TECHNISCHE MECHANIK, Band 28, Heft 2, (2008), 140 – 151
Manuskripteingang: 04. Oktober 2007

Determination of Asymptotic Waves in Maxwell Media by Double-Scale Method

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In this paper, in a thermodynamical model of a rheological medium (Maxwell) with one internal variable, derived in the framework of classical irreversible thermodynamics, the asymptotic smooth waves, studied in (1) in a more classical way, are introduced from the point of view of double scale method (see (2)). We give a physical interpretation of the new (fast) variable, related to the surface across which the derivatives of the solution vary steeply. An one-dimensional application is carried out too.

1 Introduction

The nonlinear hyperbolic partial differential equations (PDEs) describe the motion of a large number of media. Their solutions $\mathbf{U}(x^\alpha)$ are referred to as *waves*. Some of them present various types of discontinuities, some others are smooth. We deal with those smooth waves $\mathbf{U}(x^\alpha)$ called asymptotic waves. Both these types of solutions are called *nonlinear waves* because they satisfy nonlinear hyperbolic PDEs. A lot of applications to various equations from elasticity, fluid mechanics and thermodynamics and other branches of physics were carried out ((3)-(10)).
The mathematical aspects involved into the study of asymptotic waves belong to singular perturbation theory, namely the double-scale method ((11)-(21)). The multiple-scale method, and, in particular, the double-scale approach, is appropriate to phenomena which possess qualitatively distinct aspects at various scales. For instance, at some well-determined times or space coordinates, the characteristics of the motion vary steeply, while at larger scale the characteristics are slow and describe another type of motion. In addition, the scales are defined by some small parameters.
In the context of rheological media, a series of studies on linear waves were carried out in (1), (22), (23) (Ciancio-Restuccia 1985, 1987). In this paper, in a thermodynamical model of a rheological medium (Maxwell) with one internal variable, the asymptotic smooth waves are introduced from the point of view of double scale method (see (24), Georgescu 1995). To this aim a fast variable is introduced and the definition of the hyperbolicity is recalled, in view of a relationships between this variable and the internal layer occurring in the domain of motion. In Section 2, the inelastic deformations are described in the framework of classical irreversible thermodynamics and the features following the introduction of internal variables is emphasized. Corresponding governing equations are studied in Section 3 by applying the double-scale method and the involved steps as proposed in (3)-(10). In fact, the results were obtained in (1) in a more classical way. The various steps implied in determining the solution of the model of the first asymptotic approximation are described in Sections 4, 5, 6. The paper concludes with an one-dimensional example revealing the influence of the internal variable on the relaxation. These and many other results in the paper are new.

2 Asymptotic Waves as Solutions of Nonlinear PDEs Deduced by Double-Scale Method

We deal with those smooth waves $\mathbf{U}(x^\alpha)$, called the asymptotic waves, which evolve as *progressive waves*, i.e. there exists a family of hypersurfaces *S* defined by the equation $\varphi(x^\alpha) = 0$ moving in the Euclidean space E^{n+1} (consisting of points of coordinates x^α, α = 0,1,2,...,n, or, equivalently of the time $t = x^0$ and the space coordinates x^i, i=1,2,...,n)

$$\varphi(t, x^i) = \bar{\xi} = const, \tag{2.1}$$

such that **U** or their derivatives vary steeply across *S*, while along *S* their variation is slow (2). This means that around *S* there exist (asymptotic) *internal layers*, such that the order of magnitude (i.e. the scale) of some derivatives of the solution inside these layers and far away from them differ very much. Therefore, it is natural to introduce a new independent variable ξ, related to the hypersurfaces (2) S,

$$\xi = \omega\bar{\xi} = \omega\varphi(t, x^i), \tag{2.2}$$

where ξ is asymptotically fixed , i.e. $\xi = Ord(1)$ as $\omega^{-1} \to 0$, and $\omega \gg 1$ is a very large parameter, to assume that the solution depends on the old as well as the new variable, i.e. $\mathbf{U} = \mathbf{U}(x^\alpha, \xi)$, and to consider that x^α and ξ are independent.
Taking into account that $\mathbf{U}$ is sufficiently smooth, hence it has sufficiently many bounded derivatives, it follows that, except for the terms containing ω, all other terms are asymptotically fixed and the computation can proceed formally. In this way, if $x^\alpha = x^\alpha(s)$ are the parametric equations of a curve C in E^{n+1}, we have

$$\frac{d\mathbf{U}}{ds} = \omega \frac{\partial \mathbf{U}}{\partial \xi}\frac{\partial \varphi}{\partial s} + \frac{\partial \mathbf{U}}{\partial x^\alpha}\frac{dx^\alpha}{ds},$$

(throughout this paper the dummy index convention is understood). This relation shows that, indeed, along C, $\mathbf{U}$ does not vary too much if C belongs to the hypersurface $\mathcal{S}$ (in this case $\frac{d\varphi}{ds} = 0$) but has a large variation if C is not situated on $\mathcal{S}$. For these reasons, ξ is referred to as the *fast variable*. Once introduced the fast variable ξ, in order to apply the double-scale method we must define the equations to which it applies. Thus, let E^{n+1} be an Euclidean space, let $P \in E^{n+1}$ be a current point, let $\mathbf{U} = \mathbf{U}(P)$ be the unknown vector function $\mathbf{U} = (U_1, U_2, ..., U_N)$, solution of the first-order semilinear PDE

$$\mathbf{H}^\alpha(\mathbf{U}(P), P)\frac{\partial \mathbf{U}}{\partial y^\alpha} = \mathbf{h}(\mathbf{U}(P), P), \qquad (\alpha = 0, 1, 2, ..., n) \tag{2.3}$$

where $\mathbf{h} = (h_1, h_2, ..., h_N)^T$ is a column vector, y^α are the Cartesian coordinates of P and $\mathbf{H}^\alpha$ are n+1 square matrices of the $N \times N$ type. Denote by $\mathbf{H}^{\alpha A}_B$, $(A, B = 1, 2, ..., N)$ a real function defined on E^{n+1} which is a current entry of $\mathbf{H}^\alpha$. We say that (2.3) is a nonlinear hyperbolic equation if the n+1 matrices $\mathbf{H}^\alpha$ endow E^{n+1} with a hyperbolic structure at the current point $P \in E^{n+1}$, i.e. if the following two conditions are satisfied (see (9)): 1) there exists a direction $\mathbf{v} \equiv v_\alpha(P)$, such that $det\mathbf{A}^0 \neq 0$, where $\mathbf{A}^0 = \mathbf{H}^\alpha v_\alpha$; 2) if $\mathbf{v}$, $\mathbf{e}_i$, $(i = 1, 2, ..., n, \mathbf{e}_i \equiv e_{i\alpha}, v^\alpha v_\alpha = 1, v^\alpha e_{i\alpha} = 0, e_i^\alpha e_{j\alpha} = \delta_{ij})$ is an orthonormal base of E^{n+1} at P for every direction $\mathbf{n} \equiv n_j$ of the n-dimensional subspace of E^{n+1} generated by the base $\mathbf{e}_i$ (orthogonal to $\mathbf{v}$), then the matrix

