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Consideration of Reaction Forces of Holonomic Constraints as Generalized Coordinates in Approximate Determination of Lower Frequencies of Elastic Systems

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A new method for determination of lower frequencies of mechanical systems consisting of elastic bodies connected to each other is offered. The conditions of connection of bodies are written as holonomic constraints, the reactions of which are considered as generalized coordinates. Therefore the number of degrees of freedom proves to be equal to the number of constraints.

1 On the Possibility of Introducing Generalized Reaction Forces as Lagrangean Coordinates

The paper presents a development of the method suggested in the work (Yushkov and Zegzhda, 1998). The equation of frequencies obtained in this work makes it possible, if necessary, to determine any number of the system's natural frequencies for a reasonably great number N of dynamically considered vibration modes of the system elements. However as a rule it is necessary to know only several first frequencies and modes. When calculating them one can use the following approximate approach to this problem.

It has been proposed in the work (Yushkov and Zegzhda, 1998) to consider the conditions of elastic bodies connection to each other as holonomic constraints. In this case the generalized reaction forces $\Lambda_i,\ i = \overline{1,n}$, turn out to be the forces of interaction between the elastic bodies.

The potential energy of the system consisting of elastic bodies connected to each other can be represented as a positively defined quadratic form of the generalized constraint reactions introduced

$$\Pi = \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\Lambda_i\Lambda_j\,, \tag{1}$$

when considering all the natural vibration modes of the system's elements quasi-statically. The example of calculating the factors c_{ij} of this form is given in the work (Yushkov and Zegzhda, 1998). In quasistatics the deformed state of all system elements is uniquely determined by setting the quantities $\Lambda_i,\ i = \overline{1,n}$. The elastic system given comes to this state as a result of the fact that its points have obtained displacements, which can be found as linear functions of the reactions $\Lambda_i,\ i = \overline{1,n}$. Hence the position of all points of the system at the time t is uniquely determined by setting the quantities $\Lambda_i,\ i = \overline{1,n}$. Therefore they can be considered as the generalized Lagrange coordinates; and the kinetic energy of the system can be represented in the form

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\dot\Lambda_i\dot\Lambda_j\,. \tag{2}$$

Here $a_{ij},\ i,j = \overline{1,n}$, are some constants, the calculation procedure for which will be shown below through a number of examples.

Lagrange's equations of the second kind corresponding to expressions (1) and (2) are as follows

$$\sum_{j=1}^{n}(a_{ij}\ddot\Lambda_j + c_{ij}\Lambda_j) = 0\,, \quad i = \overline{1,n}\,.$$

By assuming as in the work (Yushkov and Zegzhda, 1998)

$$\Lambda_i = \widetilde{\Lambda}_i \cos(p\,t + \alpha)\,, \quad i = \overline{1, n}\,,$$

we come to the following equation of frequencies

$$\det[c_{ij} - p^2 a_{ij}] = 0\,. \tag{3}$$

When calculating the factors a_{ij} and c_{ij} of this determinant, one needs not to know the natural frequencies and natural modes of vibration of the system's elements. It is essential that these factors can be determined rather simply for the bars of variable section too.

Let us start analyzing this approach with solving the problem of approximately determining the first natural frequency and the mode of bending oscillations of the cantilever of variable cross-section.

2 Bending Vibration of the Cantilever of Variable Cross-Section

Let us assume that at the end $x = l$ the bar is rigidly clamped and that the area of cross-section and the moment of inertia of this section are defined correspondingly as follows

$$S(x) = A(\xi)S(l)\,, \quad J(x) = B(\xi)J(l)\,, \quad \xi = \frac{x}{l}\,, \quad 0 \leqslant \xi \leqslant 1\,. \tag{4}$$

Here $A(\xi)$ and $B(\xi)$ are some prescribed functions. Note that they may be step functions too.

Let us introduce into the consideration the deflection of the neutral layer of the cantilever $y(x,t)$. As the bar is rigidly clamped at the end $x = l$, then

$$y(l,t) = 0\,, \quad \left.\frac{\partial y}{\partial x}\right|_{x=l} = 0\,. \tag{5}$$

We shall consider these two conditions as holonomic constraints imposed on the motion of a free bar. The constraint reaction forces are the bending moment $M = \Lambda_1$ and the lateral force $Q = \Lambda_2$, applied to the end $x = l$ of the free bar (see Figure 1).

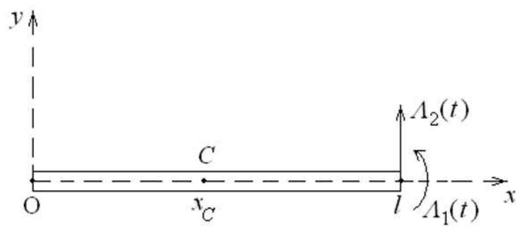

Figure 1. Constraint reaction forces applied to the cantilever.

The motion of the free bar under the action of these forces can be represented as, first, translational motion (motion of the center of mass C), secondly, rotation about the center of mass and, third, bending. This bending deformation in quasistatics can be found in the following manner.

The acceleration of the center of mass W_c and the angular acceleration $\ddot{\varphi}$ at the time t are as follows

$$W_c = \frac{\Lambda_2(t)}{\rho \int_0^l S(x)dx}\,, \qquad \ddot{\varphi} = \frac{\Lambda_1(t) + (l - x_c)\Lambda_2(t)}{\rho \int_0^l S(x)(x_c - x)^2 dx}\,. \tag{6}$$

Here ρ is the density, and x_c is the coordinate of the center of mass.

