficients E_j^* of peaks is $\omega_j^{elastic}$ ($\sum_j \omega_j^{elastic} = 1$). The peak radius and the elasticity coefficient are supposed to be independent variates.

Thus statistically the average pressure on a peak base $\langle \gamma(\epsilon_k - z_k) \cdot \bar{p}_{real,i,j}(\epsilon_k, z_k) \rangle$ is determined by the equation ($k = \overline{1, n}$)

$$\begin{aligned}
\langle \gamma(\varepsilon_k - z_k) \cdot \bar{p}_{real,i,j}(\varepsilon_k, z_k) \rangle &= \gamma(\varepsilon_k - z_k) \cdot \langle \bar{p}_{real,i,j}(\varepsilon_k, z_k) \rangle \\
&= \gamma(\varepsilon_k - z_k) \cdot \sum_i \omega_i^{radius} \cdot \left(\sum_j \omega_j^{elastic} \cdot \bar{p}_{real,i,j,k}(\varepsilon_k, z_k) \right)
\end{aligned} \tag{12}$$

7 Distribution of Spherical Segments

Let n_{z_k} be a number of peaks which have the height more than level z_k and n_{ρ_k} be a full number of peaks on the level of roughness with number k . Thus function $\varphi(z_k) = \frac{n_{z_k}}{n_{\rho_k}}$ defines the relative number of peaks which have contact with a rigid surface. This function depends on the shape of peaks. Taking into account (1) we can obtain that $\varphi(z_k)$ for spherical segments is determined by the equation (Demkin N.B., 1970; Kragelski I.V., 1968)

$$\varphi_k(z_k) = b_k \cdot \chi_k \frac{\eta_{s,k}}{z_k} = b_k \cdot \chi_k \cdot (z_k)^{\chi_k - 1} \tag{13}$$

8 Particular Case of Deformation of One-level Roughness. Generalization of Demkin-Kragelsky Theory

Let ε_k be a relative displacement of a smooth flat rigid surface in the case of its interaction with the roughness of level with number k . The increment of average pressure $d\bar{p}_k$ in the area of bases for the temporary relative level z_k ($0 \leq z_k \leq \varepsilon_k$) is defined by the average pressure applied on a peak base (12) and the increment of relative number of peaks $d\varphi(z_k)$ which have a relative height more than z_k (Figure 3). From (12), (13) we get (Demkin N.B., 1970; Kragelski I.V., 1968)

$$d\bar{p}_k = \gamma(\varepsilon_k - z_k) \cdot \langle \bar{p}_{real,i,j}(\varepsilon_k, z_k) \rangle d\varphi(z_k)$$

Thus the general average pressure on the area of peak bases for each level k is calculated by the equation

$$\begin{aligned}
\bar{p}_k &= \int_0^{\varepsilon_k} \gamma(\varepsilon_k, z_k) \cdot \langle \bar{p}_{real,i,j}(\varepsilon_k, z_k) \rangle \varphi'(z_k) dz_k \\
&= \int_0^{\varepsilon_k} \gamma(\varepsilon_k, z_k) \cdot \left(\sum_i \omega_i^{radius} \cdot \left(\sum_j \omega_j^{elastic} \cdot \bar{p}_{real,i,j}(\varepsilon_k, z_k) \right) \right) \cdot \varphi'(z_k) dz_k \\
&= \frac{b_k \cdot \chi_k \cdot (\chi_k - 1)}{2} K_{1,k} \cdot E_{rough} \cdot \int_0^{\varepsilon_k} (\varepsilon_k - z_k) \cdot \sqrt{(1 - z_k) \cdot (\varepsilon_k - z_k)} \cdot (z_k)^{\chi_k - 2} dz_k
\end{aligned} \tag{14}$$

where

$$E_{rough} = \sum_j \omega_j^{elastic} \cdot E_j^* \tag{15}$$

$$K_{1,k} = \frac{4}{3\pi} \sqrt{H_{max,k}} \left(\sum_i \frac{\omega_i^{radius}}{\sqrt{R_{i,k}}} \right) \tag{16}$$

In the general case of power χ_k the integral in the right side of equation (14) cannot be solved analytically by means of elementary functions. However, equation (1) is valid for the initial part of a relative profile of roughness. Therefore we can use the approach of minimizing the average square error for the integral approximation in (14) by a simple analytical function $K_{2,k} \cdot (\varepsilon_k)^{\chi_k + 1/2}$ on segment $\varepsilon \in [0; 0.2]$

$$\int_0^{0.2} \left(\chi_k - 1 \right) \int_0^{\varepsilon_k} (1 - \rho_k)^{1/2} (\varepsilon_k - \rho_k)^{3/2} (\rho_k)^{\chi_k - 2} d\rho_k - K_2 \cdot (\varepsilon_k)^{\chi_k + 1/2} \Big)^2 d\varepsilon_k \quad .$$

Thus

$$K_{2,k} = \frac{(\chi_k - 1) \int_0^{0.2} \left((\varepsilon_k)^{\chi_k + 1/2} \int_0^{\varepsilon_k} (1 - z_k)^{1/2} (\varepsilon_k - z_k)^{3/2} (z_k)^{\chi_k - 2} dz_k \right) d\varepsilon_k}{\int_0^{0.2} (\varepsilon_k)^{2(\chi_k + 1/2)} d\varepsilon_k} \quad .$$

The approximation segment ($\varepsilon \in [0; 0.2]$) is defined by the hypothesis that the stress of a peak base has no effect on the contact pressure (Ponomarev S.D. et al., 1958).

