

Harmonic Wavelet Analysis of Nonlinear Waves

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A method based on a multiscale (wavelet) decomposition is proposed for the analysis of nonlinear waves in hyperelastic materials. The wave solution is approximated by a discrete series expansion with respect to the harmonic wavelets, and it is compared with the solution obtained by the method of successive approximations.

1 Introduction

This paper deals with the analysis by harmonic wavelets (Cattani, 2005; Muniandy and Moroz, 1997; Newland, 1993) of the nonlinear waves arising in hyperelastic materials (see e.g. Achenbach, 1973; Cattani and Rushchitsky, 2007; Dai and Zhao, 1999; Destrade, 2005; Drumheller, 1998; Germain, 1972; Hudson, 1980; Maugin, 1988, 2007; Porubov and Maugin, 2006; Wasley, 1973; Whitam, 1974). The evolution of an hyperelastic material depends on the potential which is a function of the three basic invariants of the chosen strain tensor (Green, Almansi, Hencky or others). Thus the corresponding fundamental equations can be derived from the given potential (Mooney-Rivlin, Rivlin-Saunders, John, Signorini, etc.) quadratic in the invariants. The cubic potential, proposed by Murnaghan, (see e.g. Murnaghan, 1951; Cattani and Rushchitsky, 2007) describes the largest class of industrial materials. From the Murnaghan potential follow some nonlinear equations suitable for studying the evolution of hyperelastic materials. However, due to their complexity, these equations can be numerically solved only in a few (very special) simple problems. The most common methods of solutions, based on physical considerations, are the Van der Pol method and the method of slowly changing amplitudes (see e.g. Cattani and Rushchitsky, 2007; Ratner, 2003). However, both assume strong constraints on the evolution which are compatible with the nonlinear hypotheses only under restricted conditions.

In the following, a method based on the multiscale wavelet analysis is proposed for the solution of a nonlinear equation and compared with the method of successive approximation.

Harmonic wavelets (Cattani, 2005; Muniandy and Moroz, 1997; Newland, 1993) are infinitely differentiable functions, analytically defined, and band-limited in the Fourier domain, thus being an optimal tool for the analysis of the evolution of localized pulse function. For an initial problem the evolutionary model is completely defined by the (time) evolution of the coefficients of the series. By using the Petrov-Galerkin method the PDE is transformed into an equivalent ordinary differential system for the wavelet coefficients. The projection of the operator into a given scale is performed by using the connection coefficients (Cattani, 2005, 2006; Dahmen, 2001; Latto et. al., 1992; Lin and Zhou, 2001; Restrepo and Leaf, 1995; Romine, 1997).

The projection, which can be considered like a numerical approximation into a given scale, is one of the most used methods to describe evolutionary operators, and to find an approximate solution of PDE equations (see e.g. Amaratunga et. al., 2000; Bacry et. al., 2004; Benhaid, 2007; Cattani 2003, 2003a; Chen et. al., 2002; Muniandy and Moroz, 1997; Qian and Weiss, 1993).

Due to this multiscale approach the approximation by wavelets is the best one for at least two reasons: the minimum set of coefficients (compression) to represent the phenomenon and the direct physical interpretation of its scale dependence. In each scale the wavelet coefficients and, in particular, the detail coefficients β_k^n describe "local" oscillations. Therefore wavelets seems to be the more expedient tool for studying these problems which are localized (in time or in frequency) and/or have some discontinuities. Moreover, since the approximation by wavelets is represented by a combination of uncorrelated functions they seems to be the most suitable tool for the analysis of nonlinear problems.

2 Nonlinear Waves in Hyperelastic Materials

In nonlinear hyperelastic isotropic materials (Cattani and Rushchitsky, 2007; Dorfmann and Ogden, 2006; Germain, 1972; Hudson, 1980; Maugin, 1988,2007; Porubov and Maugin, 2006; Villaggio, 1977) the internal energy is an analytic function of the strain tensor components, and the potential can be written as an analytical function of the three basic invariants (e.g. Green).