The intensity of inertial forces caused by translational and rotational motion of the bar appears as

$$q(x,t) = -\rho(W_c + \ddot{\varphi}(x - x_c))S(x)\,. \tag{7}$$

The bending moment in section x, corresponding to the load $q(x,t)$, is equal to

$$M(x,t) = \int_0^x q(x_1,t)(x-x_1)dx_1 \,. \tag{8}$$

The deflection caused by the action of the bending moment $M(x,t)$ satisfies the equation

$$EJ(x)\frac{\partial^2 y}{\partial x^2} = M(x,t)\,.$$

This equation in dimensionless variables

$$\bar{y} = \frac{y}{l}\,, \quad \xi = \frac{x}{l}\,, \quad L(\xi,t) = \frac{M(x,t)l}{EJ(l)} \tag{9}$$

takes the form

$$B(\xi)\frac{\partial^2 \bar{y}}{\partial \xi^2} = L(\xi,t)\,. \tag{10}$$

Formulas (4), (6)–(9) imply that the dimensionless moment $L(\xi,t)$ is equal to

$$L(\xi,t) = \bar{\Lambda}_1(t)f_1(\xi) + \bar{\Lambda}_2(t)f_2(\xi)\,. \tag{11}$$

Here

$$\bar{\Lambda}_1(t) = \frac{\Lambda_1(t)l}{EJ(l)}\,, \quad \bar{\Lambda}_2(t) = \frac{\Lambda_2(t)l^2}{EJ(l)}\,,$$

$$f_1(\xi) = \int_0^\xi \frac{A(\eta)(c-\eta)}{a}\,(\xi-\eta)\,d\eta\,,$$

$$f_2(\xi) = \int_0^\xi \left(\frac{(c-\eta)A(\eta)(1-c)}{a} - \frac{A(\eta)}{b}\right)(\xi-\eta)\,d\eta\,, \tag{12}$$

$$a = \int_0^1 A(\xi)(c-\xi)^2\,d\xi\,, \quad b = \int_0^1 A(\xi)\,d\xi\,, \quad c = \frac{1}{b}\int_0^1 \xi A(\xi)\,d\xi\,.$$

Integrating (10) and taking into account the constraint equation (5) produce

$$\bar{y}(\xi,t) = \sum_{k=1}^{2}\bar{\Lambda}_k(t)h_k(\xi)\,, \quad h_k = \int_\xi^1 \frac{f_k(\eta)(\eta-\xi)}{B(\eta)}\,d\eta\,. \tag{13}$$

The potential energy of the bar

$$\Pi = \frac{1}{2}\int_0^l \frac{M^2(x,t)}{EJ(x)}\,dx$$

can be represented by using formulas (4), (9), (11) as

$$\Pi = \frac{EJ(l)}{2l}\sum_{i,j=1}^{2}\bar{c}_{ij}\bar{\Lambda}_i\bar{\Lambda}_j\,, \tag{14}$$

where

$$\bar{c}_{ij} = \int_0^1 \frac{f_i(\xi)f_j(\xi)}{B(\xi)}\,d\xi\,.$$

The kinetic energy of the system

$$T = \frac{\rho}{2}\int_0^l S(x)\left(\frac{\partial y}{\partial t}\right)^2 dx\,,$$

as follows from formulas (4), (9), (13), is

$$T = \frac{1}{2}\rho S(l)\,l^3\sum_{i,j=1}^{2}\bar{a}_{ij}\dot{\bar{\Lambda}}_i\dot{\bar{\Lambda}}_j\,, \qquad \bar{a}_{ij} = \int_0^1 A(\xi)h_i(\xi)h_j(\xi)\,d\xi\,. \tag{15}$$

Equation (3) and expressions (14), (15) imply that the dimensionless frequencies p_* related to the required frequencies p as

$$p = p_* \frac{1}{l^2}\sqrt{\frac{EJ(l)}{\rho S(l)}} \tag{16}$$

are the roots of the equation

$$\det\left[\bar{c}_{ij} - p_*^2 \bar{a}_{ij}\right] = 0\,, \qquad i,j = 1,2\,. \tag{17}$$

For vibrations with the frequencies $p_k,\ k = 1,2$, in accordance with expression (13) we obtain

$$\bar{y}_k(\xi,t) = (\tilde{\tilde{\Lambda}}_{k1} h_1(\xi) + \tilde{\tilde{\Lambda}}_{k2} h_2(\xi))\cos(p_k t + \alpha)\,, \quad k = 1,2\,.$$

The quantities $\tilde{\tilde{\Lambda}}_{k1},\ \tilde{\tilde{\Lambda}}_{k2},\ k = 1,2$, satisfy the equation

$$(\bar{c}_{21} - p_{*k}^2 \bar{a}_{21})\tilde{\tilde{\Lambda}}_{k1} + (\bar{c}_{22} - p_{*k}^2 \bar{a}_{22})\tilde{\tilde{\Lambda}}_{k2} = 0\,, \quad k = 1,2\,.$$

This yields that the fist two vibration modes of cantilever can be approximately represented as

$$Y_k(\xi) = \frac{X_k(\xi)}{X_k(0.5)}\,, \quad X_k(\xi) = h_1(\xi) - \frac{\bar{c}_{12} - p_{*k}^2 \bar{a}_{12}}{\bar{c}_{22} - p_{*k}^2 \bar{a}_{22}} h_2(\xi)\,, \quad k = 1,2\,.$$

The exact solutions for the cantilever of wedge and cone shapes have been obtained by Kirchhoff in 1879. These solutions are given in many books, in particular, in the reference book (Kamke, 1959) (Chapter IV, paragraphs 4.22, 4.24).

For the wedge, where

$$A(\xi) = \xi\,, \quad B(\xi) = \xi^3\,,$$

the natural frequencies p_* are the roots of the equation

$$J_1(\kappa)I_0(\kappa) = I_1(\kappa)J_0(\kappa)\,, \quad \kappa = 2\sqrt{p_*}\,.$$

Here $J_0(\kappa)$ and $J_1(\kappa)$ are the Bessel functions of the first kind, and $I_0(\kappa)$ and $I_1(\kappa)$ are the modified Bessel functions of the first kind. The natural modes corresponding to the natural frequencies p_* are

$$Y(\xi) = \frac{X(\xi)}{X(0.5)}\,, \quad X(\xi) = \frac{J_0(\kappa)I_1(\kappa\sqrt{\xi}) - I_0(\kappa)J_1(\kappa\sqrt{\xi})}{\sqrt{\xi}}\,.$$

