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Beams on Foundation, Winkler Bedding or Halfspace – a Comparison

F.D. Fischer, E. Gamsjäger

Beams on a foundation as rails on the ground can be analysed along Winkler's concept introducing a bedding coefficient k. In reality most beams are supported by a halfspace which simplest can be described by overall elastic properties. Exact relations between k and the elastic properties are presented for infinite beams as rails.

1 Introduction and Problem Description

Winkler (1871) presented nearly 140 years ago a book on "Der Eisenbahn-Oberbau" (which can be translated as "Constructing a Permanent Way of Railroad"). He suggested to analyse a rail as an infinite beam with the stiffness $E_r J$ (E_r Young's modulus, J moment of inertia of the cross-section of the beam) subjected to a single load P at $x = 0$ and supported by a foundation being a continuous bedding transferring a line load $-kw$ with k being the bedding coefficient and w the displacement of the rail at a position x.

Relations for the maximum moment $M_{\max,b}$ and the maximum displacement $w_{\max,b}$ can be taken from the open literature, e. g. from Hetenyi (1946) or Gummert and Reckling (1986) and read as

$$w_{\max,b} = \frac{P\alpha}{2k} = \frac{0.353P}{k\ell},\ M_{\max,b} = \frac{P}{4\alpha} = 0.353P\ell. \tag{1}$$

The quantity $\alpha = \left(k/\left(4E_r J\right)\right)^{1/4}$ has the dimension (1 / length). The "fundamental length" defined as $\ell = \left(E_r J/k\right)^{1/4} = \sqrt{2}/\left(2\alpha\right)$ can be introduced and is widely used in the open literature. It is important to note that ℓ does not include the factor 4, which has been the origin of several errors. A further result is the length $L = \pi/\alpha$ along which (in the interval $-L/2 \le x \le L/2$) the total load P is equilibrated by the bedding support. This means that outside the interval L a self-equilibrating system of compressive and tension contact forces exist. Winkler's solution has some shortcomings for the actual wheel/rail/bedding configuration, namely:

- In reality a rail, represented as an infinite beam, is supported by a half space, which means that a three-dimensional configuration exists.
- The real bedding cannot transmit tension forces.

We concentrate on the first aspect only. Finally we will relate the quantity k to E and ν and check the applicability of Winkler's theory. We would like to mention that Biot (1937) dealt with this problem more than 70 years ago. Later Hetenyi (1946) discussed Biot's results in his book. However, a generally applicable conclusion was not given in both works. One of the reasons was obviously the lack of computer facilities at this time. The main thrust of this work is to present a complete solution relating the bedding concept with the halfspace concept. This topic is of high practical relevance.

2 Problem Solution

Biot (1937) introduced a width $2b$ of the beam in the transversal horizontal direction (y-direction) and a quantity C. Depending on the stiffness of the cross-section of the beam the pressure distribution transferred by the beam in transversal direction to the halfspace varies from the case of a uniform pressure (C = 1) to a pressure distribution according to a uniform deflection (C = 1.13). Finally a "fundamental length" c was introduced as

$$c = \left(\frac{E_r J}{E'b}\right)^{1/3}, E' = \frac{E}{C\left(1-\nu^2\right)}. \tag{2}$$

Biot found by a Fourier-integral analysis the following relation for the maximum Moment $M_{\max,b}$ as

$$M_{\max,b} = Pc \cdot \frac{1}{\pi}\int_0^\infty \frac{zdz}{z^3 + \bar{\psi}(\beta)} = Pb \cdot \frac{c}{b}\frac{I_M(b/c)}{\pi}. \tag{3}$$

with $\beta = bz/c$.

The function $\bar{\psi}(\beta)$ stems from a relation between the resulting force $\bar{Q}[N/m]$, transferred by the beam to the halfspace, and the average displacement $\bar{w}$ in the interval $-b \le y \le b$ at a distinct length coordinate *x*. Here the derivation of Biot is replaced by the relation

$$\frac{\bar{w}}{\bar{Q}} = E' \cdot \frac{2}{\pi}\int_0^\infty \frac{\sin^2 \gamma d\gamma}{\gamma^2\sqrt{\gamma^2 + \beta^2}} = E' \cdot \frac{1}{f(\beta)}. \tag{4}$$

The function $f(\beta)$ can be numerically calculated and interpolated as

$$f(\beta) = 0.63662 \cdot \ln\left[\frac{0.24 + \sqrt{0.24^2 + \beta^2}}{\beta}\right] + \frac{1.76461}{(1.49276+\beta)^{1.27025}}. \tag{5}$$

This function shows a logarithmic singularity at $\beta = 0$.
The function $\bar{\psi}(\beta)$ follows then as

$$\bar{\psi}(\beta) = \left[\beta \cdot f(\beta)\right]^{-1}. \tag{6}$$

One can imagine that handling of such integrals as (4) and in particular of the integral $I_M(b/c)$ in (3) without a computer can be nearly an unsolvable task.