For hyperelastic materials the equations of motion

$$t_{ik,i} + X_k = \rho \ddot{u}_k \quad , \quad (i, k = 1, 2, 3) \quad (1)$$

are coupled with the Kirchoff stresses t_{ik} representation in terms of strain

$$t_{ik} = \frac{\partial W}{\partial u_{ik}} \quad (2)$$

with the Cauchy-Green strain tensor ε_{ik} given by the displacement $u(x, t) = \{u_1, u_2, u_3\}$ in the Lagrangian coordinates x_k , $k = 1, 2, 3$, as

$$\varepsilon_{ik} = \frac{1}{2} (u_{i,k} + u_{k,i} + u_{h,i} u_{h,k}) \quad . \quad (3)$$

In the linear elasticity theory, the stress tensor is given in the classical form of Hooke's law

$$t_{ik} = \lambda \varepsilon_{hh} \delta_{ik} + 2\mu \varepsilon_{ik}$$

and the strain tensor is linearly dependent on the $u_{i,k}$.

The nonlinear models instead depend on the choice of the potential $W = W(u_{ik})$ as a function of the invariants. By assuming the dependence on the quadratic and cubic main invariants, we have the main nonlinear quadratic and cubic models. One of the first quadratic models was proposed by Signorini in the 1940's for incompressible and compressible materials (such as rubberlike materials), by Mooney in the same years, and in the most general form in (1951) for incompressible materials by Rivlin and Saunders.

2.1 Murnaghan Potential

The Murnaghan potential is a cubic potential firstly proposed by Murnaghan (Murnaghan, 1951; Cattani and Rushchitsky, 2007) for the Green strain tensor ε_{ik} as

$$W(\varepsilon_{ik}) = \frac{1}{2} \lambda (\varepsilon_{mm})^2 + \mu (\varepsilon_{ik})^2 + \frac{1}{3} A \varepsilon_{ik} \varepsilon_{im} \varepsilon_{km} + B (\varepsilon_{ik})^2 \varepsilon_{mm} + \frac{1}{3} C (\varepsilon_{mm})^3 \quad , \quad (4)$$

or through the first algebraic invariants I_k of the tensor ε_{ik}

$$W(I_1, I_2, I_3) = \frac{1}{2} \lambda I_1^2 + \mu I_2 + \frac{1}{3} A I_3 + B I_1 I_2 + \frac{1}{3} C I_1^3 \quad .$$

Here λ, μ are the Lamé elastic constants (constants of the second order), A, B, C and are the Murnaghan elastic constants (constants of the third order).

By using (3) in (4) we get after some simplifications (Cattani and Rushchitsky, 2007)

$$\begin{aligned} W &= \frac{1}{2} \lambda (u_{m,m})^2 + \frac{1}{4} \mu (u_{i,k} + u_{k,i})^2 \\ &+ \left(\mu + \frac{1}{4} A \right) u_{i,k} u_{m,i} u_{m,k} + \frac{1}{2} (\lambda + B) u_{m,m} (u_{i,k})^2 \\ &+ \frac{1}{12} A u_{i,k} u_{k,m} u_{m,i} + \frac{1}{2} B u_{i,k} u_{k,i} u_{m,m} + \frac{1}{3} C (u_{m,m})^3 \quad , \end{aligned} \quad (5)$$

so that Eq. (2) gives

$$\begin{aligned}
t_{ik} &= \mu (u_{i,k} + u_{k,i}) + \lambda u_{k,k} \delta_{ik} \\
&+ \left(\mu + \frac{1}{4} A \right) (u_{l,i} u_{l,k} + u_{i,l} u_{k,l} + 2u_{l,k} u_{i,l}) \\
&+ \frac{1}{2} (B - \lambda) \left[(u_{m,l})^2 \delta_{ik} + 2u_{i,k} u_{l,l} \right] + \frac{1}{4} A u_{k,l} u_{l,i} \\
&+ B (u_{l,m} u_{m,l} \delta_{ik} + 2u_{k,i} u_{l,l}) + C (u_{l,l})^2 \delta_{ik}
\end{aligned}$$

By taking into account only the quadratic nonlinearities, the motion equations (in the hypothesis $X_k = 0$), according to (1) and to the simplified form of Murnaghan potential (5), are

$$\left\{ \begin{array}{l} \rho u_{1,tt} - (\lambda + 2\mu) u_{1,11} = N_1 u_{1,11} u_{1,1} + N_2 (u_{2,11} u_{2,1} + u_{3,11} u_{3,1}), \\ \rho u_{2,tt} - \mu u_{2,11} = N_2 (u_{2,11} u_{1,1} + u_{1,11} u_{2,1}), \\ \rho u_{3,tt} - \mu u_{3,11} = N_2 (u_{3,11} u_{1,1} + u_{1,11} u_{3,1}), \end{array} \right. \quad (6)$$

with

$$N_1 = 3 [(\lambda + 2\mu) + 2(A + 3B + C)], \quad N_2 = \lambda + 2\mu + \frac{1}{2} A + B.$$