In the case of a cone, where

$$A(\xi) = \xi^2\,, \quad B(\xi) = \xi^4\,,$$

the equation of frequencies and functions $X(\xi)$ take the form

$$\kappa(J_0(\kappa)I_1(\kappa) + I_0(\kappa)J_1(\kappa)) = 4J_1(\kappa)I_1(\kappa)\,,$$

$$X(\xi) = \frac{I_1(\kappa)[J_1(\kappa\sqrt{\xi}) - \frac{\kappa\sqrt{\xi}}{2}J_0(\kappa\sqrt{\xi})]}{\xi\sqrt{\xi}} + \frac{J_1(\kappa)[I_1(\kappa\sqrt{\xi}) - \frac{\kappa\sqrt{\xi}}{2}I_0(\kappa\sqrt{\xi})]}{\xi\sqrt{\xi}}\,.$$

By using the approximate approach suggested we obtain
— for the wedge

$$p_{*1} = 5.3187\,, \qquad p_{*2} = 17.3006\,,$$

$$h_1(\xi) = 1 - \frac{5\xi}{2} + 2\xi^2 - \frac{\xi^3}{2}\,, \quad h_2(\xi) = \frac{1}{6} - \frac{\xi}{2} + \frac{\xi^2}{2} - \frac{\xi^3}{6}\,,$$

— for the cone

$$p_{*1} = 8.73521\,, \qquad p_{*2} = 25.1813\,,$$

$$h_1(\xi) = \frac{7}{6} - 3\xi + \frac{5\xi^2}{2} - \frac{2\xi^3}{3}\,, \quad h_2(\xi) = \frac{1}{6} - \frac{\xi}{2} + \frac{\xi^2}{2} - \frac{\xi^3}{6}\,.$$

The exact values of the first two frequencies are as follows
— for the wedge: $p_{*1} = 5.3151$, $\quad p_{*2} = 15.2072$,
— for the cone: $p_{*1} = 8.71926$, $\quad p_{*2} = 21.1457$.

The second frequency error for the wedge as well as for the cone is great enough. Therefore this approximation method can be used only for the determination of the first frequency and the first vibration mode for the cantilever of variable cross-section.

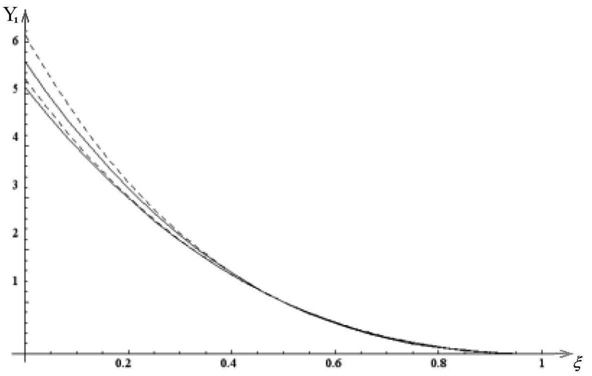

Figure 2. First natural modes.

The first natural modes for the wedge and the cone are shown in Figure 2. The solid curves correspond to the approximate solution, and the dashed curves over them correspond to the exact solution. For visualization of differences between the curves depicted, the deflection at $\xi = 1/2$ is taken as a unit of measurement for each of them. The cone is a more flexible bar than the wedge and thus the first natural mode of the cone for $\xi < 1/2$ is located higher than the corresponding curve for the wedge.

For the cone the mass per unit of length decreases while approaching to the end by a quadratic law, and for the wedge the linear law is applied. For the bar of constant cross-section the mass per unit of length is constant. Pay attention to the following fact. The first frequency error for the cone is equal to 0.2%, for the wedge it is equal to 0.07%. For the bar of constant cross-section we have
— approximately: $p_{*1} = 3.516035$, $\quad p_{*2} = 22.7125$,
— exactly: $p_{*1} = 3.516015$, $\quad p_{*2} = 22.0345$.

Thus, the first frequency error makes up only $5.7 \cdot 10^{-4}\%$. Upon comparison of the given above errors for the cone, the wedge and the bar of constant cross-section, we can expect that for the bar of constant cross-section with the mass localized at the end the approximate solution will become practically the exact one. In fact, in this case we obtain

$$A(\xi) = 1 + \gamma\delta(\xi)\,, \quad B(\xi) = 1\,, \quad \gamma = \frac{m_2}{m_1}\,.$$

Here $\delta(\xi)$ is the Dirac delta-function, m_2 is the load mass, m_1 is the bar mass.

For $\gamma = 1$ we have
— approximately: $p_{*1} = 1.5572990$, $\quad p_{*2} = 16.6203$,
— exactly: $p_{*1} = 1.5572976$, $\quad p_{*2} = 16.2501$.

We see that the first frequency error decreased six times relative to the case when $\gamma = 0$.

For the cantilever with the disk at its end we obtain the solution quite accurately if we consider the presence of the disk at the end as the third and the forth holonomic constraints. This system with four degrees of freedom will makes it possible to determine to a rather high accuracy not only the first frequency but the second and the third ones. So let's analyze the following problem.

3 Determination of the Lower Natural Frequencies of Bending Vibrations of the Cantilever of Variable Cross-Section with a Disk at its End

In the rotor dynamics, the urgent problem is accurate determination of first two critical speeds of the cantilever shaft with a disk at its end. We remind that the values of these critical speeds are proportional to natural frequencies of the cantilever with a disk. Actually, as for instance in the case of marine screw (water propeller) or airscrew, there is not a disk at the shaft end but a body of rather complicated shape. There are methods allowing us to determine the moment of inertia of this body relative to the axis that is perpendicular to the shaft axis. Let us assume that this moment is set in the form

$$I = m_2 R^2,$$

where $m_2 = \gamma \rho l S(l)$ is mass of the body, and $R = rl$ is its radius of inertia. Note that with given functions $A(\xi)$ and $B(\xi)$ the required natural frequencies p_* will depend on two parameters γ and r.

