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Asymptotic Analysis of Strongly Anisotropic Solids

I.V. Andrianov, G.A. Starushenko

An asymptotic approach is proposed to describe the strongly anisotropic solids. We simplified the input boundary value problem using ratios of the elastic constants as small parameters. The obtained results can be used for the investigation of the composite materials.

1 Introduction

Problems involving material anisotropy are in general more difficult to solve than the isotropic ones. The problem of an anisotropic composite material can be examined on the assumption that the matrix is only slightly anisotropic (Lekhnitskii, 1963, 1968; Sachenkov and Daragan, 1972). On the other hand, as it has been shown by Kosmodamianskii for an anisotropic plane with two identical elliptic holes (Kosmodamianskii, 1966, 1976), strong anisotropy leads to the possibility of sufficient simplifications of the governing boundary value problems. Independently Manevitch, Pavlenko and Shamrovskii (1970, 1971) and Everstine and Pipkin (1971, 1973) (see also Spencer, 1974; Sanchez and Pipkin, 1978; Pipkin, 1979, 1984; Christensen, 1979), beginning with elasticity theory and treating the extensibility of the material in a preferred direction as a small parameter, used the singular perturbation method to obtain approximate boundary value problems. The governing plane problem is reduced to that of solving two Laplace equations and, if higher-order approximations are wanted, a number of Poisson's equations. The comparison of approximate solutions with the exact anisotropic elastic solutions showed satisfactory agreement (Kosmodamianskii, 1976; Manevitch, Pavlenko and Shamrovskii, 1970, 1971; Everstine and Pipkin, 1971, 1973; Spencer, 1974; Pipkin, 1979, 1984; Sanchez and Pipkin, 1978; Christensen, 1979). In the paper by Everstine and Pipkin (1971) a cantilever beam with end load was analysed. Spencer (1974) studied the problem of a crack parallel to the fibres in shear, as well as a crack normal to the fibre direction opened by internal pressure. Bogan (1981, 1994) considered the stress distribution within the elliptic region, and the contact problem for the half-plane. Sanchez and Pipkin (1978) showed how to compute the elastic stress factor at a crack tip from the force in the singular fiber passing throw the tip. The efficiency of the asymptotic approach for solving of plane anisotropic contact problems was shown in the papers (Manevitch and Pavlenko, 1975, 1982; Manevitch, Pavlenko and Koblik, 1979; Manevitch, 2001; Pavlenko, 1981; Andrianov, Awrejcewicz and Manevitch, 2004). A generalization for the axially symmetric case with a cylindrically orthotropic matrix was proposed in the papers by Pavlenko (1980), Manevitch and Pavlenko (1991). The papers by Kagadii and Pavlenko (1989, 1992) are devoted to the fibre reinforced composites with a viscoelastic orthotropic matrix.

In all above mentioned papers, approximated governing equations and boundary conditions are used. In our paper we propose to simplify only boundary conditions.

2 The Elliott Ansatz

Elliott (1948) (see also Lekhnitskii, 1963) proposed to search the solution of axially the symmetric static problem for transversally-isotropic media as a sum of two functions, which functions are solutions of the following equations:

$$\nabla_i^2 \varphi_i = 0\,, \quad i = 1, 2, \tag{1}$$

where:

$$\nabla_i^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\,, \quad s_i^2 = \frac{a + (-1)^{i+1}\left(a^{-2} - 2b\right)^{1/2}}{b}\,, \quad i = 1, 2,$$

$$a=\left[\varepsilon^{-2}-2\nu_1(1+\nu)\right]l,\ \ b=2\left(1-\nu^2\right)ll_1^{-1},\ \ \varepsilon^2=G_1/E_1\ ,\ \ l=1/\left(1-\nu_1^2 l_1\right),\ \ l_1=E/E_1\ ,$$

E, E_1 are the Young's moduli in r and z directions, G, G_1 are the shear coefficients, $G=0.5E/(1-\nu^2)$, ν, ν_1 are the Poisson's coefficients.