2.2 Signorini Potential

The potential named after Signorini has been given by him in terms of the Almansi tensor as follows

$$W(\tilde{\varepsilon}_{ik}) = \sqrt{\frac{G}{g}} \left\{ c I_2(\tilde{\varepsilon}) + \frac{1}{2} \left(\lambda + \mu - \frac{c}{2} \right) (I_1(\tilde{\varepsilon}))^2 + \left(\mu + \frac{c}{2} \right) (I - I_1(\tilde{\varepsilon})) \right\} - \left(\mu + \frac{c}{2} \right).$$

By referring this potential to the Lagrangian coordinates, we can easily obtain (like in the Murnaghan case) the fundamental equations (see e.g. Cattani and Rushchitsky, 2007 and references therein)

$$\left\{ \begin{array}{l} \rho u_{1,tt} - (\lambda + 2\mu) u_{1,11} = \frac{1}{2} (-\lambda + 5c) u_{1,11} u_{1,1} + \frac{1}{2} c (u_{2,11} u_{2,1} + u_{3,11} u_{3,1}) \\ \rho u_{2,tt} - \mu u_{2,11} = 2 \left(\lambda + \mu + \frac{c}{2} \right) (u_{2,11} u_{1,1} + u_{1,11} u_{2,1}) + 4c u_{2,11} u_{2,1} \\ \rho u_{3,tt} - \mu u_{3,11} = 2 \left(\lambda + \mu + \frac{c}{2} \right) (u_{3,11} u_{1,1} + u_{1,11} u_{3,1}) + 4c u_{3,11} u_{3,1} \end{array} \right. \quad (7)$$

where c is the Signorini constant related to Murnaghan constants by

$$c = \frac{1}{5} [7\lambda + 12\mu + 12(A + 3B + C)].$$

Let us show that the plane longitudinal wave equations for Murnaghan and Signorini potentials coincide (while they do not for other plane transverse waves). In fact, if we consider a problem where at the entrance into the medium $x = 0$ only a longitudinal wave is given, i.e. $u_2(0, t) = u_3(0, t) = 0$ then as a solution of (6)_{2,3}, as well as (7)_{2,3}, we have

$$u_2(x, t) = 0 \quad , \quad u_3(x, t) = 0.$$

So that systems (6),(7) reduces to the only nontrivial equation

$$\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} &= N_1 \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} && \text{(Murnaghan)} \\
\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} &= \frac{1}{2} (-\lambda + 5c) \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} && \text{(Signorini)}.
\end{aligned}$$

After a division by $\lambda + 2\mu$ and defining the non dimensional time $\sqrt{\frac{\lambda + 2\mu}{\rho}} t \rightarrow t$, we obtain the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = N_1 \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \quad (8)$$

with

$$N_1 := \begin{cases} \frac{N_1}{\lambda + 2\mu} & \text{(Murnaghan)} \\ \frac{-\lambda + 5c}{2(\lambda + 2\mu)} & \text{(Signorini)}. \end{cases}$$

In the following we will analyze equation (8) in the harmonic wavelet basis and compare the results with those obtained by the method of successive approximation.

3 Harmonic Wavelets

Wavelets are some special functions $\psi_k^n(x)$ which depend on two parameters: n is the scale (refinement, compression, or dilation) parameter and k is the localization (translation) parameter. These functions fulfill the fundamental axioms of multiresolution analysis so that by a suitable choice of the scale and translation parameter one is able to easily and quickly approximate any function (even tabular) with decay to infinity.

The dilated and translated instances of the harmonic scaling and wavelet function are (Cattani, 2005; Muniandy and Moroz 1997; Newland 1993)

$$\begin{cases} \varphi_k^n(x) \stackrel{\text{def}}{=} 2^{n/2} \frac{e^{2\pi i(2^n x - k)} - 1}{2\pi i(2^n x - k)} \\ \psi_k^n(x) \stackrel{\text{def}}{=} 2^{n/2} \frac{e^{4\pi i(2^n x - k)} - e^{2\pi i(2^n x - k)}}{2\pi i(2^n x - k)} \end{cases} \quad (9)$$

with $n, k \in \mathbb{Z}$.