In the case of the bar of constant cross-section the exact values of the frequencies p_* are found from the equation

$$\det \begin{bmatrix} V(x) + \gamma x U(x) & S(x) + \gamma x V(x) \\ S(x) - \gamma r^2 x^3 T(x) & T(x) - \gamma r^2 x^3 U(x) \end{bmatrix} = 0 , \quad x = \sqrt{p_*} . \tag{18}$$

Here

$$S(x) = \frac{1}{2}(\cosh x + \cos x) , \quad T(x) = \frac{1}{2}(\sinh x + \sin x) ,$$
$$U(x) = \frac{1}{2}(\cosh x - \cos x) , \quad V(x) = \frac{1}{2}(\sinh x - \sin x)$$

are the Krylov functions.

In an approximate determination of the frequencies p_* we shall consider the conditions of rigid fixing (5) as two holonomic constraints as before. We shall denote now their reaction forces: the bending moment $M(t)$ and lateral force $Q(t)$, by $\Lambda_1(t)$ and $\Lambda_2(t)$ as before.

The condition that the deflection $y(0,t)$ is equal to the displacement of mass m_2, and the angle of rotation of the bar's end

$$\varphi = \left. \frac{\partial y}{\partial x} \right|_{x=0}$$

is equal to the angle of the body rotation will be considered as two holonomic constraints imposed on motion of the free bar. The reaction forces of these constraints are the lateral force $\Lambda_3(t)$ and bending moment $\Lambda_4(t)$. They are applied to the bar at the section $x = 0$. Positive directions of reactions applied to the bar are shown in Figure 3.

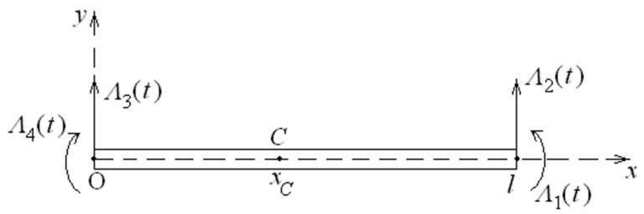

Figure 3. Constraint reaction forces applied to the bar.

Formulas (6) in this case will take the form

$$W_c = \frac{\Lambda_2(t) + \Lambda_3(t)}{\rho \int_0^l S(x) dx} , \qquad \ddot{\varphi} = \frac{\Lambda_1(t) - \Lambda_4(t) + (l - x_c)\Lambda_2(t) - x_c \Lambda_3(t)}{\rho \int_0^l S(x)(x_c - x)^2 dx} .$$

The intensity of inertial forces $q(x,t)$ will be calculated by formula (7) as before; formula (8) will take the form

$$M(x,t) = \Lambda_4(t) + x\Lambda_3(t) + \int_0^x q(x_1,t)(x - x_1)\, dx_1 .$$

When going to dimensionless variables we obtain

$$L(\xi,t)=\sum_{k=1}^{4}\bar{\Lambda}_k(t)f_k(\xi)\,.$$

Here

$$\bar{\Lambda}_1(t)=\frac{\Lambda_1(t)l}{EJ(l)}\,,\quad \bar{\Lambda}_2(t)=\frac{\Lambda_2(t)l^2}{EJ(l)}\,,$$

$$\bar{\Lambda}_3(t)=\frac{\Lambda_3(t)l^2}{EJ(l)}\,,\quad \bar{\Lambda}_4(t)=\frac{\Lambda_4(t)l}{EJ(l)}\,.$$

The functions $f_1(\xi)$ and $f_2(\xi)$ are set by formulas (12), and the functions $f_3(\xi)$ and $f_4(\xi)$ are as follows

$$f_3(\xi)=\xi+\int_0^{\xi}\left(\frac{(\eta-c)cA(\eta)}{a}-\frac{A(\eta)}{b}\right)(\xi-\eta)\,d\eta\,,$$

$$f_4(\xi)=1+\int_0^{\xi}\frac{A(\eta)(\eta-c)}{a}(\xi-\eta)\,d\eta\,.$$

Formulas (13), (14), (17) remain valid, but their indices i, j and k run now from 1 to 4.

When calculating the kinetic energy it is necessary to take into account the kinetic energy of the disk, therefore the factors $\bar{a}_{ij}$ of determinant (17) in this case are

$$\bar{a}_{ij}=\int_0^1 A(\xi)h_i(\xi)h_j(\xi)\,d\xi+\gamma h_i(0)h_j(0)+\gamma r^2\varphi_i(0)\varphi_j(0)\,,\quad i,j=\overline{1,4}\,. \tag{19}$$

Here

$$\varphi_i(\xi)=\frac{dh_i(\xi)}{d\xi}\,,\quad i=\overline{1,4}\,.$$

In the case of the bar of constant cross-section equation (18) allows us to calculate the natural frequencies exactly and so estimate an error of this approximate method.

The radius of inertia for the thin disk R is equal to $R_1/2$, where R_1 is the radius of the disk and therefore $R_1=2lr$.

If the shaft of radius r_1 and the disk of thickness h are made of the same material then for $r_1=l/20$ and $h=R_1/20$ we obtain

$$\gamma=160\,r^3\,. \tag{20}$$

Assuming that γ and r are related to each other with this expression and r varies within the range from 0 to 1/2, let us follow the change in error for the first, second and third frequencies. Upon calculations we obtain the following values for the error in percentage terms (%)

$r=0.000$	$1.5\cdot 10^{-4}$	$5.6\cdot 10^{-1}$	2.67
$r=0.125$	$3.7\cdot 10^{-5}$	$9/5\cdot 10^{-2}$	0.85
$r=0.250$	$1.4\cdot 10^{-6}$	$3.7\cdot 10^{-4}$	0.40
$r=0.500$	$-\,9.0\cdot 10^{-6}$	$3.9\cdot 10^{-5}$	0.35

The first column corresponds to the first frequency, the second column corresponds to the second frequency and the third one corresponds to the third frequency. We see that the higher frequency, the greater error.

For $r\geqslant 0.125$ the error for the first frequency is close to the limits of accuracy which is provided by the software package "Mathematica 5.2". In this regard one can say that this method permits to determine the first frequency exactly. Therefore it may be used in both the rotor dynamics and for testing the programs for analysis of complicated mechanical systems.