$$\mathbf{A}_n = \mathbf{A}^{0^{-1}}\mathbf{H}^\alpha e_{i\alpha} n_i \tag{2.4}$$

possesses N linearly-independent left and right eigenvectors $\mathbf{l}_A$ and $\mathbf{r}_A$, respectively, corresponding to the real eigenvalue λ^A of multiplicity m^A, i.e.

$$(\mathbf{A}_n(\mathbf{U}) - \lambda^{(A)} I)\mathbf{d}_A = 0, \qquad \mathbf{l}_A(\mathbf{A}_n(\mathbf{U}) - \lambda^{(A)} I) = 0. \tag{2.5}$$

Since $\mathbf{A}_n$ depends on $\mathbf{U}$ and $\mathbf{n}$ ($\mathbf{n}$ being arbitrary in E^n), this means that the eigenvalues and eigenvectors also depend on $\mathbf{U}$ and $\mathbf{n}$. This is why we write e.g. $\lambda^{(A)}(\mathbf{U}, \mathbf{n})$. The index n of $\mathbf{A}_n$ remainds the vector $\mathbf{n}$ and not the dimension of the Euclidean space E^n. The superscript "A" in $\lambda^{(A)}$ is not related to the matrix A but to the multiplicity m^A.
The above definition of nonlinear hyperbolicity generalizes the definition from the case of linear or affine hyperbolic PDEs, where $\mathbf{H}^\alpha$ and $\mathbf{h}$ do not depend on $\mathbf{U}$. It preserves the fact that the Cauchy problem for (2.3) is well-posed. Indeed, condition 1) ensures the invertibility of $\mathbf{A}^0$ and, therefore, the possibility to write (2.3) in the form

$$\mathbf{U}_t + \mathbf{A}^i \mathbf{U}_{x^i} = \mathbf{B}(\mathbf{U}, x^\alpha),. \tag{2.6}$$

where

$$x^0 = t \equiv y^\alpha v_\alpha,\ x^i = y^\alpha e_{i\alpha},\ \mathbf{A}^i(\mathbf{U}, x^\alpha) = \mathbf{A}^{0^{-1}}\mathbf{H}^\alpha e_{i\alpha},\ \mathbf{B}(\mathbf{U}, x^\alpha) = \mathbf{A}^{0^{-1}}\mathbf{h}. \tag{2.7}$$

The form (2.6) is obtained by multiplying (2.3) by $\mathbf{A}^{0^{-1}}$ (which exists, by virtue of condition 1)).
In applications, one encounters equations of the form (2.3). Therefore, in order to be sure that the machinery of hyperbolic PDEs's theory works, it is necessary to show that all geometric and analytic quantities occurring in the definition of the nonlinear hyperbolic PDEs can be defined for each concrete situation.
In Section 3 we present a particular form of equation (2.3). It occurs in thermodynamics of rheological media, namely the Maxwell media. In this case, $x^0 = t$, $\mathbf{A}^0$, i.e. the matrix coefficient of $\frac{\partial \mathbf{U}}{\partial t}$, is $\mathbf{A}^0 = \mathbf{I}$ (the unit matrix), $N = 10$, $n = 3$, x_1, x_2, x_3 are the space variables, $\mathbf{v} = \mathbf{e}_0$ and has the coordinates $e_{0\alpha}$, $\mathbf{e}_i$ have the coordinates $e_{i\alpha}$, $e_{\alpha\beta} = \delta_{\alpha\beta}$, $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$; $\alpha, \beta = 0, 1, 2, 3$; $i = 1, 2, 3$. In this way, $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis in the Euclidean space E^4, and, in the Euclidean space E^3, it corresponds to the

canonical basis. Hence, for our concrete situations, (2.6) is a nonlinear hyperbolic equation.
The matrix A_n follows defined by $\mathbf{A}_n = \mathbf{A}^{0^{-1}}\mathbf{H}^1 n_1 + \mathbf{A}^{0^{-1}}\mathbf{H}^2 n_2 + \mathbf{A}^{0^{-1}}\mathbf{H}^3 n_3$, or, taking into account (2.7) equivalently by $\mathbf{A}_n = \mathbf{A}^1 n_1 + \mathbf{A}^2 n_2 + \mathbf{A}^3 n_3$. Moreover, in (1) it was found that, the eigenvalues are real, indeed, while the eigenvectors are linearly independent. Further $\mathbf{n}$ is chosen to be just the unit vector normal to that hypersurface $\mathcal{S}$ (improperly called wavefront)

$$\varphi(t, x_1, x_2, x_3) = 0, \tag{2.8}$$

which was supposed to be characterized by the fact that the solution $\mathbf{U}$ of (2.6) varies steeply across it. Then (2.8) implies that along $\mathcal{S}$ we have $\frac{d\varphi}{dt} = 0$, implying $\frac{\partial\varphi}{\partial t} + \mathbf{v} \cdot grad\varphi = 0$, or equivalently, $\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|} + \mathbf{v} \cdot \frac{grad\varphi}{|grad\varphi|} = 0$. Obviously, $\frac{grad\varphi}{|grad\varphi|} = \mathbf{n}$, such that the previous equality reads
$\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|} + \mathbf{v} \cdot \mathbf{n} = 0$. Introducing the notations

$$\lambda = -\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|}, \quad \Lambda^i(\mathbf{U}, \mathbf{n}) = \frac{\partial\Psi}{\partial\varphi^i}, \quad where \quad \varphi^i = \frac{\partial\varphi}{\partial x^i}$$

and

$$\Psi(x^\alpha, \frac{\partial\varphi}{\partial x^\alpha}) \equiv \frac{\partial\varphi}{\partial t} + \lambda|grad\varphi|, \tag{2.9}$$

we have

$$\lambda(\mathbf{U}, \mathbf{n}) = \mathbf{v} \cdot \mathbf{n}, \quad such\ that \quad \lambda = v^i\Lambda_i, \tag{2.10}$$

where λ is called the *velocity normal to the progressive wave* and $\mathbf{\Lambda}$, of coordinates Λ^i, the *radial velocity*.