If we look for a bedding coefficient *k*, which would lead to the same maximum moment as $M_{\max,b}$, (3), then we have to equalize $(1)_2$ with (3) after inserting of (5), (6) and obtain

$$\frac{k}{E'} = 1.522\frac{b}{c} \cdot \frac{1}{\left(I_M(b/c)\right)^4}. \tag{7}$$

The corresponding relation for $w_{\max}$ can be taken again from Biot after omitting some mistakes as

$$w_{\max} = \frac{P}{bE'} \cdot \frac{1}{\pi}\int_0^\infty \frac{dz}{z^4 + z\bar{\psi}(\beta)} = \frac{P}{bE'} \cdot \frac{I_w(b/c)}{\pi}. \tag{8}$$

The function $I_w(b/c)$ is a further integral which Biot and his team tried to evaluate by hand.
One can now calculate the ratio $w_{\max}/w_{\max,b}$ with $w_{\max,b}$ being the maximum displacement according to the Winkler concept, see $(1)_1$, with *k* expressed by E' and depending on b/c due to (7). After some analysis we find

$$\frac{w_{\max}}{w_{\max,b}} = \frac{1}{(0.353\pi)^{1/4}} \cdot \frac{I_w(b/c)}{\left(I_M(b/c)\right)^3}. \tag{9}$$

3 Discussion and Conclusion

Biot (1937) and later Hetenyi (1946) have come to the conclusion that *k* can be replaced by *E* to obtain the same value for the moment $M_{\max}$ either by the Winkler concept with *k* as bedding coefficient or that for an elastic halfspace with *E* as Young's modulus. This operation makes necessary an evaluation of the integral $I_M\left(b/c\right)$ and a following fitting of $I_M\left(b/c\right)$. Biot reported the relation $I_M = 1.043\cdot\left(c/b\right)^{0.831}$, which is, however, not a correct fitting! Specifically for small values of the ratio (*b*/*c*) Biot's fitting would run to infinity. We performed a fitting and found $I_M = 1.005\left(b/c\right)^{0.08907}$. Furthermore, the quantity *b* has to be replaced by *c* in equation (37) of Biot (1937).

Winkler's concept and the halfspace model is compared by means of the following figures. Figure 1 shows the relation k/E' as function of *b*/*c* according to (7). k/E' obtains the value 1 for *b*/*c* ~ 0.80. Since the integral $I_M\left(b/c\right)$ tends to a constant value of 1.18208, the rather simple relation $k/E' \sim 0.79\cdot b/c$ can be used for larger values of *b*/*c*. The ratio $w_{\max}/w_{\max,\mathrm{b}}$ obtains already for *b*/*c* << 1 the value of ~ 1.0 and increases rather slowly with increasing *b*/*c* to 2.0 for *b*/*c* = 10. The function $w_{\max}/w_{\max,b}$ is depicted in Fig. 2. In other words, the halfspace reacts softer compared to the Winkler bedding for a given value of the maximum moment $M_{\max,\mathrm{b}}$.

Figure 2.:

If one investigates a rail, e.g. the rail UIC 60, one finds for $E_\mathrm{r}J = 6.38.10^6$ Nm^2. Typical values of E,ν for the halfspace are reported from the literature in Fischer et al. (2005) and Fischer et al. (2008) as $E \sim 130.10^6$ N/m^2, $\nu = 0.2$. For a sleeper of the length *b* = 2.4 m and *C* ~ 1 one finds the "fundamental length" *c*, see (2), as 0.28 m and *b*/*c* ~ 8.5. For a much weaker ground with $E \sim 13.10^6$ N/m^2 the ratio *b*/*c* obtains ~ 4.0. The ratio k/E' ranges between 3.5 and 6.5 and does obtain the value of ~ 1.0, as one would assume from Biot's and Hetenyi's work.
If one is interested in the spring constant of a wheel/rail configuration, see e.g. Fischer et al. (2005) and (2008), one finds that for a given maximum moment $M_{\max,\mathrm{b}}$ in the rail the Winkler bedding concept yields a 1.5 to 1.8 times higher spring constant than the halfspace concept.