In rotor engineering it is important to have the analytical dependence of the first natural frequency on the system's parameters. This method based on the consideration of four holonomic constraints does not allow us to do that as it leads to the solution of the algebraic equation of fourth order. But if we limit ourselves to consideration of only two constraints at the end where the disk is located, then the required first frequency will be determined in analytical form as a root of biquadratic equation.

Let us prove, that this simple solution also makes it possible to find the first frequency accurately enough. When getting this solution it is reasonable to measure the coordinate of the bar cross-section not from the free end but from the rigidly clamped one. Formulas (4) and (17) remain valid, but now $S(l)$ and $J(l)$ will correspond not to the rigidly clamped end, but to the place of disk fixation.

The bending moment $\Lambda_1(t)$ and lateral force $\Lambda_2(t)$, applied to the end $x = l$, are constraint reactions and considered in this problem as the generalized coordinates. Their positive directions, as well as the positive direction of the moment $M(x,t)$ applied to the cross-section x, are shown in Figure 4.

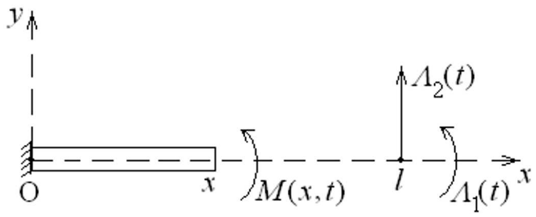

Figure 4. The bending moment and constraint reaction forces.

The dimensionless bending moment $L(\xi,t)$ introduced by formula (9) is equal in this case to

$$\begin{gathered} L(\xi,t) = \bar{\Lambda}_1(t) f_1(\xi) + \bar{\Lambda}_2(t) f_2(\xi)\,, \\ f_1(\xi) = 1\,, \quad f_2(t) = 1-\xi\,, \quad \bar{\Lambda}_1(t) = \frac{\Lambda_1 l}{EJ(l)}\,, \quad \bar{\Lambda}_2(t) = \frac{\Lambda_2 l^2}{EJ(l)}\,. \end{gathered} \tag{21}$$

Expression (11), as can be seen, survives and therefore the potential energy will be written in the form (14).

Integrating equation (10) and taking into account that

$$\bar{y}(0,t) = \left.\frac{\partial \bar{y}}{\partial \xi}\right|_{\xi=0} = 0\,,$$

imply expression (13), where now

$$h_k(\xi) = \int_0^{\xi} \frac{f_k(\eta)(\eta-\xi)}{B(\eta)}\,d\eta\,, \quad k = 1,2\,. \tag{22}$$

As the deflection is represented in the same form (13), the kinetic energy will be written in the same form (15) too. The factors $\bar{a}_{ij}$ in this case should be calculated by formulas (19), but now $h_i(0)$ should be replaced with $h_i(1)$, and $\varphi_i(0)$ should be replaced with $\varphi_i(1)$.

When calculating for the bar of constant cross-section the error of the first and second frequencies in percentage terms (%) for the same relation (20) between γ and r, we obtain

$r = 0.000$	0.47	58
$r = 0.125$	$8.4 \cdot 10^{-2}$	15.6
$r = 0.250$	$2.6 \cdot 10^{-3}$	0.21
$r = 0.500$	$1.1 \cdot 10^{-5}$	$1.1 \cdot 10^{-3}$

For $r \geqslant 0.25$ we can say that for the first frequency we obtain the exact value. Notice, however that for $r = 0.25$ the disk diameter is equal to the shaft length, and for $r = 0.5$ it is two times greater. For such relation between these quantities for the assumed values $r_1 = l/20$ and $h = R_1/20$ this disk can not be regarded as a perfectly rigid body. It is necessary to take into account the influence of its compliance on the natural frequencies of the system. It is feasible but it will require additional calculations, the basic framework of which will be shown through the example of the cantilever with a flexible bar at its end. This example will require no new mathematical apparatus. It is reduced to the same calculations as above.

4 Determination of the First Three Frequencies of the Cantilever with a Flexible Bar at its End

Let us analyze the problem, when the bar that is executing longitudinal oscillations in the mechanical system considered in the work (Yushkov and Zegzhda, 1998) is absent (see Figure 5). Within the frames of such problem we have three constraints and three reaction forces, correspondingly. The bending moment $\Lambda_1(t)$ and the lateral force $\Lambda_2(t)$ are applied to the cantilever as is shown in Figure 4. The third reaction force is the lateral force $\Lambda_3(t)$ applied to the bar which is perpendicular to the cantilever.

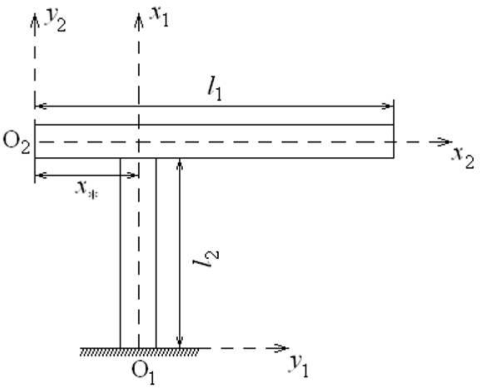

Figure 5. The system of two bars.

Both the kinetic and the potential energy of the cantilever are determined by the formulas given above. Therefore it is necessary to take into account only the second bar. When released from the constraints it becomes free and similar to the bar shown in Figure 1, but now the bending moment $M(t) = \Lambda_1(t)$ and lateral force $Q(t) = \Lambda_3(t)$ are applied not to the end of the bar but to the cross-section $x_* = zl$. Therefore the equations will be written in the form

$$y(x_*, t) = 0\,, \qquad \left.\frac{\partial y}{\partial x}\right|_{x=x_*} = 0\,. \tag{23}$$

We shall not provide the parameters of the second bar with indices when considering the question how the deflection curve will change depending on the place of application of the reactions. We shall do that upon obtaining the expressions for the potential energy of its deflection and for the deflection curve.

Formulas (6) in this case will appear as

$$W_c = \frac{\Lambda_3(t)}{\rho\int_0^l S(x)\,dx}\,, \qquad \ddot{\varphi} = \frac{\Lambda_1(t) + (x_* - x_c)\Lambda_3(t)}{\rho\int_0^l S(x)(x_c - x)^2 dx}\,,$$

and formula (7) remains valid.