These quantities play an important role in applying the double-scale method.

Since the closed-form solutions of nonlinear PDEs are rare, usually the solution is looked for in the form of an asymptotic expansion with respect to an asymptotic sequence of powers of some small parameter. This expansion is called the asymptotic solution of the PDE. In particular, in nonlinear hyperbolic equations ω is related to the thickness of internal layers, across which the solution varies steeply. Correspondingly, a new independent (fast) variable ξ is defined that models just this fast variation across the internal layers situated near some surfaces $\mathcal{S}$ and the slow variation along $\mathcal{S}$. The asymptotic method involving ξ is known as the double-scale method (2). The solution in the form of an asymptotic series it yields is referred to as the *asymptotic wave*. In this way through the fast variable ξ, the mathematical treatment of asymptotic waves relates facts in singular perturbation theory (to which the double-scale method belongs) to hyperbolic PDEs theory. Apart from this peculiarity, the application of the double-scale method is standard: the solution $\mathbf{U}(x^\alpha, \xi)$ is written as an asymptotic power series of the small parameter, the coefficients $\mathbf{U}^i$ being functions of x^α and ξ. Introducing the series in the equations, after the matching of the series in the right-and left-hand sides, the equations for $\mathbf{U}^i$, with $i \geq 1$ are obtained. They are called equations of order i and $\mathbf{U}^i$ are the asymptotic approximations of order i. A special meaning must be done for $\mathbf{U}^0$. It is taken as the initial, unperturbed state, where no small parameter occurs, and, so, no $\mathcal{S}$ exists.
In the following sections we deal with an application of double scale method to the study of asymptotic waves in a particular rheological medium which undergoes elastic and anelastic deformations.

3 Inelastic Deformations

In some previous papers (25)-(31), a theory for mechanical phenomena, which is based on the thermodynamics of irreversible processes, was developed. In particular in (27), it was assumed that several microscopic phenomena occur, which give rise to inelastic deformations, such that the tensor of the total strain $\varepsilon_{\alpha\beta}$, reads $\varepsilon_{\alpha\beta} = \varepsilon^{el}_{\alpha\beta} + \varepsilon^{in}_{\alpha\beta}$, where the tensors $\varepsilon^{el}_{\alpha\beta}$ and $\varepsilon^{in}_{\alpha\beta}$ describe the elastic and inelastic strains, respectively. Contrary to the elastic strains, the inelastic deformations are due to the effects of the lattice defects (e.g. slip, dislocations) and to the influence of microscopic stress fields, surrounding imperfections in the medium and giving rise to memory effects on the mechanical and thermodynamic behavior of the medium. Experiments show that there exist several types of such independent and simultaneous contributions to the inelastic strain, so that, assuming that they are of n different types, then

$$\varepsilon^{in}_{\alpha\beta} = \sum_{k=1}^{n} \varepsilon^{(k)}_{\alpha\beta}. \tag{3.1}$$

Remark that n is arbitrary as postulated by Kluitenberg (27).

The contributions $\varepsilon_{\alpha\beta}^{(k)}$ can be introduced as tensorial *internal variables* in the expression of the specific entropy of the system $s = s(e, \varepsilon_{\alpha\beta}^{(el)}, \varepsilon_{\alpha\beta}^{(1)}, ..., \varepsilon_{\alpha\beta}^{(n)})$, where e is the specific internal energy. In this theory $\varepsilon_{\alpha\beta}$ is assumed to be small, i. e. $\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial}{\partial x^k}u_i + \frac{\partial}{\partial x^i}u_k\right)$, where u_i is the i-th component of the displacement field **u** and x^i is the i-th component of the position vector **x** in Eulerian coordinates in a Cartesian reference frame. The first law of thermodynamics reads

$$\rho\frac{de}{dt} = -div\mathbf{J}^{(q)} + \sum_{\alpha,\beta=1}^{3}\tau_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}, \tag{3.2}$$

where ρ is the mass density of the body, $\mathbf{J}^{(q)}$ is the heat flux and $\tau_{\alpha\beta}$ is the Cauchy mechanical stress tensor. The tensors $\tau_{\alpha\beta}$, $\varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha\beta}^{(el)}$, $\varepsilon_{\alpha\beta}^{(k)}$ $(k = 1, 2, ..., n)$ are assumed to be symmetric. According to the usual procedure of non equilibrium thermodynamics, by virtue of the entropy principle, in (27) the phenomenological equations for anisotropic and isotropic media were obtained. In particular, in the case of isotropic media, assuming that the equations of state may be linearized and the phenomenological coefficients may be regarded as constants, an explicit form for the stress strain relation was derived, which has the form of a linear relation among the deviators of the mechanical stress tensor $\tau_{\alpha\beta}$, the first n derivatives with respect to time of this tensor, the tensor of total strain $\varepsilon_{\alpha\beta}$ and the first $n+1$ derivatives with respect to the time of the tensor of total strain, where n is the number of phenomena that give rise to inelastic deformations. The well-known Burgers equation is a special case of this relation for $n = 2$, i.e. when only two internal variables of mechanical origin are taken into consideration. The rheological relations for ordinary viscous fluids, for thermoelastic media and for Maxwell, Kelvin (Voigt), Jeffreys, Poyting-Thomson, Prandtl-Reuss, Bingham, Saint Venant and Hooke media are special cases of these more general mentioned above relations too (27)-(31).