The coefficient $K_{2,k}$ does not depend on ε_k . It is defined by the parameters of a relative profile. We get

$$\bar{p}_k = \frac{b_k \cdot \chi_k}{2} K_{1,k} \cdot K_{2,k} \cdot E_{rough} \cdot (\varepsilon_k)^{\chi_k + 1/2} \quad . \quad (17)$$

Thus we can define relative displacements by the average pressure on the base length

$$\varepsilon_k = (C_{0,k} \cdot \bar{p}_k)^{2/(2\chi_k + 1)} \quad (18)$$

where

$$C_{0,k} = \frac{2}{b_k \cdot \chi_k \cdot K_{1,k} \cdot K_{2,k} \cdot E_{rough}} \quad .$$

Obviously, equations (17) and (18) generalise the Demkin-Kragelski theory to the case of natural scattering of reduced elastic moduli (15) and curvature radii of peaks. Equations (16) – (18) show that the Demkin-Kragelski theory is valid in the case of small scattering of the reduced elastic moduli (15) and the curvature radii of the peaks.

9 Two-level Roughness. Composition of Two Height Distribution

Let us consider a two-level roughness. It is assumed that v is the height of a section which refers to the peak with general maximal height $H_{max,1..2}$. In accordance with the well known approach (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968), we assume that (Figure 4)

$$v = v_1 + v_2 \quad , \quad (19)$$

where v_1 is a height of a section of the first level roughness, v_2 is a height of a section of the second level roughness. In addition $v_k \in [0, v]$ and $v < H_{max,k}, k = 1..2$ (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968).

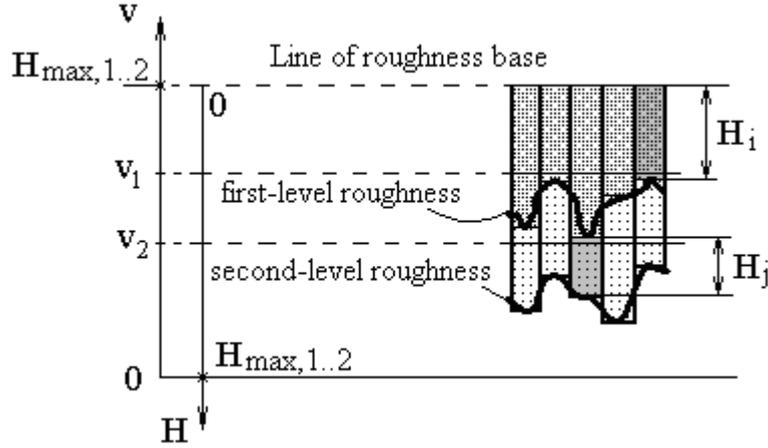


Figure 4. Diagram of sections of two level roughness

Let us consider a set of heights $\{H_{k,i}\}_{i=0}^{N-1}$ of level of roughness with number k . The system of inequalities is assumed to be valid

$$H_{1,i} \geq H_{\max,1} - v_1, \quad (20)$$

$$H_{2,j} \geq H_{\max,2} - v_2$$

Hence the inequality is valid for a general height $H_{1,i} + H_{2,j}$ of two surface deviations of different levels (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968) (19), (20)

$$H_{1,i} + H_{2,j} \geq (H_{\max,1} + H_{\max,2}) - v \geq H_{\max,1..2} - v$$

It means that the compound surface deviation with height $H_{1,i} + H_{2,j}$ participates in setting up a real contact area. (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968).

Let us consider the first level of roughness ($k=1$). It is known (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968) that the probability of a case when the height $H_{i,1}$ with number i is higher than $H_{\max,1} - v_1$ is defined by the equation: $P(H_{i,1} \geq H_{\max,1} - v_1) = \eta_{s,1}$. The same equation can be obtained for the height with number j of the second level of roughness: $P(H_{j,2} \geq H_{\max,2} - v_2) = \eta_{s,2}$. The increment of a relative area of deviations with heights corresponding to dv_k is defined by $d\eta_{s,k}$. Taking into account equation $\varepsilon_k = v_k / H_{\max,k}$ ($k = \overline{1,2}$) and (1) we obtain the equation for defining the relative area of composition section of two distributions of heights (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968)

$$\eta_{s,1..2} = \int_0^v \left(\int_0^{v-v_1} d\eta_{s,2} \left(\frac{v_2}{H_{\max,2}} \right) \right) d\eta_{s,1} \left(\frac{v_1}{H_{\max,1}} \right) = \frac{1}{H_{\max,1}} \int_0^v \eta_{s,2} \left(\frac{v-v_1}{H_{\max,2}} \right) \cdot \eta_{s,1}' \left(\frac{v_1}{H_{\max,1}} \right) dv_1 \quad (21)$$