The bending moment $M(x,t)$ applied to the left of the cross-section $x = x_*$ is set by expression (8), and the bending moment applied to the right of cross-section takes the form

$$M(x,t) = \int_x^l q(x_1,t)(x_1 - x)\,dx_1\,, \quad x_* < x < l\,.$$

Hence the bar is divided into two sections and the deflections of its left and right parts have to be calculated independently. Denoting the bending moment $M(x,t)$ for $0 < x < x_*$ by $M_1(x,t)$, and for $x_* < x < l$ by $M_2(x,t)$, and going to dimensionless variables (9), we obtain

$$L_n(\xi,t) = \bar{\Lambda}_1^{(2)}(t) f_{1n}(\xi) + \bar{\Lambda}_3^{(2)}(t) f_{3n}(\xi)\,, \quad n = 1, 2\,.$$

Here

$$\bar{\Lambda}_1^{(2)}(t) = \frac{\Lambda_1(t)l}{EJ(t)}\,, \quad \bar{\Lambda}_3^{(2)}(t) = \frac{\bar{\Lambda}_3(t)l^2}{EJ(t)}\,,$$

$$\begin{aligned}
f_{11}(\xi) &= \int_0^{\xi} \frac{(c-\eta)A(\eta)}{a}\,(\xi-\eta)\,d\eta\,, \quad 0 \leqslant \xi \leqslant z\,,\\
f_{31}(\xi) &= \int_0^{\xi} \left(\frac{(c-\eta)(z-c)A(\eta)}{a} - \frac{A(\eta)}{b}\right)(\xi-\eta)\,d\eta\,, \quad 0 \leqslant \xi \leqslant z\,,\\
f_{12}(\xi) &= \int_{\xi}^{1} \frac{A(\eta)(\eta-c)}{a}\,(\xi-\eta)\,d\eta\,, \quad z \leqslant \xi \leqslant 1\,,\\
f_{32}(\xi) &= \int_{\xi}^{1} \left(\frac{(\eta-c)(z-c)A(\eta)}{a} + \frac{A(\eta)}{b}\right)(\xi-\eta)\,d\eta\,, \quad z \leqslant \xi \leqslant 1\,.
\end{aligned} \tag{24}$$

We remind that the values a, b, c, included in these expressions are calculated by formulas (12). The index "2" of the quantities $\bar{\Lambda}_1^{(2)}(t)$ and $\bar{\Lambda}_3^{(2)}(t)$ means that transition to the dimensionless variables corresponds to the parameters l, E and $J(l)$ of the second bar (see Figure 5). The functions $A(\xi)$, $B(\xi)$ and the values a, b, c, should be also provided with index "2" hereinafter, but for the sake of simplicity they are omitted.

Integrating equation (10) for $L(\xi,t) = L_1(\xi,t)$, and then for $L(\xi,t) = L_2(\xi,t)$, and taking into account the constraint equations (23) imply

$$\begin{aligned}
\bar{y}(\xi,t) &= \bar{\Lambda}_1^{(2)}(t)h_{11}(\xi) + \bar{\Lambda}_3^{(2)}(t)h_{31}(\xi)\,, \quad 0 \leqslant \xi \leqslant z\,,\\
\bar{y}(\xi,t) &= \bar{\Lambda}_1^{(2)}(t)h_{12}(\xi) + \bar{\Lambda}_3^{(2)}(t)h_{32}(\xi)\,, \quad z \leqslant \xi \leqslant 1\,,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
f_{k1}(\xi) &= \int_{\xi}^{z} \frac{f_{k1}(\eta)(\eta-\xi)}{B(\eta)}\,d\eta\,, \quad 0 \leqslant \xi \leqslant z\,,\\
f_{k2}(\xi) &= \int_{z}^{\xi} \frac{f_{k2}(\eta)(\xi-\eta)}{B(\eta)}\,d\eta\,, \quad z \leqslant \xi \leqslant 1\,, \quad k = 1,3\,.
\end{aligned}$$

By using the unit function

$$U(x) = \begin{cases} 1\,, & x \geqslant 0\,,\\ 0\,, & x < 0\,, \end{cases}$$

we represent expressions (25) as

$$\bar{y}(\xi,t) = \bar{\Lambda}_1^{(2)}(t)h_1(\xi) + \bar{\Lambda}_3^{(2)}(t)h_3(\xi)\,, \quad 0 \leqslant \xi \leqslant 1\,. \tag{26}$$

Here

$$h_k(\xi) = h_{k1}(\xi)U(z-\xi) + h_{k2}(\xi)U(\xi-z)\,. \tag{27}$$

The potential energy of deformation of the second bar has to be calculated independently for its right and left sections. Calculating and summing these energies produce

$$\Pi = \frac{EJ(l)}{2l}\left(\bar{c}_{11}^{(2)}(\bar{\Lambda}_1^{(2)}(t))^2 + 2\bar{c}_{13}^{(2)}\bar{\Lambda}_1^{(2)}(t)\bar{\Lambda}_3^{(2)}(t) + \bar{c}_{33}^{(2)}(\bar{\Lambda}_3^{(2)}(t))^2\right), \tag{28}$$

where

$$\begin{aligned}
\bar{c}_{kk}^{(2)} &= \int_0^z \frac{f_{k1}^2(\xi)}{B(\xi)}\,d\xi + \int_z^1 \frac{f_{k2}^2(\xi)}{B(\xi)}\,d\xi\,, \quad k = 1,3\,,\\
\bar{c}_{13}^{(2)} &= \int_0^z \frac{f_{11}(\xi)f_{31}(\xi)}{B(\xi)}\,d\xi + \int_z^1 \frac{f_{12}(\xi)f_{32}(\xi)}{B(\xi)}\,d\xi\,.
\end{aligned}$$

Adding the potential energy in bending of the first bar to potential energy (28), we represent their sum in the form

$$\Pi = \frac{E_1J_1(l_1)}{2l_1}\sum_{i,j=1}^{3}\bar{c}_{ij}\bar{\Lambda}_i^{(1)}\bar{\Lambda}_j^{(1)}\,. \tag{29}$$

Here index "1" means that this quantity corresponds to the first bar. The dimensionless variables $\bar{\Lambda}_i^{(1)}$, $i = \overline{1,3}$, are introduced by the formulas:

$$\bar{\Lambda}_1^{(1)} = \frac{\Lambda_1 l_1}{E_1 J_1(l_1)}\,, \quad \bar{\Lambda}_k^{(1)} = \frac{\Lambda_k l_1^2}{E_1 J_1(l_1)}\,, \quad k = 2, 3\,.$$

Notice that here $J_1(l_1)$ corresponds not to the place of rigid fixing of bar, as it was in the beginning of the paper, but to the point where the first bar is connected to the second one (see Figure 5).