The corresponding Fourier transforms $\widehat{\varphi}(\omega) = \widehat{\varphi(x)} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} dx$ are:

$$\begin{cases} \widehat{\varphi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(2\pi + \omega/2^n) \\ \widehat{\psi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(\omega/2^n) \end{cases} \quad (10)$$

being $\chi(\omega)$ the characteristic function defined as

$$\chi(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & , \quad 2\pi \leq \omega \leq 4\pi , \\ 0 & , \quad \text{elsewhere} . \end{cases} \quad (11)$$

From the definition of the inner (or scalar or dot) product of two functions $f(x), g(x)$, and taking into account the Parseval equality

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} d\omega = 2\pi \langle \widehat{f}, \widehat{g} \rangle , \quad (12)$$

it can be easily shown (Cattani 2005; Newland 1993) that harmonic wavelets are orthonormal functions in the sense that

$$\begin{cases} \langle \varphi_k^n(x), \varphi_h^m(x) \rangle = \delta^{nm} \delta_{kh} & , \quad \langle \overline{\varphi}_k^n(x), \overline{\varphi}_h^m(x) \rangle = \delta^{nm} \delta_{kh} & , \quad \langle \varphi_k^n(x), \overline{\varphi}_h^m(x) \rangle = 0 , \\ \langle \psi_k^n(x), \psi_h^m(x) \rangle = \delta^{nm} \delta_{kh} & , \quad \langle \overline{\psi}_k^n(x), \overline{\psi}_h^m(x) \rangle = \delta^{nm} \delta_{kh} & , \quad \langle \psi_k^n(x), \overline{\psi}_h^m(x) \rangle = 0 , \\ \langle \varphi_k^n(x), \overline{\psi}_h^m(x) \rangle = 0 & , \quad \langle \overline{\varphi}_k^n(x), \psi_h^m(x) \rangle = 0 & , \end{cases} \quad (13)$$

where the bar stands for complex conjugation and δ^{nm} (δ_{hk}) is the Kronecker symbol.

Some simple technical calculations also show that the harmonic scaling function and the harmonic wavelets fulfill the multiresolution conditions

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1 \quad , \quad \int_{-\infty}^{\infty} \psi_k^n(x) dx = 0 .$$

4 Multiscale Representation of Functions

Let us consider the class of functions $f(x)$ such that the integrals

$$\left\{ \begin{array}{l} \alpha_k = \langle f(x), \varphi_k^0(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{\varphi_k^0(x)} dx \\ \alpha_k^* = \langle f(x), \overline{\varphi_k^0(x)} \rangle = \int_{-\infty}^{\infty} f(x) \varphi_k^0(x) dx \\ \beta_k^n = \langle f(x), \psi_k^n(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{\psi_k^n(x)} dx \\ \beta_k^{*n} = \langle f(x), \overline{\psi_k^n(x)} \rangle = \int_{-\infty}^{\infty} f(x) \psi_k^n(x) dx \end{array} \right. \quad (14)$$

exist and are finite for any value of the two parameters n, k .

For a complex function its reconstruction in terms of harmonic wavelets can be obtained by the formula (Newland, 1993)

$$f(x) = \left[\sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \right] + \left[\sum_{k=-\infty}^{\infty} \alpha_k^* \overline{\varphi_k^0(x)} + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^{*n} \overline{\psi_k^n(x)} \right] \quad (15)$$

which involves the basis and (for a complex function) its conjugate basis.