The following equation is valid

$$v = H_{\max,1..2} \cdot \varepsilon \quad (22)$$

Hence we can obtain the following equation from (21) and (22) (Demkin N.B., 1970; Kragelski I.V., 1968):

$$\eta_{s,1..2} = \frac{K_{3,1..2} \cdot (H_{\max,1..2})^{\chi_1 + \chi_2}}{(H_{\max,1})^{\chi_1} \cdot (H_{\max,2})^{\chi_2}} b_1 b_2 (\varepsilon)^{\chi_1 + \chi_2} \quad (23)$$

where

$$K_{3,1..2} = \chi_1 \int_0^1 (1-\tau)^{\chi_2} \cdot (\tau)^{\chi_1-1} d\tau$$

It is to be emphasized that equation (21) and the familiar result of investigation of interaction of two surfaces with different one-level roughnesses (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968) are of the same structure.

10 Application of the Self-consistent Method for Defining the Average Elasticity of Two-level Roughness

The self-consistent method is widely applied in the theory of composite materials (Shermergor T.D., 1977). The average parameters of inhomogeneous bodies are calculated with the help of this method. In the case of contacting of two-level roughness and a rigid surface, we have two deformable compounds with different characteristics of Abbott curves and radii of peak curvatures.

We have the joint distribution of heights of the two-level roughness (23). The general displacement of the two-level roughness is defined by equation (22). Taking into account (23) we have to solve two separate problems under the supposition that one level of roughness is not deformable. After that we can define the average elastic parameter for the two-level roughness with the help of a self-consistent condition. This condition assures that the pressure is constant for both levels.

Let the peaks of the second level be rigid. Hence the displacement v is reached by deforming the first level roughness i.e. $v_1 = v$ (2)

$$\varepsilon = \frac{H_{\max,1}}{H_{\max,1..2}} \varepsilon_1, \quad (24)$$

Taking into account (23), (24) we obtain

$$\eta_{s,1..2} = \left(\frac{H_{\max,1}}{H_{\max,2}} \right)^{\chi_2} K_{3,1..2} \cdot b_1 \cdot b_2 \cdot (\varepsilon_1)^{\chi_1 + \chi_2} \quad (25)$$

Let us define the pressure which is necessary for deforming the system of one deformable and one rigid roughness up to the relative displacement $\varepsilon_1 = \frac{v}{H_{\max,1}}$ (24). Taking into account (13), (25) the distribution function of spherical segments for the joint distribution of heights can be defined by the equation

$$\varphi_1(z_1) = \left(\frac{H_{\max,1}}{H_{\max,2}} \right)^{\chi_2} K_{3,1..2} \cdot b_1 \cdot b_2 \cdot (\chi_1 + \chi_2) \cdot (z_1)^{\chi_1 + \chi_2 - 1} \quad (26)$$

Further we define pressure \bar{p}_1 which is necessary for deforming the system with the second rigid level of roughness up to the absolute displacement $v_1 = v$ (section 8 of the paper)

$$\bar{p}_1 = \frac{b_1 \cdot b_2 \cdot (\chi_1 + \chi_2)}{2 \cdot (H_{\max,1})^{\chi_1 + 1/2} \cdot (H_{\max,2})^{\chi_2}} K_{1,1} \cdot K_{2,1} \cdot E_{rough} \cdot K_{3,1..2} \cdot (v)^{\chi_1 + \chi_2 + 1/2} \quad (27)$$

The second type of problem is connected with the inverse hypothesis about absolute rigidity of the first level and elasticity of the second level. In the same manner we define pressure \bar{p}_2 which is necessary for deforming the new system up to the absolute displacement $v_2 = v$

$$\bar{p}_2 = \frac{b_1 \cdot b_2 \cdot (\chi_1 + \chi_2)}{2 \cdot (H_{\max,1})^{\chi_1} \cdot (H_{\max,2})^{\chi_2 + 1/2}} K_{1,2} \cdot K_{2,2} \cdot E_{rough} \cdot K_{3,1..2} \cdot (v)^{\chi_1 + \chi_2 + 1/2} \quad (28)$$