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Address: Prof. F.D. Fischer, Doz. Ernst Gamsjäger, Institut für Mechanik, Montanuniversität Leoben, A-8700 Leoben. email: mechanik@unileoben.ac.at, e.gamsjaeger@unileoben.ac.at

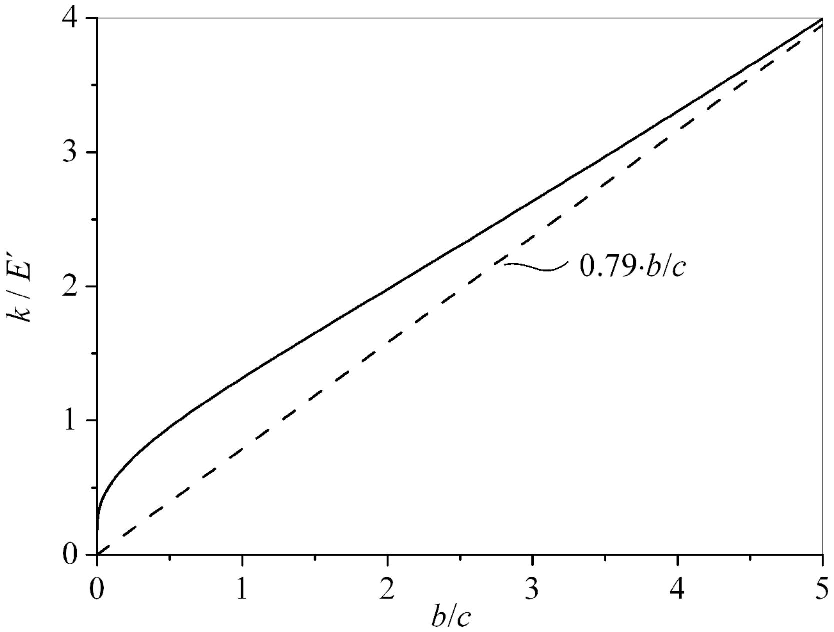

Figure 1. The Winkler bedding constant k, calculated from the elastic constants of a halfspace for a given maximum moment of a bedded beam subjected to a single load.

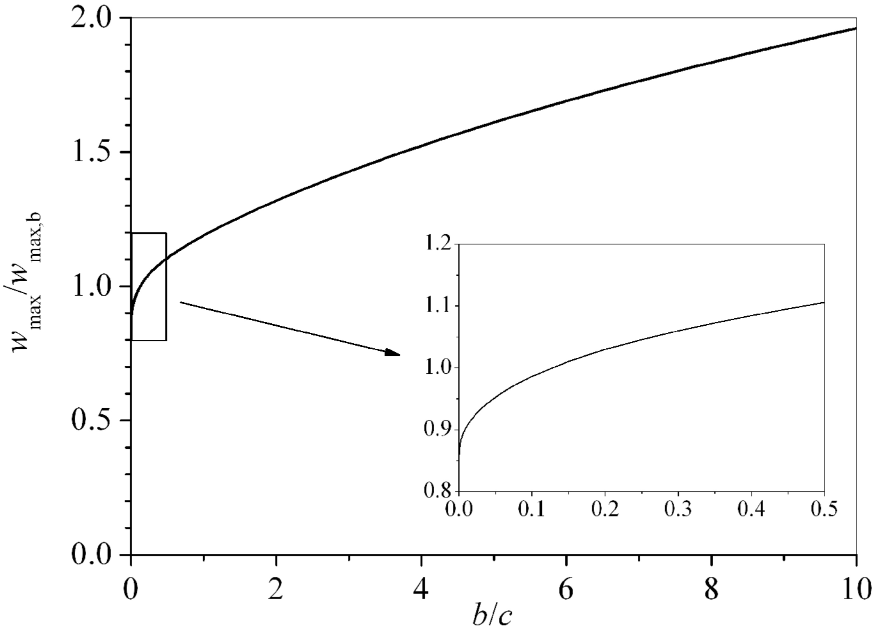

Figure 2. Ratio of the maximum deflection w_{max} (calculated from the halfspace concept) to $w_{max,b}$ (calculated from Winkler's concept) for a given maximum moment of a bedded beam subjected to a single load.