4 Equations Governing the Motion in Maxwell Media

Assume that only one microscopic phenomenon gives rise to inelastic strain, Then, in the isotropic case, the stress strain relations describing the behaviour of anelastic media of order one ($n = 1$, i.e. when only one tensorial internal variable of mechanical origin is taken into consideration) without memory (Maxwell media), can be written in the following form (see (1)).

$$R_{(d)_0}^{(\tau)}\tilde{P}_{ik} + \frac{d}{dt}\tilde{P}_{ik} + R_{(d)_1}^{(\varepsilon)}\frac{d}{dt}\tilde{\varepsilon}_{ik} = 0, \qquad R_{(v)_0}^{(\tau)}P' + \frac{d}{dt}P' + R_{(v)_1}^{(\varepsilon)}\frac{d}{dt}\varepsilon = 0, \tag{4.1}$$

where $\tilde{P}_{ik}$ and P are the deviator and the scalar part of the mechanical pressure tensor P_{ik} and $\tilde{\varepsilon}_{ik}$ and ε are the deviator and the scalar part of the strain tensor ε_{ik}, respectively. We define P_{ik} in terms of the symmetric Cauchy stress tensor $P_{ik} = -\tau_{ik}$ $(i, k = 1, 2, 3)$, and

$$\tilde{P}_{ik} = P_{ik} - \frac{1}{3}P_{ss}\delta_{ik}, \quad P = \frac{1}{3}P_{ss}, \quad P_{ss} = trP,$$

$$P_{ik} = \tilde{P}_{ik} + P\delta_{ik}, \quad \tilde{P}_{ss} = 0, \quad P' = P - P_0 = -(\tau - \tau_0),$$

where τ_0 and P_0 are the scalar parts of τ_{ik} and P_{ik}, respectively, in a state of thermodynamic equilibrium. The coefficients in (4.1) satisfy the relations

$$R_{(d)_0}^{(\tau)} = a^{(1,1)}\eta_s^{(1,1)} \geq 0, \quad R_{(d)_1}^{(\varepsilon)} = a^{(1,1)} \geq 0,$$
$$R_{(v)_0}^{(\tau)} = b^{(1,1)}\eta_v^{(1,1)} \geq 0, \quad R_{(v)_1}^{(\varepsilon)} = b^{(1,1)} \geq 0, \tag{4.2}$$

where $a^{(1,1)}$ and $b^{(1,1)}$ are scalar constants which occur in the equation of state, while the coefficients $\eta_s^{(1,1)}$ and $\eta_v^{(1,1)}$ are called the *phenomenological coefficients* and represent fluidities.

In (1) the propagation of asymptotic waves was studied using a general method devoted to oscillatory approximate solutions for first order quasilinear hyperbolic systems. In particular, the evolution equation for the first perturbation term was derived and the equation of the wavefront was determined. In this paper some results are revised from the point of view of the double-scale method, some others are new. An application to the one-dimensional case is carried out.

The balance equations for the mass density and momentum in the case of Maxwell media read

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \qquad (i = 1, 2, 3) \tag{4.3}$$

$$\rho \left(\frac{\partial}{\partial t} v_i + v_k \frac{\partial}{\partial x^k} v_i \right) + \frac{\partial}{\partial x^k} \tilde{P}_{ik} + \frac{\partial}{\partial x_i} P = 0. \tag{4.4}$$

Here $v_i = \frac{du_i}{dt}$ is the i-th component of the velocity field. Then, equations (4.1) become

$$a^{(1,1)} \eta_{(s)}^{(1,1)} \tilde{P}_{ik} + \frac{\partial}{\partial t} \tilde{P}_{ik} + v_p \frac{\partial}{\partial x_p} \tilde{P}_{ik} + \frac{1}{2} a^{(1,1)} \left(\frac{\partial}{\partial x_k} v_i + \frac{\partial}{\partial x_i} v_k \right) - \frac{1}{3} a^{(1,1)} \frac{\partial}{\partial x_p} v_p \delta_{ik} = 0, \tag{4.5}$$

$$b^{(1,1)} \eta_{(v)}^{(1,1)} P' + \frac{\partial}{\partial t} P' + v_p \frac{\partial}{\partial x_p} P' + \frac{1}{3} b^{(1,1)} \frac{\partial}{\partial x_p} v_p = 0, \tag{4.6}$$

where the relation $\frac{d\varepsilon_{ik}}{dt} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$ is used.

Eqs. (4.3)-(4.6) form a system of ten quasi-linear first order PDEs for mass density, three components of the velocity field, five independent components of $\tilde{P}_{ik}$ and the scalar part of the mechanical pressure tensor. Let

$$\begin{aligned}
\mathbf{U} &= \left(\begin{array}{cccccccccc} \rho, & v_1, & v_2, & v_3, & \tilde{P}_{11}, & \tilde{P}_{12}, & \tilde{P}_{13}, & \tilde{P}_{22}, & \tilde{P}_{23}, & P' \end{array} \right)^T, \\
\mathbf{B} &= \left(\begin{array}{cccccccccc} 0, & 0, & 0, & 0, & \tilde{P}^*_{11}, & \tilde{P}^*_{12}, & \tilde{P}^*_{13}, & \tilde{P}^*_{22}, & \tilde{P}^*_{23}, & P'^* \end{array} \right)^T, \\
\mathbf{U}_\alpha &= \frac{\partial \mathbf{U}}{\partial x_\alpha}, \qquad \tilde{P}^*_{ik} = -a^{(1,1)} \eta_{(s)}^{(1,1)} \tilde{P}_{ik}, \qquad P'^* = -b^{(1,1)} \eta_{(v)}^{(1,1)} P',
\end{aligned} \tag{4.7}$$

where $\alpha = 0, 1, 2, 3$ and $i, k = 1, 2, 3$. Then, the system (4.3)-(4.6) becomes a particular form of (2.6), namely

$$\mathbf{A}^\alpha(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x^\alpha} = \mathbf{B}(\mathbf{U}), \qquad (\alpha = 0, 1, 2, 3) \tag{4.8}$$

where n=3, $x_0 = t$, $\mathbf{A}^0(\mathbf{U}) = \mathbf{I}$ is the identity matrix, while the 10×10 square matrices $A^1(\mathbf{U})$, $A^2(\mathbf{U})$ and $A^3(\mathbf{U})$ are

$$A^1 = \begin{pmatrix}
v_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & \frac{1}{\rho} \\
0 & 0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
0 & \frac{2}{3}a & 0 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}a & 0 & 0 & v_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & v_1 & 0 & 0 & 0 \\
0 & -\frac{1}{3}a & 0 & 0 & 0 & 0 & 0 & v_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & 0 \\
0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1
\end{pmatrix},$$

$$A^2 = \begin{pmatrix}
v_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_2 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & v_2 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & \frac{1}{\rho} \\
0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\
0 & 0 & -\frac{1}{3}a & 0 & v_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}a & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3}a & 0 & 0 & 0 & 0 & v_2 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & v_2 & 0 \\
0 & 0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & 0 & v_2
\end{pmatrix},$$

$$
A^3 = \begin{pmatrix}
v_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_3 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\
0 & 0 & 0 & v_3 & -\frac{1}{\rho} & 0 & 0 & -\frac{1}{\rho} & 0 & \frac{1}{\rho} \\
0 & 0 & 0 & -\frac{1}{3}a & v_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3}a & 0 & 0 & 0 & v_3 & 0 & 0 \\
0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & 0 & v_3 & 0 \\
0 & 0 & 0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & v_3
\end{pmatrix}.
$$

Consequently the motion in Maxwell media is governed by the Cauchy problem for the system (2.3), where $\mathbf{U}$ and $\mathbf{B}$ are defined by (4.7).