However, both levels are deformed simultaneously. Therefore we assume that in this case the pressure \bar{p} which is necessary for deforming the two-level roughness up to absolute displacement v is defined by the equation

$$\bar{p} = (1 - \alpha) \cdot \bar{p}_1 + \alpha \cdot \bar{p}_2 = \frac{1}{C_{0,1..2}} (\varepsilon)^{\chi_1 + \chi_2 + 1/2} \quad (29)$$

where

$$C_{0,1..2} = \frac{(H_{\max,1})^{\chi_1 + 1/2} \cdot (H_{\max,2})^{\chi_2 + 1/2}}{(1 - \alpha) \cdot K_{1,1} \cdot K_{2,1} \cdot (H_{\max,2})^{1/2} + \alpha \cdot K_{1,2} \cdot K_{2,2} \cdot (H_{\max,1})^{1/2}} \times \\ \times \frac{2}{b_1 \cdot b_2 \cdot (\chi_1 + \chi_2) \cdot E_{rough} \cdot K_{3,1..2} \cdot (H_{\max,1..2})^{\chi_1 + \chi_2 + 1/2}} .$$

The coefficient α averages the coefficients in (27) and (28). This coefficient redistributes the deformation proportionally between roughness levels in accordance with the elastic properties of levels and it is defined by the self-consistent condition, i.e. the pressure is constant for each level

$$(1 - \alpha) \cdot p_1 - \alpha \cdot p_2 = 0 \quad (30)$$

Directly from (27), (28), (30) we obtain

$$\alpha = \frac{(H_{\max,2})^{1/2} \cdot K_{1,1} \cdot K_{2,1}}{(H_{\max,2})^{1/2} \cdot K_{1,1} \cdot K_{2,1} + (H_{\max,1})^{1/2} \cdot K_{1,2} \cdot K_{2,2}} .$$

Taking into account (29), a function for defining the relative displacement ε of two-level roughness by the value of the pressure \bar{p} is defined by the equation

$$\varepsilon = (C_{0,1..2} \cdot \bar{p})^{1/(\chi_1 + \chi_2 + 1/2)} \quad (31)$$

It has to be noted that the structures of (18) and (31) are similar.

11 Multi-level Roughness. Composition of n Height Distribution

Let us consider a roughness which consists of n levels. Let us assume the following approach stated in section 9 that displacement v can be defined as

$$v = v_1 + v_2 + \dots + v_n , \quad (32)$$

where v_k is the height of a section of roughness level with number k . In addition, $v_k \in [0, v]$ and $v < H_{\max,k}, k = 1..n$. It is assumed that the system of inequalities is valid for some set of heights $\{H_{k,i}\}_{k=1}^n$ of peaks from different levels

$$H_{k,i} \geq H_{\max,k} - v_k , \quad k = 1..n \quad (33)$$

Hence

$$\sum_{k=1}^n H_{k,i} \geq \sum_{k=1}^n H_{\max,k} - v \geq H_{\max,1..n} - v$$

It means that the height $\sum_{k=1}^n H_{k,i}$ of the compound surface deviation participates in setting up a real contact area of the composition of the height distribution (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968).

The probability of a case when the height $H_{k,i}$ with number i is higher than $H_{\max,k} - v_k$ is defined by the equation $P(H_{k,i} \geq H_{\max,k} - v_k) = \eta_{s,k}$. The increment of the relative area of deviations with heights

corresponding to dv_k is defined by $d\eta_{s,k}$. Taking into account the equation $\varepsilon_k = v_k / H_{\max,k}$ and (1) we obtain the equation for defining the relative area of section of composition of n height distributions (Demkin N.B., 1970; Kragelski I.V., 1948; Kragelski I.V., 1968)

$$\begin{aligned} \eta_{s,1..n} &= \int_0^v \left(\int_0^{v-v_1} \left(\int_0^{v-v_1-v_2} \dots \int_0^{v-v_1-\dots-v_{n-1}} d\eta_{s,n} \left(\frac{v_n}{H_{\max,n}} \right) \dots d\eta_{s,3} \left(\frac{v_3}{H_{\max,3}} \right) \right) d\eta_{s,2} \left(\frac{v_2}{H_{\max,2}} \right) \right) d\eta_{s,1} \left(\frac{v_1}{H_{\max,1}} \right) \\ &= \frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n (H_{\max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot (H_{\max,1..n})^{\mu_{1..n}} \cdot (\varepsilon)^{\mu_{1..n}} \end{aligned} \quad (34)$$

where

$$\mu_{1..n} = \left(\sum_{k=1}^n \chi_k \right)$$

$$K_{3,1..n} = \int_0^1 \left(\int_0^{1-\tau_1} \left(\int_0^{1-\tau_1-\tau_2} \dots \int_0^{1-\tau_1-\dots-\tau_{n-1}} d(\tau_n)^{\chi_n} \dots d(\tau_3)^{\chi_3} \right) d(\tau_2)^{\chi_2} \right) d\eta_{s,1}(\tau_1)^{\chi_1}.$$

12 Penetration of Rigid Multi-level Roughness into a Perfectly Plastic Body with a Flat Contact Surface

It is assumed that the roughness is not deformed when penetrating into a perfectly plastic body. We can solve the problem using (34) and assuming that the average pressure for each penetrated peak is constant and equal to $3 \cdot \sigma_s$, where σ_s is a yield stress of a plastic body (or thick coating). Hence (34)

$$\bar{p} = 3 \cdot \sigma_s \cdot \eta_{1..n} = 3 \cdot \sigma_s \cdot \frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n (H_{\max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot (H_{\max,1..n})^{\mu_{1..n}} \cdot (\varepsilon)^{\mu_{1..n}} \quad (35)$$

The results of the numerical analysis show that the relative area of contact and the relative penetration of a rough rigid surface into a plastic body with a flat surface significantly depend on the parameters of the Abbott curve of each level (1) (Figure 5).