The wavelet coefficients, according to (12), can be computed in the Fourier domain

$$\left\{ \begin{array}{l} \alpha_k = 2\pi \langle \widehat{f(x)}, \widehat{\varphi_k^0(x)} \rangle = \int_{-\infty}^{\infty} \widehat{f}(\omega) \overline{\widehat{\varphi_k^0(\omega)}} d\omega = \int_0^{2\pi} \widehat{f}(\omega) e^{i\omega k} d\omega \\ \alpha_k^* = 2\pi \langle \widehat{f(x)}, \widehat{\overline{\varphi_k^0(x)}} \rangle = \dots = \int_0^{2\pi} \widehat{f}(\omega) e^{-i\omega k} d\omega \\ \beta_k^n = 2\pi \langle \widehat{f(x)}, \widehat{\psi_k^n(x)} \rangle = \dots = 2^{-n/2} \int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega) e^{i\omega k/2^n} d\omega \\ \beta_k^{*n} = \langle \widehat{f(x)}, \widehat{\overline{\psi_k^n(x)}} \rangle = \dots = 2^{-n/2} \int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega) e^{-i\omega k/2^n} d\omega , \end{array} \right. \quad (16)$$

being $\widehat{\overline{f(x)}} = \overline{\widehat{f(x)}}$. The approximation up to the scale $N \leq \infty$ and to a finite translation $M \leq \infty$ is

$$f(x) \cong \Pi_M^N f(x) = \left[\sum_{k=0}^M \alpha_k \varphi_k^0(x) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^n \psi_k^n(x) \right] + \left[\sum_{k=0}^M \alpha_k^* \overline{\varphi_k^0(x)} + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^{*n} \overline{\psi_k^n(x)} \right] . \quad (17)$$

For a real function ($f(x) = \overline{f(x)}$) it is $\alpha_k^* = \alpha_k$, $\beta_k^{*n} = \beta_k^n$.

The wavelet approximation, i.e. the projection into the $N \times M$ wavelet space, is obtained by fixing an upper limits in the series expansion (15), so that with $N < \infty$, $M < \infty$ we have

$$f(x) \cong \left[\sum_{k=0}^M \alpha_k \varphi_k^0(x) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^n \psi_k^n(x) \right] + \left[\sum_{k=0}^M \alpha_k^* \overline{\varphi_k^0(x)} + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^{*n} \overline{\psi_k^n(x)} \right] . \quad (18)$$

Since wavelets are localized, they can capture with a few terms the main features of functions defined in a short range interval. By increasing the scale N it is possible to quickly approximate the peaks of a function while, in order to approximate the tails, higher values of the translation parameter M are needed. In any case, due to the localization and compression property of wavelets only a few terms (in the series (18)) may give us a good approximation.

However the reconstruction of a function $f(x)$ it is impossible when the integrals (14) are not defined, therefore the function (to be reconstructed) must be concentrated around the origin (like a pulse) and should rapidly decay to zero. The reconstruction can also be done for periodic functions or functions localized in a point different from zero: $x_0 \neq 0$, by using the periodized harmonic wavelets (Muniandy and Moroz, 1997; Newland, 1993).

It should be noticed that, for a non trivial function $f(x) \neq 0$ the corresponding wavelet coefficients (16), in general, vanish when either

$$\widehat{f}(\omega) = 0, \quad \forall k \quad \text{or} \quad \widehat{f}(\omega) = Cnst., \quad k \neq 0.$$

In particular, it can be seen that the wavelet coefficients (14) trivially vanish when

$$\begin{cases} f(x) = \sin(2k\pi x) & , \quad k \in \mathbb{Z} \\ f(x) = \cos(2k\pi x) & , \quad k \in \mathbb{Z} \quad (k \neq 0). \end{cases} \quad (19)$$

For instance from (14)₁ for $\cos(2k\pi x)$ it is

$$\begin{aligned} \alpha_k &= \int_{-\infty}^{\infty} \cos(2k\pi x) \overline{\varphi}_k^0(x) dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (e^{-2ih\pi x} + e^{2ih\pi x}) \overline{\varphi}_k^0(x) dx \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-2ih\pi x} \overline{\varphi}_k^0(x) dx + \int_{-\infty}^{\infty} e^{2ih\pi x} \overline{\varphi}_k^0(x) dx \right] \end{aligned}$$

from where by the change of variable $2\pi x = \xi$ and taking into account that

$$e^{\pi i n} = \begin{cases} 1, & n = 2k, & k \in \mathbb{Z} \\ -1, & n = 2k + 1, & k \in \mathbb{Z}, \end{cases} \quad (20)$$

there follows

$$\alpha_k = \frac{1}{2} \left[\widehat{\overline{\varphi}_k^0(x)} + \widehat{\varphi}_k^0(x) \right]_{x=2\pi h}.$$