Before performing the series expansion of the solution let us determine the matrix $\mathbf{A}_n$ and its eigenvalues and eigenvectors.

5 The Eigenvalues and Eigenvectors of the Matrix $\mathbf{A}_n$

The expression (2.4) for $\mathbf{A}_n(\mathbf{U})$ becomes

$$
\mathbf{A}_n(\mathbf{U}) = \mathbf{A}^i n_i =
$$

$$
\begin{pmatrix}
v_n & \rho n_1 & \rho n_2 & \rho n_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_n & 0 & 0 & \frac{n_1}{\rho} & \frac{n_2}{\rho} & \frac{n_3}{\rho} & 0 & 0 & \frac{n_1}{\rho} \\
0 & 0 & v_n & 0 & 0 & \frac{n_1}{\rho} & 0 & \frac{n_2}{\rho} & \frac{n_3}{\rho} & \frac{n_2}{\rho} \\
0 & 0 & 0 & v_n & -\frac{n_3}{\rho} & 0 & \frac{n_1}{\rho} & -\frac{n_3}{\rho} & \frac{n_2}{\rho} & \frac{n_3}{\rho} \\
0 & \frac{2}{3}\, a^{(1,1)} n_1 & -\frac{a^{(1,1)}}{3} n_2 & -\frac{a^{(1,1)}}{3} n_3 & v_n & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a^{(1,1)}}{2} n_2 & \frac{a^{(1,1)}}{2} n_1 & 0 & 0 & v_n & 0 & 0 & 0 & 0 \\
0 & \frac{a^{(1,1)}}{2} n_3 & 0 & \frac{a^{(1,1)}}{2} n_1 & 0 & 0 & v_n & 0 & 0 & 0 \\
0 & -\frac{a^{(1,1)}}{3} n_1 & \frac{2}{3}\, a^{(1,1)} n_2 & -\frac{a^{(1,1)}}{3} n_3 & 0 & 0 & 0 & v_n & 0 & 0 \\
0 & 0 & \frac{a^{(1,1)}}{2} n_3 & \frac{a^{(1,1)}}{2} n_2 & 0 & 0 & 0 & 0 & v_n & 0 \\
0 & \frac{b^{(1,1)}}{3} n_1 & \frac{b^{(1,1)}}{3} n_2 & \frac{b^{(1,1)}}{3} n_3 & 0 & 0 & 0 & 0 & 0 & v_n
\end{pmatrix}.
$$

In (30) the eigenvalues of $\mathbf{A}_n(\mathbf{U})$ were found to be

$$
\lambda_1 = \mathbf{v}\cdot\mathbf{n} = v_n, \quad \lambda_2^{(\pm)} = v_n \pm \sqrt{\frac{a^{(1,1)}}{2\rho}}, \quad \lambda_3^{(\pm)} = v_n \pm \sqrt{\frac{2a^{(1,1)} + b^{(1,1)}}{3\rho}}, \tag{5.1}
$$

where the multiplicity of λ_1 is equal to 4, and the multiplicity of each among $\lambda_2^{(+)}$ and $\lambda_2^{(-)}$ is equal to 2. Obviously $\lambda_3^{(\pm)}$ are simple eigenvalues. Moreover, the discontinuity waves which are propagated with velocities satisfying $(5.1)_1$ and $(5.1)_2$ obey the Lax-Boillat exceptionality condition, while the discontinuty waves whose velocities of propagation $\lambda_3^{(\pm)}$ do not possess this property after some time generate shock waves ((30)).
Now, in this paper we see that the left eigenvectors $\mathbf{l}$ and the right eigenvectors $\mathbf{r}$ corresponding to $\lambda_2^{(\pm)}$ or $\lambda_3^{(\pm)}$ have the form

$$
\mathbf{l} \equiv \Bigg(0, \frac{6(v_n-\lambda)^2\rho - 3a}{n_1 n_3(a+2b)} - \frac{n_2^2+n_3^2}{n_1 n_3}, \frac{n_2}{n_3}, 1, \frac{\{-6(v_n-\lambda)^2\rho + 3a\}}{\rho n_3(a+2b)(v_n-\lambda)} + \frac{n_2^2+2n_3^2}{\rho(v_n-\lambda)n_3},
$$

$$
\frac{n_2\{-6(v_n-\lambda)^2\rho+3a\}}{\rho n_1 n_3(a+2b)(v_n-\lambda)} + \frac{n_2(n_2^2+n_3^2-n_1^2)}{\rho n_1 n_3(v_n-\lambda)},
$$

$$
\frac{n_3\{-6(v_n-\lambda)^2\rho+3a\}}{\rho n_1 n_3(a+2b)(v_n-\lambda)} + \frac{(n_2^2+n_3^2-n_1^2)}{\rho n_1(v_n-\lambda)}, \frac{n_3^2-n_2^2}{\rho n_3(v_n-\lambda)},
$$

$$-\frac{2n_2}{\rho(v_n-\lambda)}, -\frac{n_1\{6(v_n-\lambda)^2\rho-3a\}}{\rho n_1 n_3(a+2b)(v_n-\lambda)}\Bigg), \tag{5.2}$$