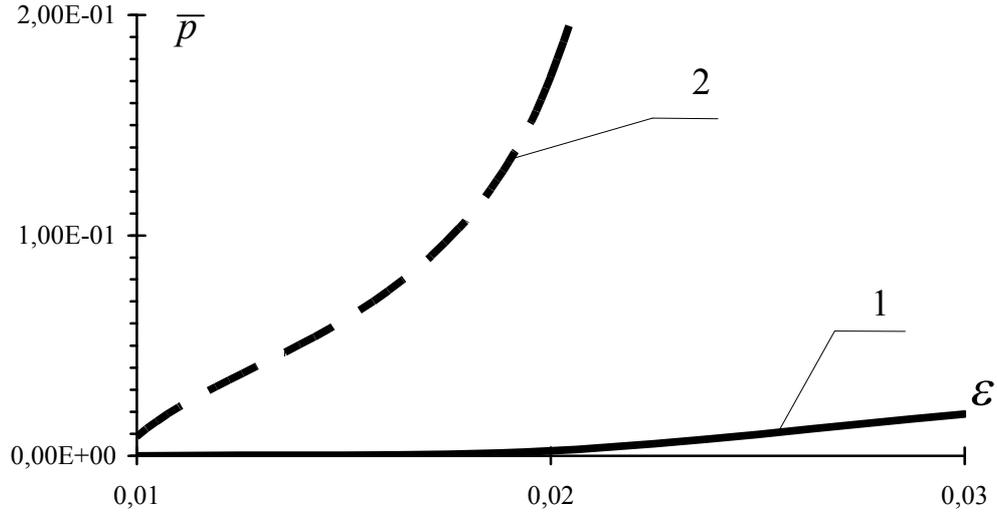


Figure 5. The relative penetration of multi-level rigid roughness into a perfectly plastic body. The number of levels n is equal to 3. The pressures are calculated with the help of (35)

($\sigma_s = 7 \cdot 10^8 \text{ N/m}^2$, $H_{\max,1..3} = 3 \cdot 10^{-6} \text{ m}$, $H_{\max,1} = 1.1 \cdot 10^{-6} \text{ m}$, $H_{\max,2} = 1.5 \cdot 10^{-6} \text{ m}$, $H_{\max,3} = 0.7 \cdot 10^{-6} \text{ m}$):

	b_1	b_2	b_3	b_4	b_5	b_6
For curve 1	3	2.5	1.7	2	2.1	1.1
For curve 2	2	1.7	1.5	1.5	1.7	1.1

13 Deformation of Multi-level Roughness. Application of Self-consistent Method for Defining the Average Elasticity

It is supposed that the general displacement v for the joint deformation of the n -level roughness is found by the equation: $v = H_{\max,1..2} \cdot \varepsilon$.

Taking into account (34) we have to solve a set of separate problems under the supposition that only one level of roughness with number k is deformable and all other levels are rigid. After that we can define the average elastic parameter for n -level roughness with the help of self-consistent condition (Shermergor T.D., 1977). This condition assures that the pressure is constant for all levels. Taking into account this supposition the general displacement v of the multilevel roughness is equal to the displacement v_k of a level with number k , i.e.

$\varepsilon = \frac{H_{\max,k}}{H_{\max,1..n}} \varepsilon_k$. Hence we can obtain

$$\eta_{s,1..n} = \frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n (H_{\max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot (H_{\max,k})^{\mu_{1..n}} \cdot (\varepsilon_k)^{\mu_{1..n}}, k=1..n. \quad (36)$$

Taking into account (13), (36) the distribution of spherical segments for the level with number k is defined by the equation

$$\varphi_k(z_k) = \frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n (H_{\max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot (H_{\max,k})^{\mu_{1..n}} \cdot \mu_{1..n} \cdot (z_1)^{\mu_{1..n}-1}, k=1..n \quad (37)$$

Further, we obtain the pressure \bar{p}_k which is used for deforming a system with one deformable level with number k . In this case we have a displacement $v = H_{\max,1..n} \cdot \varepsilon$

$$\bar{p}_k = \frac{K_{1,k} \cdot K_{2,k}}{(H_{\max,k})^{1/2}} \cdot \frac{E_{rough} \cdot \prod_{k=1}^n b_k}{2 \cdot \prod_{k=1}^n (H_{\max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot \mu_{1..n} \cdot (H_{\max,1..n})^{\mu_{1..n}+1/2} \cdot (\varepsilon)^{\mu_{1..n}+1/2}, \quad k=1..n \quad (38)$$