For displacements and stresses one has (Lekhnitskii, 1963)

$$U_r=\frac{\partial}{\partial r}(\varphi_1+\varphi_2),\ \ U_z=\frac{\partial}{\partial z}(k_1\varphi_1+k_2\varphi_2), \tag{2}$$

$$\sigma_r=-\frac{k_3}{r}U_r-G_1\frac{\partial^2}{\partial z^2}\left[(1+k_1)\varphi_1+(1+k_2)\varphi_2\right], \tag{3}$$

$$\sigma_\theta=\frac{k_3}{r}U_r+\frac{\partial^2}{\partial z^2}(k_4\varphi_1+k_5\varphi_2),\ \ \sigma_z=\frac{\partial^2}{\partial z^2}(k_4\varphi_1+k_5\varphi_2), \tag{4}$$

$$\tau_{rz}=G_1\frac{\partial^2}{\partial r\partial z}\left[(1+k_1)\varphi_1+(1+k_2)\varphi_2\right], \tag{5}$$

where:

$$k_i=\frac{G_1a_1s_i^{-2}-1}{G_1a_2+1},\ \ i=1,2,$$

$$k_3=\frac{(1-\nu)E}{(1+\nu)\left(1-\nu-2\nu_1^2l_1^2\right)},\ \ k_4=k_1a_3-a_2s_1^{-2},\ \ k_5=k_2a_3-a_2s_2^{-2},$$

$$a_1=\frac{E\left(1-\nu_1^2l_1\right)}{(1+\nu)\left(1-\nu-2\nu_1^2l_1\right)},\ \ a_2=\frac{\nu_1E_1}{(1-\nu)l_1^{-1}-2\nu_1^2},\ \ a_3=\frac{E_1^2(1-\nu)}{E\left[(1-\nu)l_1^{-1}-2\nu_1^2\right]}.$$

3 Asymptotic Simplification

As small parameter we will use the parameter ε. If ε << 1, the following asymptotic estimations are valid:

$$s_1\sim\varepsilon^{-1},\ \ s_2\sim\varepsilon,\ \ k_1\sim\varepsilon^4,\ \ k_2\sim\varepsilon^{-2}. \tag{6}$$

So, in the first approximation one can suppose

$$\sigma_z\approx k_5\frac{\partial^2\varphi_2}{\partial z^2}. \tag{7}$$

Let us suppose the boundary value problem for a half-space with the following boundary conditions:

$$\sigma_z=\varphi(r),\ \ \tau_{rz}=0\ \ \text{for}\ \ z=0. \tag{8}$$

Then one can solve in the first approximation the following boundary value problems

$$\nabla_2^2\varphi_2=0, \tag{9}$$

$$k_5\frac{\partial^2\varphi_2}{\partial z^2}=\varphi(r)\ \ \text{for}\ \ z=0, \tag{10}$$

and

$$\nabla_1^2\varphi_1=0, \tag{11}$$

$$\frac{\partial \varphi_1}{\partial z} = -\frac{1+k_2}{1+k_1}\frac{\partial \varphi_2}{\partial z} \quad \text{for} \quad z = 0\,. \tag{12}$$

For example let us solve the Boussinesq problem for a transversally-isotropic half-space. In this case one has

$$\delta(r) = 0.5P\delta(r)/(\pi\, r),$$

where: $\delta(r)$ is the Dirac delta-function.

The exact solution of this problem can be written as follows (Lekhnitskii, 1963):

$$\sigma_z = \frac{P}{\pi\sqrt{2b}}\frac{z}{s_1 - s_2}\left[\frac{1}{\left(r^2 + s_1^2 z^2\right)^{3/2}} - \frac{1}{\left(r^2 + s_2^2 z^2\right)^{3/2}}\right], \tag{13}$$

$$\tau_{rz} = \frac{P}{\pi\sqrt{2b}}\frac{r}{s_1 - s_2}\left[\frac{1}{\left(r^2 + s_1^2 z^2\right)^{3/2}} - \frac{1}{\left(r^2 + s_2^2 z^2\right)^{3/2}}\right]. \tag{14}$$

Now we will test the approximate solutions based on the simplified boundary value problems (9), (10) and (11), (12). First of all we solve the equation

$$\nabla_2^2 \varphi_2^{(1)} = 0 \tag{15}$$

with the boundary condition

$$\frac{\partial^2 \varphi_2^{(1)}}{\partial z^2} = \frac{A\delta(r)}{r} \quad \text{for} \quad z = 0\,, \tag{16}$$

where: A is the unknown constant, which will be obtained further.