$$\mathbf{r} \equiv \Bigg(\frac{3\rho\{-2\rho(v_n-\lambda)^2+a\}}{n_1(a+2b)(v_n-\lambda)}, 1, \frac{\{6\rho(v_n-\lambda)^2-3a\}}{n_1n_2(a+2b)} - \frac{(n_1^2+n_3^2)}{n_1n_2}, \frac{n_3}{n_1},$$
$$-\frac{an_1}{v_n-\lambda} + \frac{\{2a\rho(v_n-\lambda)^2-a^2\}}{n_1(a+2b)(v_n-\lambda)}, \frac{a(a+2b)(n_3^2+n_1^2-n_2^2)-6a\rho(v_n-\lambda)^2+3a^2}{2n_2(a+2b)(v_n-\lambda)},$$
$$-\frac{an_3}{v_n-\lambda}, \frac{2a\{-2\rho(v_n-\lambda)^2+a\}}{n_1(a+2b)(v_n-\lambda)} + \frac{a(n_3^2+n_1^2)}{n_1(v_n-\lambda)},$$
$$\frac{3an_3\{-2\rho(v_n-\lambda)^2+a\}}{2n_1n_2(a+2b)(v_n-\lambda)} + \frac{an_3(n_3^2-n_2^2+n_1^2)}{2n_1n_2(v_n-\lambda)}, \frac{b\{-2\rho(v_n-\lambda)^2+a\}}{n_1(a+2b)(v_n-\lambda)}\Bigg) \tag{5.3}$$

such that they satisfy the relation

$$\mathbf{l}\cdot\mathbf{r} = \frac{2}{n_3n_1}. \tag{5.4}$$

This condition ensures the hyperbolity of the system (2.6) in its particular form (4.8) corresponding to our case of interest. Here, a and b stand for $a^{(1,1)}$ and $b^{(1,1)}$.
We are interested only in the progressive fast longitudinal wave traveling to the right with velocity $\lambda_3^{(+)}$. Denoting $\gamma = \sqrt{\frac{2a^{(1,1)}+b^{(1,1)}}{3\rho}}$ we have
$\lambda_3^{(+)} = v_n + \gamma$ and the corresponding eigenvectors (5.2) and (5.3) read

$$\mathbf{l}_3^{(+)} = \frac{1}{\gamma}\left(0, \gamma\frac{\mathbf{n}}{n_3}, \frac{(n_1^2-n_3^2)}{\rho n_3}, 2\frac{n_1n_2}{\rho n_3}, 2\frac{n_1}{\rho}, \frac{n_2^2-n_3^2}{\rho n_3}, 2\frac{n_2}{\rho}, \frac{1}{\rho n_3}\right), \tag{5.5}$$

$$\mathbf{r}_3^{(+)} = \frac{1}{\gamma}\left(\frac{\rho}{n_1},\ \gamma\frac{\mathbf{n}}{n_1},\ \frac{a}{3}\frac{(3n_1^2-1)}{n_1},\ an_2,\ an_3,\ \frac{a}{3}\frac{(3n_2^2-1)}{n_1},\ a\frac{n_2n_3}{n_1},\ \frac{b}{3n_1}\right). \tag{5.6}$$

6 Equations of First and Second Asymptotic Approximation

Let us now apply the double-scale method to (2.3) written in its particular form (4.8). To this aim all the quantities, depending on x^α, are considered as depending on x^α and ξ. Consequently, the derivative $\frac{\partial}{\partial x^\alpha}$ must be replaced by $\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x^\alpha}$, i. e. by $\frac{\partial}{\partial x^\alpha} + \frac{1}{\omega}\frac{\partial}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha}$. Then, let us choose $\mathbf{U} = \mathbf{U}(x^\alpha, \xi)$ in the form of an asymptotic series with respect to the asymptotic sequence $1, \omega^{-1}, \omega^{-2}, ...$, as $\omega^{-1} \to 0$, namely

$$\mathbf{U}(x^\alpha,\xi) \sim \mathbf{U}^0(x^\alpha,\xi) + \omega^{-1}\mathbf{U}^1(x^\alpha,\xi) + \omega^{-2}\mathbf{U}^2(x^\alpha,\xi) + ... \tag{6.1}$$

and introduce (6.1) in (4.8) to obtain the following equations of first and second asymptotic approximation of (4.8)

$$A_0^\alpha \frac{\partial\varphi}{\partial x^\alpha}\frac{\partial \mathbf{U}^1}{\partial\xi} = 0, \tag{6.2}$$

$$A_0^\alpha \frac{\partial\varphi}{\partial x^\alpha}\frac{\partial \mathbf{U}^2}{\partial\xi} = -\left[A_0^\alpha\frac{\partial\mathbf{U}^1}{\partial x^\alpha} + \nabla A_0^\alpha \mathbf{U}^1\left(\frac{\partial \mathbf{U}^1}{\partial\xi}\frac{\partial\varphi}{\partial x^\alpha}\right) - (\nabla\mathbf{B}_0)\mathbf{U}^1\right], \tag{6.3}$$

where $\mathbf{U}^0 = \left(\ \rho^0, 0, 0, 0, 0, 0, 0, 0, 0, P^0\ \right)$ is the unperturbed constant solution, $\mathbf{A}_0^\alpha = \mathbf{A}^\alpha(\mathbf{U}^0)$ and $\mathbf{B}_0 = \mathbf{B}(\mathbf{U}^0)$. Equation (6.2) also reads

$$(\mathbf{A}_{0n} - \lambda I)\frac{\partial \mathbf{U}^1}{\partial\xi} = 0, \quad where \quad \mathbf{A}_{0n} = \mathbf{A}_n(\mathbf{U}^0), \tag{6.4}$$

and shows that $\frac{\partial \mathbf{U}^1}{\partial\xi}$ can be taken as equal to the right-eigenvector **r** of $\mathbf{A}_{0n}$, corresponding to some eigenvalue λ.

By integration, it follows that $\mathbf{U}^1(x_\alpha,\xi)$ has the form

$$\mathbf{U}^1(x_\alpha,\xi) = u(x_\alpha,\xi)\ \mathbf{r}(\mathbf{U}^0,\mathbf{n}) + \mathbf{v}^1(x_\alpha). \tag{6.5}$$

It can be proved that $\mathbf{v}^1$ can be taken zero (see (7)). Consequently, in order to determine $\mathbf{U}^1$ we must determine u.