Let all levels of roughness be deformable. In accordance with the above approach (section 10 of the paper) it is assumed that the pressure which is applied to the multi-level roughness can be defined by the equation

$$\bar{p} = \sum_{k=1}^n \alpha_k \cdot \bar{p}_k = \frac{1}{C_{0,1..n}} (\varepsilon)^{\mu_{1..n}+1/2} \quad (39)$$

where

$$C_{0,1..n} = \frac{2}{E_{rough}} \cdot \frac{\prod_{k=1}^n (H_{\max,k})^{\chi_k}}{K_{3,1..n} \cdot (H_{\max,1..n})^{\mu_{1..n}+1/2} \cdot \mu_{1..n} \cdot \prod_{k=1}^n b_k} \cdot \left(\sum_{k=1}^n \frac{\alpha_k \cdot K_{1,k} \cdot K_{2,k}}{(H_{\max,k})^{1/2}} \right)^{-1}$$

The coefficients α_k ($k=1..n$) average the elastic properties of all levels of roughness. These coefficients are determined by the system of equations

$$\sum_{k=1}^n \alpha_k = 1 \quad (40)$$

$$\alpha_k \cdot \bar{p}_k - \alpha_{k+1} \cdot \bar{p}_{k+1} = 0, \quad k=1..n-1 \quad (41)$$

Taking into account (41) we obtain

$$\alpha_k = \frac{K_{1,k} \cdot K_{2,k}}{(H_{\max,k})^{1/2}} \cdot \frac{(H_{\max,n})^{1/2}}{K_{1,n} \cdot K_{2,n}} \alpha_n, \quad k=1..n-1.$$

Hence (40)

$$\alpha_n = \left(1 + \frac{(H_{\max,n})^{1/2}}{K_{1,n} \cdot K_{2,n}} \cdot \sum_{k=1}^{n-1} \frac{K_{1,k} \cdot K_{2,k}}{(H_{\max,k})^{1/2}} \right)^{-1}$$

Taking into account (39) we get

$$\varepsilon = (C_{0,1..n} \cdot \bar{p})^{1/(\mu_{1..n}+1/2)} \quad (42)$$

The relative displacement of a multi-level rough surface for interaction with rigid flat surface significantly depends on parameters of the Abbott curve of each level of roughness (1) (Figure 6).