From (15), (16) one obtains

$$\sigma_z^{(2)} = -\frac{Az}{\left(r^2 + s_2^2 z^2\right)^{3/2}}\,. \tag{17}$$

Then we solve the boundary value problem

$$\nabla_1^2 \varphi_1^{(1)} = 0\,, \tag{18}$$

$$\frac{\partial \varphi_1^{(1)}}{\partial z} = -\frac{1+k_2}{1+k_1}\frac{\partial \varphi_2^{(1)}}{\partial z} \quad \text{for} \quad z = 0\,. \tag{19}$$

From (18), (19) one obtains

$$\sigma_z^{(1)} = -\frac{Az}{\left(r^2 + s_1^2 z^2\right)^{3/2}}\,. \tag{20}$$

So,

$$\sigma_z = Az\left[\frac{1}{\left(r^2 + s_1^2 z^2\right)^{3/2}} - \frac{1}{\left(r^2 + s_2^2 z^2\right)^{3/2}}\right], \tag{21}$$

$$\tau_{rz} = Ar\left[\frac{1}{\left(r^2 + s_1^2 z^2\right)^{3/2}} - \frac{1}{\left(r^2 + s_2^2 z^2\right)^{3/2}}\right]. \tag{22}$$

If one finds the constant A from condition

$$\frac{\partial^2}{\partial z^2}\left(k_4\varphi_1 + k_5\varphi_2\right) = \frac{P\delta(r)}{2\pi r} \quad \text{for} \quad z = 0\,,$$

then the expressions (21), (22) coincides with the exact solutions (13), (14).

4 Conclusions

The obtained results can be used for the investigation of composite fracture. The proposed solution can be applied in Civil Engineering to model the behavior of piles or piers embedded in soil media which exhibit a linear elastic response in the working-load range. Analytic solutions presented in this paper can be useful in evaluating results calculated by boundary elements and finite element methods.

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References

Andrianov, I.V.; Awrejcewicz, J.; Manevitch, L.I.: *Asymptotical mechanics of thin-walled structures: a handbook.* Springer-Verlag, Berlin (2004).

Bogan, Yu.A.: Distribution of stresses in elastic strongly anisotropic materials. *J. Appl. Mech. Tech. Phys.,* 35(3), (1994), 476-480.

Bogan, Yu.A.: Second boundary value problem of the theory of elasticity for a significantly anisotropic plate with an elliptic hole. *Soviet Appl. Mech.,* 17(9), (1981), 821-825.

Christensen, R.M.: *Mechanics of Composite Materials.* Wiley Interscience, New York (1979).

Elliott, H.A.: Three-dimensional stress distributions in hexagonal aelotropic crystals. *Proc. Cambridge Philosoph. Soc.,* 44(4), (1948), 522-533.

Everstine, G.C.; Pipkin, A.C.: Boundary layers in fiber-reinforced materials. *J. Appl. Mech.,* 40, (1973), 518-522.

Everstine, G.C.; Pipkin, A.C.: Stress channeling in transversely isotropic elastic composites. *Zeitschrift für Angewandte Mathematik und Physik,* 22(5), (1971), 825-834.

Kagadii, T.S.; Mossakovskaya, L.V.; Pavlenko, A.V.: The perturbation method in a spatial problem of the linear viscoelasticity of anisotropic bodies. *J. Appl. Math. Mech.,* 56 (1), (1992), 147-151.

Kagadii, T.S.; Pavlenko, A.V.: The axisymmetric problem of transferring a load to a viscoelastic orthotropic body by means of an elastic rod. *J. Appl. Math. Mech.,* 53(5), (1989), 619-622.