Remark that u is a function which can be determined from the equation of the second approximation (6.3). More precisely, let us write (6.3) in the form

$$\mathbf{A}_0^\alpha \frac{\partial \varphi}{\partial x^\alpha} \frac{\partial \mathbf{U}^2}{\partial \xi} = -\mathbf{s}_1, \quad or, \quad equivalently, \quad (\mathbf{A}_{0n} - \lambda I) \frac{\partial \mathbf{U}^2}{\partial \xi} = -\mathbf{s}_1,$$

and multiply this last equation by the left eigenvector $\mathbf{l}^0$ (corresponding to $\mathbf{U}^o$) to obtain

$$0 = -\mathbf{l}^0 \cdot \mathbf{s}_1. \tag{6.6}$$

Since $\mathbf{s}_1$ depends on $\mathbf{U}^1$, therefore on u, it follows that equation (6.6) yields u. However, defined as the right-hand side of (6.3), s_1 depends on φ, i. e. it depends on the equation of the wavefront $\mathcal{S}$, which, so far is not determined. This is why, first we must determine a first approximation for $\mathcal{S}$. This was minutely done in (1) by using the Boillat method (7). Among the 10 eigenvalues λ, we choose $\lambda_3^{(+)}$, which is simple, corresponds to the fastest progressive longitudinal wave traveling to the right. Eq. (6.4) shows that $\frac{\partial \mathbf{U}^1}{\partial \xi}$ is a right eigenvector of $\mathbf{A}_{0n}$ corresponding to the eigenvalue $\lambda_3^{(+)}$. For various other physical and related mathematical aspects of asymptotic waves the reader is kindly referred to (31)-(35).

7 First Approximation of Wavefront and of U

Let the superscript "o " stand for values taken for $\mathbf{U}^0$ and denote $\phi^i = \frac{\partial \varphi}{\partial x^i}$. Then for the PDE $\Psi^0 = 0$ (where Ψ is defined by (2.9)), it follows the characteristic equations $\frac{dx^i}{d\sigma} = \frac{\partial \Psi^0}{\partial \phi^i}$, $\frac{d\phi^i}{d\sigma} = -\frac{\partial \Psi^0}{\partial x^i}$, $i = 1, 2, 3$, where $\sigma = x^0 = t$ is the time along the rays. Then taking into account the expression of $\mathbf{U}^0$ from (4.7) and λ from (2.9), it follows that ϕ^i and so, $\mathbf{n}$, are constant along the characteristic rays. Next, from the equality (2.9) and taking into account that

$$\Lambda_i(\mathbf{U}, \mathbf{n}) = \frac{\partial \Psi}{\partial \phi_i} = \lambda n_i + \frac{\partial \lambda}{\partial n_i} - \left(\mathbf{n} \cdot \frac{\partial \lambda}{\partial \mathbf{n}}\right) n_i = \lambda n_i + v_i - (n_k v_k) n_i,$$

hence, $\boldsymbol{\Lambda}(\mathbf{U}, \mathbf{n}) = \mathbf{v} - (v_n - \lambda)\mathbf{n}$.
Then $\mathbf{x} = \mathbf{x}|_{t=0} - \boldsymbol{\Lambda}_0 t$, and, since the Jacobian $J = \theta^2$ of the transformation $\mathbf{x} \to \mathbf{x}|_{t=0}$ is nonvanishing, it follows that, in the first approximation, $\varphi(t, x_i) = \varphi^0(x_i - \Lambda_i^0 t)$, where φ^0 is the value of φ at t = 0.
The equation (6.6) reads

$$\frac{\partial u}{\partial \sigma} + (\nabla \Psi \cdot \mathbf{r})^0 u \frac{\partial u}{\partial \xi} + \frac{1}{\theta} \frac{\partial}{\partial \sigma} u = \nu^0 u, \tag{7.1}$$

where

$$\nu = \frac{\mathbf{l} \cdot \nabla \mathbf{B} \cdot \mathbf{r}}{\mathbf{l} \cdot \mathbf{r}}. \tag{7.2}$$

Using the general expressions (5.2) and (5.3) for $\mathbf{l}$ and $\mathbf{r}$ we obtain for all these eigenvectors ν is given by

$$\nu = -\frac{\left(2(a^{(1,1)})^2 \eta_s^{(1,1)} + (b^{(1,1)})^2 \eta_v^{(1,1)}\right) n_1}{2(2a^{(1,1)} + b^{(1,1)})}, \tag{7.3}$$

and, furthermore, it is independent on the unperturbed state $\mathbf{U}^0$.
For $\lambda = \lambda_3^{(+)}$ it follows that $\boldsymbol{\Lambda}_0 = \gamma_0 \mathbf{n}^0$ and in (7.1) straightforward computations give

$$\left(\nabla \Psi \cdot \mathbf{r}_3^{(+)}\right)^0 = |grad\varphi|^0 \left(\nabla \lambda_3^{(+)} \cdot \mathbf{r}_3^{(+)}\right)^0 = |grad\varphi|^0 \frac{1}{2n_1},$$

$$where \quad \nabla \lambda_3^{(+)} = \frac{\partial \lambda_3^{(+)}}{\partial \mathbf{U}} \equiv \left(-\frac{1}{2\rho}\gamma, n_1, n_2, n_3, 0, 0, 0, 0, 0, 0\right).$$

Let $w = \nu^0 \sigma$, $u = \frac{v}{\theta} e^w$ and denote $k = \int_0^\sigma 0.5 |grad\varphi|^0 \frac{e^w}{\theta} d\sigma$.
By using this transformation of variables eq. (7.1) can be reduced to an equation of the type $\frac{\partial v}{\partial k} + v\frac{\partial v}{\partial \xi} = 0$.
Therefore, for the initial condition $u|_{t=0} = F(x_i^0, \xi^0)$, where $\xi^0 = \omega \varphi^0(x_i^0)$ it follows $v = F(x_i^0, \xi - vk)$. This ends the determination of u and, so, of $\mathbf{U}^1$ in the caxe of $\lambda = \lambda_3^{(+)}$.

8 One-Dimensional Case

Consider the system of equations (4.3)-(4.6). Assume that $v_2 = v_3 = 0$, $x_2 = x_3 = 0$ and that the involved physical quantities depend only on x_1, denoted by x. Denote $v_1(x,t)$ by v. Denote by D_{ik} and P the components of the deviator and the scalar part of the mechanical pressure tensor P_{ik}, respectively. Finally, let a and b stand for $a^{(1,1)}$and $b^{(1,1)}$ and let η^1 and η^2 stand for $\eta_s^{(1,1)}$ and $\eta_v^{(1,1)}$, respectively. Then the system (4.3)-(4.6) becomes

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \tag{8.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial D_{11}}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0, \tag{8.2}$$

$$\frac{\partial D_{21}}{\partial x} = 0, \tag{8.3}$$

$$\frac{\partial D_{31}}{\partial x} = 0, \tag{8.4}$$

$$\frac{\partial D_{11}}{\partial t} + v\frac{\partial D_{11}}{\partial x} + \frac{2}{3}a\frac{\partial v}{\partial x} + a\eta^1 D_{11} = 0, \tag{8.5}$$

$$\frac{\partial D_{12}}{\partial t} + v\frac{\partial D_{12}}{\partial x} + a\eta^1 D_{12} = 0, \tag{8.6}$$

$$\frac{\partial D_{13}}{\partial t} + v\frac{\partial D_{13}}{\partial x} + a\eta^1 D_{13} = 0, \tag{8.7}$$

$$\frac{\partial D_{22}}{\partial t} + v\frac{\partial D_{22}}{\partial x} - \frac{1}{3}a\frac{\partial v}{\partial x} + a\eta^1 D_{22} = 0, \tag{8.8}$$

$$\frac{\partial D_{23}}{\partial t} + v\frac{\partial D_{23}}{\partial x} + a\eta^1 D_{23} = 0, \tag{8.9}$$

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial x} + \frac{1}{3}b\frac{\partial v}{\partial x} + b\eta^2 P = 0, \tag{8.10}$$

where $D_{ik} = D_{ki}$.