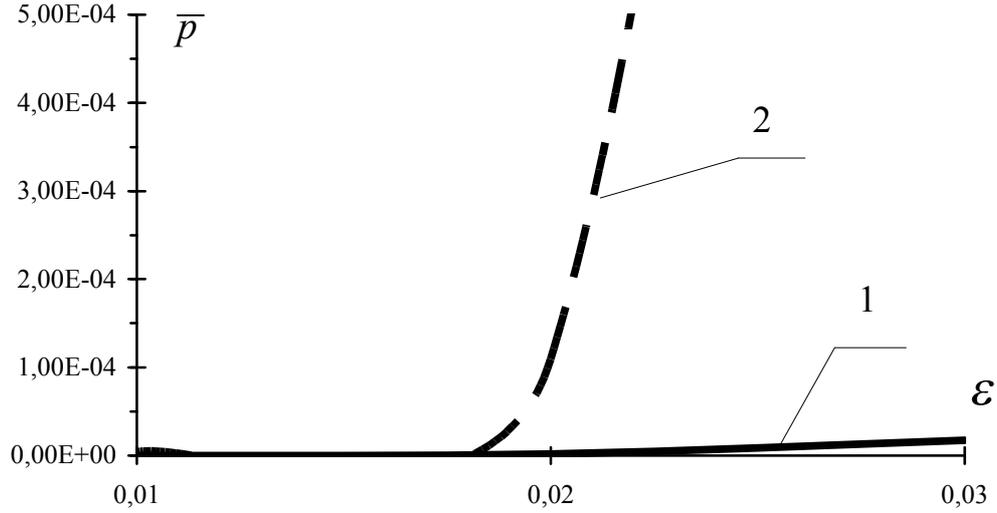


Figure 6. The dependence of the nominal contact pressure on the relative displacement (elastic roughness) (42) ($H_{max,1,3} = 3 \cdot 10^{-6} m$, $H_{max,1} = 1.1 \cdot 10^{-6} m$, $H_{max,1} = 1 \cdot 10^{-6} m$, $H_{max,2} = 1.5 \cdot 10^{-6} m$, $H_{max,3} = 0.7 \cdot 10^{-6} m$; the array of rates and the radii of curvature: $\left\{ \left(\omega_i^{radius}; R_i \right) \right\}_{i=1}^6 = \left\{ \left(0.3; 3 \cdot 10^{-4} m \right), \left(0.2; 2 \cdot 10^{-4} m \right), \left(0.1; 1 \cdot 10^{-4} m \right), \left(0.2; 4 \cdot 10^{-4} m \right), \left(0.1; 7 \cdot 10^{-4} m \right), \left(0.1; 6 \cdot 10^{-4} m \right) \right\}$; the array of rates and the reduced moduli of elasticity $\left\{ \left(\omega_j^{elastic}; E_j^* \right) \right\}_{j=1}^7 = \left\{ \left(0.1; 1.8 \cdot 10^{11} N/m^2 \right), \left(0.15; 1.9 \cdot 10^{11} N/m^2 \right), \left(0.05; 2.0 \cdot 10^{11} N/m^2 \right), \left(0.3; 1.5 \cdot 10^{11} N/m^2 \right), \left(0.15; 1.6 \cdot 10^{11} N/m^2 \right), \left(0.2; 1.7 \cdot 10^{11} N/m^2 \right), \left(0.05; 1.4 \cdot 10^{11} N/m^2 \right) \right\}$):

	b ₁	b ₂	b ₃	b ₄	b ₅	b ₆
For curve 1	3	2.5	1.7	2	2.1	1.1
For curve 2	2	1.7	1.5	1.5	1.7	1.1

14 Interaction of a Multi-level Elastic Roughness and a Perfectly Plastic Thick Coating

It is assumed that the coating is sufficiently thick. In this case the influence of the undercoat deformation is small. Let us consider an elastic roughness which consists of n sublevels. Let the level with the smallest step of deviations has number n . It is assumed that the peak of this level does not penetrate into the coating when the inequality $\bar{p}_{real,i,j}(\varepsilon_n, z_n) \leq 3 \cdot \sigma_s$ is valid for the average pressure $\bar{p}_{real,i,j}(\varepsilon_n, z_n)$ of the peak (here σ_s is the yield stress of coating). It means that the multi-level roughness does not penetrate into the coating either (Kravchuk A.S. *et al.*, 2005). Taking into account scattering of reduced elastic moduli and radii of peaks (4), one can define the criterion of guaranteed non-penetration of a peak into the coating by means of following inequality

$$\max_{i,j} \left\{ \bar{p}_{real,i,j}(\varepsilon_n, 0) \right\} \leq 3 \cdot \sigma_s, \quad (43)$$

Let us estimate the relative deformation $\varepsilon_{n,elastic}$ of the highest peak of the level with number n taking into account (4), (43)

$$\frac{4}{3\pi} \max_j \left\{ E_j^* \right\} \sqrt{\frac{\delta_n(\varepsilon_{n,elastic}, 0)}{\min_i \{ R_{i,n} \}}} = 3 \cdot \sigma_s.$$

Hence we can obtain

$$\varepsilon_{n,elastic} = \frac{\min_i \{ R_{i,n} \}}{H_{max,n}} \cdot \left(\frac{9}{4} \pi \cdot \frac{\sigma_s}{\max_j \{ E_j^* \}} \right)^2.$$

When the relative displacement ε_n exceeds $\varepsilon_{n,elastic}$, then (Kravchuk A.S. *et al.*, 2005) we can define a relative number of peaks $\gamma(\varepsilon_n)$ of level n which have plastic penetration into the coating in the whole number of peaks which contact with the coating (36)

$$\gamma(\varepsilon_n) = \frac{\eta_{s,1..n}(\varepsilon_n - \varepsilon_{n,elastic})}{(\varepsilon_n - \varepsilon_{n,elastic})} \cdot \frac{\varepsilon_n}{\eta_{s,1..n}(\varepsilon_n)} = \left(1 - \frac{\varepsilon_{n,elastic}}{\varepsilon_n}\right)^{\mu_{1..n}-1}, \quad (\varepsilon_n \geq \varepsilon_{n,elastic}).$$

The numerical experiments show that $\gamma(\varepsilon_n)$ significantly depends on the average characteristics of the multi-level roughness and characteristics of sublevel with number n .

The minimal pressure $\bar{p}_{elastic}$ for deformation $\varepsilon_{n,elastic}$ can be estimated by means of (38) and under the assumption that the other levels of roughness are rigid

$$\bar{p}_{elastic} = K_{1,n} \cdot K_{2,n} \cdot \frac{E_{rough} \cdot \prod_{k=1}^n b_k}{2 \cdot \prod_{k=1}^n (H_{max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot \mu_{1..n} \cdot (H_{max,n})^{\mu_{1..n}} \cdot (\varepsilon_{n,elastic})^{\mu_{1..n}+1/2}. \quad (44)$$

If the multi-level elastic roughness has no plastic penetration into the coating, the general elastic relative displacement $\varepsilon_{elastic}$ (in the case of elasticity of all levels) is defined by (42) and (44)

$$\varepsilon_{elastic} = \left(C_{0,1..n} \cdot K_{1,n} \cdot K_{2,n} \cdot \frac{E_{rough} \cdot \prod_{k=1}^n b_k}{2 \cdot \prod_{k=1}^n (H_{max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot \mu_{1..n} (H_{max,n})^{\mu_{1..n}} \right)^{1/(\mu_{1..n}+1/2)} \cdot \varepsilon_{n,elastic}$$