In formulas (24) and (28) all quantities refer to the second bar. Introduction of the parameters

$$\alpha = \frac{E_1 J_1(l_1) l_2^3}{E_2 J_2(l_2) l_1^3}\,, \quad \beta = \frac{l_2}{l_1}$$

allows us to represent the potential energy (28) of the second bar as

$$\Pi_2 = \alpha \frac{E_1 J_1(l)}{2 l_1} (\bar{c}_{11}^{(2)} (\bar{\Lambda}_1^{(1)}(t))^2 \beta^{-2} + 2 \bar{c}_{13}^{(2)} \bar{\Lambda}_1^{(1)}(t) \bar{\Lambda}_3^{(1)}(t) \beta^{-1} + \bar{c}_{33}^{(2)} (\bar{\Lambda}_3^{(1)}(t))^2)\,.$$

This implies that the factors $\bar{c}_{ij}$ in expression (29) are as follows

$$\bar{c}_{11} = \bar{c}_{11}^{(1)} + \alpha\beta^{-2}\bar{c}_{11}^{(2)}\,, \quad \bar{c}_{12} = \bar{c}_{12}^{(1)}\,,$$
$$\bar{c}_{13} = \alpha\beta^{-1}\bar{c}_{13}^{(2)}\,, \quad \bar{c}_{22} = \bar{c}_{22}^{(1)}\,, \quad \bar{c}_{23} = 0\,, \quad \bar{c}_{33} = \alpha\bar{c}_{33}^{(2)}\,.$$

Here in accordance with formulas (14), (21)

$$\bar{c}_{11}^{(1)} = \int_0^1 \frac{d\xi}{B_1(\xi)}\,, \quad \bar{c}_{12}^{(1)} = \int_0^1 \frac{(1-\xi)\,d\xi}{B_1(\xi)}\,, \quad \bar{c}_{22}^{(1)} = \int_0^1 \frac{(1-\xi)^2 d\xi}{B_1(\xi)}\,.$$

The kinetic energy of the first bar will be represented by using expressions (15), (21), (22) in the form

$$T_1 = \frac{1}{2}\rho_1 S_1(l_1)\, l_1^3 \sum_{i,j=1}^{2} \bar{a}_{ij}^{(1)} \dot{\bar{\Lambda}}_i^{(1)} \dot{\bar{\Lambda}}_j^{(1)}\,,$$
$$\bar{a}_{ij}^{(1)} = \int_0^1 A_1(\xi) h_i^{(1)}(\xi) h_j^{(1)}(\xi)\, d\xi\,,$$
$$h_1^{(1)}(\xi) = \int_0^\xi \frac{(\xi-\eta)\,d\eta}{B_1(\eta)}\,, \quad h_2^{(1)}(\xi) = \int_0^\xi \frac{(1-\eta)(\xi-\eta)\,d\eta}{B_1(\eta)}\,.$$

Let us calculate the kinetic energy of the second bar now. The assumption that the amplitude of the oscillations of the bars under consideration is small allows us to calculate the kinetic energy of the translational motion of the second bar along the axis independently from the kinetic energy of its motion in the direction that is perpendicular to its axis.

The kinetic energy of the translational motion of the second bar is

$$T_{21} = \frac{m_2 l_1^2}{2} (h_1^{(1)}(1) \dot{\bar{\Lambda}}_1^{(1)} + h_2^{(1)}(1) \dot{\bar{\Lambda}}_2^{(1)})^2\,,$$

$$m_2 = \rho_2 S_2(l_2)\, l_2 \int_0^1 A_2(\xi)\, d\xi\,.$$

Displacements of the cross-sections of the second bar in the direction perpendicular to the bar axis are caused, firstly, by rotation of the bar about the cross-section $x_* = z l_2$, and, secondly, by the deflection defined by expression (26). Therefore we have

$$y_2(\xi, t) = l_2(\psi(t)(z-\xi) + \bar{\Lambda}_1^{(2)}(t) h_1^{(2)}(\xi) + \bar{\Lambda}_3^{(2)}(t) h_3^{(2)}(\xi))\,.$$

Here

$$\psi(t) = \varphi_1(1)\bar{\Lambda}_1^{(1)}(t) + \varphi_2(1)\bar{\Lambda}_2^{(1)}(t)\,, \quad \varphi_k(1) = \left.\frac{dh_k^{(1)}}{d\xi}\right|_{\xi=1}, \quad k = 1, 2\,.$$

Index "2" of the functions $h_1^{(2)}(\xi)$ and $h_3^{(2)}(\xi)$ means that these functions defined by expressions (27) are calculated for the parameters of the second bar.