Thus, equs. (8.3) and (8.4) show that

$$D_{21} = f(t), \qquad D_{31} = f_1(t),$$

where f and f_1 are functions of t. Therefore, we have

$$D_{12} = e^{-a\eta^1 t} + D_{12}^0, \qquad D_{13} = e^{-a\eta^1 t} + D_{13}^0.$$

Remark that, due to the presence of a tensorial internal variable, there is a response time of the medium possessing mechanical relaxation properties. Then, the remained system reads

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{B},$$

where

$$\mathbf{U} = (\rho, v, D_{11}, D_{22}, D_{23}, P)^T, \quad \mathbf{B} = (0, 0, -a\eta^1 D_{11}, -a\eta^1 D_{22}, -a\eta^1 D_{23}, -b\eta^2 P),$$

$$A=\begin{pmatrix} v & \rho & 0 & 0 & 0 & 0\\ 0 & v & \frac{1}{\rho} & 0 & 0 & \frac{1}{\rho}\\ 0 & \frac{2}{3}a & v & 0 & 0 & 0\\ 0 & -\frac{1}{3}a & 0 & v & 0 & 0\\ 0 & 0 & 0 & 0 & v & 0\\ 0 & \frac{b}{3} & 0 & 0 & 0 & v \end{pmatrix}$$

which has the eigenvalues:

- $\lambda_1 = v$ (of multiplicity equal to 4);

- the simple eigenvalues $\lambda_3^{(\pm)} = v \pm \gamma$. The right eigenvectors corresponding to $\lambda_3^{(\pm)}$ can be taken as

$$\mathbf{r}_3^{(\pm)} = (\rho, -(v-\lambda_3^{(\pm)}),\ \frac{2}{3}a, -\frac{a}{3},\ 0,\ (v-\lambda_3^{(\pm)})^2\rho - \frac{2}{3}a)^T. \tag{8.11}$$

The left eigenvectors are taken as

$$\mathbf{l}_3^{(\pm)} = \left(0, -\left(v-\lambda_3^{(\pm)}\right),\ \frac{1}{\rho},\ 0,\ 0,\ \frac{1}{\rho}\right). \tag{8.12}$$

This last result formally follows from (5.2) but multiplied by n_3 which is null. Therefore, direct computations were necessary to be done to derive eqs.(8.11) and (8.12). But, formally it is possible to obtain eq. (8.11) from (5.3) multiplied by $v_n - \lambda_3^{(\pm)}$.

Expressions (8.11) and (8.12) are true for every λ.
Let us consider only the longitudinal wave traveling in the right direction and the case where the propagation is in a constant state $\mathbf{U}^0$, i. e.

$$\lambda_3^{(+)} = v + \gamma_0 \quad and \quad \mathbf{U}^0 = (\rho_0,\ 0,\ 0,\ 0,\ 0,\ P_0),$$

where ρ_0 and P_0 are constants and $\gamma_0 = \gamma(\mathbf{U}^0)$. The characteristic rays are

$$x = \sigma, \quad x = x_0 + \lambda_3^{(+)}(\mathbf{U}^0)\sigma, \tag{8.13}$$

whence the wave front is

$$\varphi(t,x) = x(t) - \lambda_3^{(+)}\sigma + x_0, \tag{8.14}$$

implying $\varphi_x = 1$. Eqs. (8.13) and (8.14) are true for all λ.

In order to compute the terms in (7.1) we start with

$$\nabla\Psi \cdot r_3^{(+)} = \varphi_x(\nabla\lambda_3^{(+)} \cdot r_3^{(+)}).$$

Taking into account the expression of $\lambda_3^{(+)}$ for $\mathbf{U}^0$, we have

$$\nabla\lambda_3^{(+)} = \left(\frac{\partial\lambda_3^{(+)}}{\partial\rho}, \frac{\partial\lambda_3^{(+)}}{\partial v}, \frac{\partial\lambda_3^{(+)}}{\partial D_{11}}, \frac{\partial\lambda_3^{(+)}}{\partial D_{22}}, \frac{\partial\lambda_3^{(+)}}{\partial D_{23}}, \frac{\partial\lambda_3^{(+)}}{\partial P}\right) \tag{8.15}$$

$$(\nabla\lambda_3^{(+)})^0 = \left(-\frac{\gamma_0}{2\rho_0}, 1, 0, 0, 0, 0\right), \tag{8.16}$$

where $\gamma_0 = \sqrt{\frac{2a+b}{3\rho_0}}$.
Hence,

$$\left(\nabla\lambda_3^{(+)} \cdot \mathbf{r}_3^{(+)}\right)^0 = \frac{1}{2}\gamma_0$$

and, so,

$$(\nabla\Psi \cdot \mathbf{r}_3^{(+)})^0 = \frac{1}{2}\gamma_0.$$

Further, a direct easy computation gives

$$\left(l_3^{(+)} \cdot \nabla \mathbf{B} \cdot r_3^{(+)}\right)^0 = -\frac{1}{3\rho_0}[2(a)^2\eta^1 + (b)^2\eta^2]$$

and

$$\left(\mathbf{l}_3^{(+)} \cdot \mathbf{r}_3^{(+)}\right)^0 = 2\gamma_0,$$

implying

$$\nu^0 = -\frac{[2(a)^2\eta^1 + (b)^2\eta^2]}{2(2a+b)},$$

which in accordance with the result (7.3).

This example, in spite of its simplicity, shows just profond features of the influence of a tensorial internal variable on the behaviour of the medium motion.

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