Kosmodamianskii, A.S.: Stress state of a plate with strongly anisotropy weakened by two elliptical openings. *Soviet Appl. Mech.,* 11(1), (1966), 59-65.

Kosmodamianskii, A.S.: *The stressed state of anisotropic medias with holes or cavities.* Visha Schkola, Kiev–Donezk (1976).

Lekhnitskii, S.G.: *Anisotropic plates.* Gordon and Breach, New York (1968).

Lekhnitskii, S.G.: *Theory of elasticity of an anisotropic elastic bodies.* Golden-Day Inc., San Francisco (1963).

Manevitch, L.I.: *On one asymptotic method in contact problems for anisotropic solids.* In: Vorovich, I.I; Alexandrov, V.M., editors. Mechanics of contact interactions. Fizmatlit, Moscow (2001), 55-72.

Manevitch, L.I.; Pavlenko, A.V.: *Asymptotic methods in micromechanics of composite materials.* Naukova Dumka, Kiev (1991).

Manevitch, L.I.; Pavlenko, A.V.: Taking into account the structural inhomogeneity of a composite material in estimating adhesive strength. *J. Appl. Mech. Tech. Phys.,* 23(3), (1982), 434-439.

Manevitch, L.I.; Pavlenko, A.V.: Transmission of a dynamic load by a longitudinal stiffener on an elastic orthotropic plate. *Mech. Solids,* 10(2), (1975), 100-106.

Manevitch, L.I.; Pavlenko, A.V.; Koblik, S.G.: *Asymptotic methods in the theory of elasticity of orthotropic body.* Visha Schkola, Kiev–Donetzk (1979).

Manevitch, L.I.; Pavlenko, A.V.; Shamrovskii, A.D.: *Application of the group theory methods to the dynamical problems for orthotropic plates.* In: Proceedings of the 7th all-union conference on plates and shells (Dnepropetrovsk, 1969). Nauka, Moscow (1970), 408-412.

Manevitch, L.I.; Pavlenko, A.V.; Shamrovskii, A.D.: The approximate solution of the theory of elasticity contact problem for an orthotropic strip reinforced by ribs. *Hydroaeromechanics and Theory of Elasticity,* 13, (1971), 102-112.

Pavlenko, A.V.: Application of asymptotic method to three-dimensional problem of elasticity for composite materials. *Mech. Solids,* 15(3), (1980), 47-57.

Pavlenko, A.V.: Transfer of a load from a rod to an elastic anisotropic half-plane. *Mech. Solids,* 16(6), (1981), 88-95.

Pipkin, A.C.: Stress analysis for fibre-reinforced materials. *Advances in Appl. Mech.,* 19, (1979), 1-51.

Pipkin, A.C.: *Stress channeling and boundary layers in strongly anisotropic solids.* In: Spencer AJM, editor. Continuum theory of the mechanics of fibre-reinforced composites. Springer Verlag, Wien, New York (1984), 123-145.

Sachenkov, A.V.; Daragan, V.I.: *Method of small parameter in plane problem of the theory of elasticity of anisotropic bodies.* In: Investigation in the theory of plates and shells YIII. Kazan State University, Kazan (1972), 77-95.

Sanchez, V.; Pipkin, A.C.: Crack-tip analysis for elastic materials reinforced with strong fibers. *Quarterly J. Mech. Appl. Math.,* 31, (1978), 349-362.

Spencer, A.J.M.: Boundary layers in highly anisotropic plane elasticity. *Int. J. Solids and Structures,* 10(10), (1974), 1103-1123.

Addresses: Prof. Dr. Sc. Igor V. Andrianov (corresponding author), Institute of General Mechanics RWTH Aachen, Templergraben 64, D-52056, Aachen
email: igor_andrianov@hotmail.com

Dr. Galina A. Starushenko, Dnipropetrovsk Regional Institute of State Management of National Academy of State Management at the President of Ukraine, 29 Gogolya Str., UA-49631, Dnipropetrovsk, Ukraine
email: gala_star@km.ru