Taking into account that

$$\bar{\Lambda}_1^{(2)} = \alpha\beta^{-2}\bar{\Lambda}_1^{(1)}\,, \quad \bar{\Lambda}_3^{(2)} = \alpha\beta^{-1}\bar{\Lambda}_3^{(1)}\,,$$

the kinetic energy

$$T_{22} = \frac{1}{2}\rho_2 \int_0^{l_2} S_2(x)\left(\frac{\partial y_2}{\partial t}\right)^2 dx$$

will be represented as

$$T_{22} = \frac{1}{2}\rho_2 S_2(l_2) l_2^3 \int_0^1 A_2(\xi)((\varphi_1(1)\dot{\bar{\Lambda}}_1^{(1)} + \varphi_2(1)\dot{\bar{\Lambda}}_2^{(1)})(z-\xi)+ \\ +\alpha\beta^{-2}\dot{\bar{\Lambda}}_1^{(1)} h_1^{(2)}(\xi) + \alpha\beta^{-1}\dot{\bar{\Lambda}}_3^{(1)} h_3^{(2)}(\xi))^2 d\xi\,.$$

Introducing into consideration the third parameter

$$\gamma = \frac{\rho_2 S_2(l_2) l_2}{\rho_1 S_1(l_1) l_1}\,,$$

the total kinetic energy of the second bar appears as

$$T_2 = \frac{\gamma}{2}\rho_1 S_1(l_1)\, l_1^3 \sum_{i,j=1}^{3} \bar{a}_{ij}^{(2)} \dot{\bar{\Lambda}}_i^{(1)} \dot{\bar{\Lambda}}_j^{(1)}\,.$$

Analytic expressions for the factors $a_{ij}^{(2)}$, dependent on the functions $A_2(\xi)$ and parameters α and β, are rather intricate and thus not given here. Note that they are easily found with the software package "Mathematica 5.2".

The kinetic energy of both bars is

$$T = \frac{1}{2}\rho_1 S_1(l_1)\, l_1^3 \sum_{i,j=1}^{3} \bar{a}_{ij} \dot{\bar{\Lambda}}_i^{(1)} \dot{\bar{\Lambda}}_j^{(1)}\,, \quad \bar{a}_{ij} = \bar{a}_{ij}^{(1)} + \bar{a}_{ij}^{(2)}\,.$$

We find the required natural frequencies p_* by solving equation (17). Note that in formula (16) of the transition to dimensional frequencies all quantities correspond to the first bar at the point of its connection to the second bar.

5 Comparison with the Bars of Constant Cross-Section

The problem of bars of constant cross-section has been solved exactly by the methods of mathematical physics. As this takes place the equation of the frequencies is obtained by equating the determinant of sixth order to zero. Its elements are the Krylov functions, the arguments of which depend on the parameters α, γ and z. This intricate transcendental equation, the computational solution of which was a matter of some difficulty even for modern computers, has been used for testing the method suggested in (Zegzhda and Yushkov, 1999). Note that the calculation of first three frequencies by using this suggested method does not create any difficulties.

As stated above, the problem under investigation is a particular case of the problem discussed in (Yushkov and Zegzhda, 1998). When the bar executing logitudinal vibration is absent, the frequency determinant is a determinant of third order. Comparing the roots of the transcendental equation with the roots of the frequency equation, which was obtained by the technique suggested in (Yushkov and Zegzhda, 1998), shows that the first frequency is determined with four valid significant digits in the second approximation, the second frequency is determined with the same accuracy in the fourth approximation, and the third one is obtained with the same accuracy in the sixth approximation.

If the first and the second bar is made of the same material and have the same cross-sections, then at $z = 1/2$ the solution depends on a single parameter $\beta = l_2/l_1$, for in this case $\gamma = \beta$ and $\alpha = \beta^3$. The calculations show that for the first frequency the error decreases as β rises, and for $\beta = 1/8,\ 1/4,\ 1/2,\ 2$ it is equal to $0.22,\ 0.12,\ 0.056,\ 0.0022$ percent (%) correspondingly. Note that in this example in case $\beta \leqslant 0.25$, it is reasonable to consider the second bar as a concentrated mass located at the end of the cantilever beam and to use the method presented in the beginning of the paper.

Let us discuss briefly the errors of the method under consideration for the second and third frequencies. Let us examine this problem through the example of the bars, differing only in length.

For $\alpha = \beta = \gamma = 1$ and $z = 1/2$ the exact and approximate values of first three dimensionless frequencies p_* are

$$1.44851\,, \quad 6.20782\,, \quad 14.0641\,,$$

$$1.44876\,, \quad 6.24235\,, \quad 14.1204\,.$$

The errors in percentage terms (%) are equal correspondingly to

$$0.017\,, \qquad 0.56\,, \qquad 0.40\,.$$

If the second bar is symmetrically positioned in relation to the first one, there exists a vibration mode such that the first bar does not oscillate, and both halves of the second bar oscillate as a cantilever of length $l = \beta l_1/2$. The first frequency of the cantilever oscillation in the dimensionless variables is

$$p_* = 3.516\frac{4}{\beta^2}\,. \tag{30}$$

This frequency in the series of frequencies of the system consisting of two bars has the number n. This number increases as β decreases. For example, for $\beta = 1/4$ it will be the ninth frequency, and the third root of equation (17) will correspond to it. Let us find this root in the explicit form.

When the second bar does not oscillate, then the bending moment Λ_1 and the lateral force Λ_2 applied to the end of the first bar vanish. Therefore in this vibration mode only the lateral force Λ_3 applied to the middle of the second bar is not equal to zero. Under the action of this force the second bar moves translationally and bends so that the application point of the force Λ_3 is immovable. In quasistatics the intensity of inertial forces is constant in this case, therefore either the second or the third root of equation (17) at $z = 1/2$ is equal to

$$p_* = \sqrt{\frac{c}{a}\,\frac{4}{\beta^2}}\,, \quad c = \int_0^1 f^2(\xi)\,d\xi\,, \quad f(\xi) = \int_0^\xi (\xi - \eta)\,d\eta\,,$$
$$a = \int_0^1 h^2(\xi)\,d\xi\,, \quad h(\xi) = \int_\xi^1 f(\eta)\,(\xi - \eta)\,d\eta\,.$$

When calculating we obtain

$$p_* = 3.530\,\frac{4}{\beta^2}\,. \tag{31}$$

This frequency exceeds its exact value obtained by formula (30) by $0.40\,\%$.

For $\beta = 1/2$ the frequency approximately defined by expression (31) corresponds to the exact value of the fourth frequency, for $\beta = 1$ and $\beta = 2$ it corresponds to the third frequency, and for $\beta = 4$ it corresponds to the second frequency.

Hence this approximate method makes it possible to determine the first frequency for any values of the system parameters with a rather high degree of accuracy, and for some values of the parameters it allows us to define the second and the third frequencies as well.

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