Hence after reducing we obtain

$$\varepsilon_{elastic} = \left(K_{1,n} \cdot K_{2,n} \cdot \frac{(H_{max,n})^{\mu_{1..n}}}{(H_{max,1..n})^{\mu_{1..n}+1/2}} \cdot \left(\sum_{k=1}^n \frac{\alpha_k \cdot K_{1,k} \cdot K_{2,k}}{(H_{max,k})^{1/2}} \right)^{-1} \right)^{1/(\mu_{1..n}+1/2)} \cdot \varepsilon_{n,elastic}$$

The general relative displacement of elastic multi-level roughness in its contact interaction with a plastic coating can be defined by equations (35) (42) (Kravchuk A.S. *et al.*, 2005)

$$\varepsilon = \begin{cases} (C_{0,1..n} \cdot p)^{1/(\mu_{1..n}+1/2)}, & p \in [0, p_{elastic}), \\ \varepsilon_{elastic} + \left[\frac{1}{3 \cdot \sigma_s \cdot b_{1..n}} (p^{1/(\mu_{1..n}+1/2)} - (p_{elastic})^{1/(\mu_{1..n}+1/2)})^{\mu_{1..n}+1/2} \right]^{1/\mu_{1..n}}, & p \in [p_{elastic}, +\infty). \end{cases}$$

where

$$b_{1..n} = \frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n (H_{max,k})^{\chi_k}} \cdot K_{3,1..n} \cdot (H_{max,1..n})^{\mu_{1..n}}.$$

The application of $\varepsilon_{elastic}$ and the solutions of nonlocal problems (Kravchuk A.S., 2005; Kravchuk A.S. *et al.*, 2004) allow for theoretically defining a yield stress σ_s of a coating when roughness does not scabble it. The macro parameters of contacted bodies, the characteristics of a surface, and the value of a force are the predetermined data in this solution.

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16 Conclusions

The most convenient way of creating the theory of deformation of multi-level roughness is to apply the methods of mechanics of composite materials. The application of a self-consistent method is preferable since it allows for defining the effective (average) elastic parameters for all levels. By means of this approach we can obtain the equation of deformation of multi-level roughness which has the same structure as the equation of deformation of one-level roughness in the Demkin-Kragelski theory. It has been supposed that the radii of peaks for each level and reduced elastic modulus have scattering. This is the generalisation of Demkin-Kragelski theory. The analytical equations for defining the relative displacement with the help of the average pressure have been obtained. The solution of the problem is based on the approximation of the initial part of the Abbott curve for any level of roughness by means of power functions.

The analytical equation for defining the relative displacement of multi-level roughness by means of an average pressure has been obtained. It significantly depends on parameters of the Abbott curve of each level of roughness.

References

- Demkin N.B.: *Engagement of Rough Surface [In Russian]*. Nauka, Moscow (1970).
- Greenwood J.A., Wu J.J.: Surface Roughness and Contact: An Apology. *Meccanica*, 36, (2001), 617–630.
- Johnson K.L.: *Contact Mechanics*. Cambridge University Press, Cambridge (1985).
- Kragelski I.V. Friction of Two Rough Surfaces [In Russian]. *Proceedings of the Academy of Sciences of USSR, Branch of Technical Science*, 10, (1948), 1621 - 1625.
- Kragelski I.V. *Friction and Wear [In Russian]*. Mashinostroenie, Moskow (1968).
- Kravchuk A., Buzio R., Valbusa U., Rymuza Z.: Initial Penetration of an Elastic Axially Symmetric Indenter into a Rigid-Perfectly-Plastic Half Space. *Technische Mechanik*, 25, 2, (2005), 81-89.
- Kravchuk A.S., Chigarev A.V., Buzio R., Valbusa U.: Penetration of Elastic Roughness into Thick Plastic Coating. *Journal of Friction and Wear*, 25, 5, (2005), 455-459.
- Kravchuk A.S., Kravchuk A.I., Remuza Z.: Contact Displacements of Spatial Contact of Rough Body and Body with Coating. *Technische Mechanik*, 24, 2, (2004), 116-124.
- Kravchuk A.S.: Nonlocal Contact of Rough Bodies with Elliptical Contact Area. *Mechanics of Solids*, 3, (2005), 42-52.
- Persson B.N.J.: Elastoplastic Contact between Randomly Rough Surfaces. *Physical Review Letters*, 87, 11 (2001), 116101.

Ponomarev S.D., Biderman V.L., Likharev K.K., Makushyn V.M., Malinin N.N., Fedosiev V.I.: Strength Calculus in Engineering. Vol. 2: *Some Problems of Applied Theory of Elasticity [In Russian]*. Mashgiz, Moscow (1958).

Shermergor T.D.: *Theory of Elasticity of Micro-Inhomogeneous Bodies [In Russian]*. Nauka, Moscow (1977).

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