According to (10) it is

$$\widehat{\varphi}_k^0(2\pi h) = \frac{1}{2} e^{-i2\pi h k} \chi(2\pi + 2\pi h) \stackrel{(20)}{=} \frac{1}{2} \chi(2\pi + 2\pi h)$$

and, because of (11)

$$\chi(2\pi + 2\pi h) = 1 \quad , \quad 0 < h < 1$$

so that

$$\widehat{\varphi}_k^0(2\pi h) = 0 \quad , \quad \forall h \neq 0.$$

There it follows that $\alpha_h = 0$, as well as the remaining wavelet coefficients of $\cos(2k\pi x)$ (with $k \in \mathbb{Z}$ and $k \neq 0$) are trivially vanishing. Analogously, it can be shown that all wavelet coefficients of $\sin(2k\pi x)$ ($\forall k \in \mathbb{Z}$) are zero.

As a consequence, a given function $f(x)$, for which the coefficients (14) are defined, admits the same wavelet coefficients of

$$f(x) + \sum_{h=0}^{\infty} [A_h \sin(2h\pi x) + B_h \cos(2h\pi x)] - B_0, \quad (21)$$

or (by a simple tranformation) in terms of complex exponentials,

$$f(x) - C_0 + \sum_{h=-\infty}^{\infty} C_h e^{2ih\pi x}, \quad (22)$$

so that the wavelet coefficients of $f(x)$ are defined unless an additional Fourier series (the coefficients A_h, B_h, C_h being constant) as in (21). This property can be extended to any frequency (i.e. different from $1/(2\pi)$) by using a scaling factor in the definition (9) of the harmonic wavelets.

5 Connection Coefficients for Harmonic Wavelets

In order to apply the Galerkin method we must deal with the differential properties of the basis and, in particular, with the computation of their derivatives through the connection coefficients (Latto, 1992; Cattani, 2005).

The differential properties of wavelets are based on the knowledge of the following inner products

$$\left\{ \begin{array}{l} \lambda^{(\ell)}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \varphi_k^0(x), \varphi_h^0(x) \right\rangle, \quad \Lambda^{(\ell)m}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \varphi_k^0(x), \psi_h^m(x) \right\rangle \\ \gamma^{(\ell)nm}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \psi_k^n(x), \psi_h^m(x) \right\rangle, \quad \zeta^{(\ell)n}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \psi_k^n(x), \varphi_h^0(x) \right\rangle, \end{array} \right. \quad (23)$$

and the corresponding inner products with conjugate functions

$$\left\{ \begin{array}{l} \bar{\lambda}^{(\ell)}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \bar{\varphi}_k^0(x), \bar{\varphi}_h^0(x) \right\rangle, \quad \bar{\Lambda}^{(\ell)m}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \bar{\varphi}_k^0(x), \bar{\psi}_h^m(x) \right\rangle \\ \bar{\gamma}^{(\ell)nm}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \bar{\psi}_k^n(x), \bar{\psi}_h^m(x) \right\rangle, \quad \bar{\zeta}^{(\ell)n}_{kh} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \bar{\psi}_k^n(x), \bar{\varphi}_h^0(x) \right\rangle. \end{array} \right. \quad (24)$$

The coefficients (23),(24) can be easily computed in the Fourier domain (Cattani, 2005) and it can be shown that, if we define the *sign*-function

$$\mu(x) = \begin{cases} -1 & , x < 0 \\ 1 & , x > 0 \\ 0 & , x = 0 \end{cases}$$

then the following theorem is valid.

Theorem 1 *The only non trivial connection coefficients $\lambda^{(\ell)}_{kh}$, $\gamma^{(\ell)nm}_{kh}$ are given by*

$$\left\{ \begin{array}{l} \lambda^{(\ell)}_{kh} = \frac{1}{2\pi} \left[i^\ell (1 - |\mu(h-k)|) \frac{(2\pi)^{\ell+1}}{\ell+1} \right. \\ \quad \left. - i \mu(h-k) \sum_{s=1}^{\ell} (-1)^{[1+\mu(h-k)](2\ell-s+1)/2} \frac{\ell! (2i\pi)^{\ell-s+1}}{(\ell-s+1)! |h-k|^s} \right] \\ \gamma^{(\ell)nm}_{kh} = \frac{2^{-(n+m)/2}}{2\pi} \left[(1 - |\mu(h-k)|) \frac{i^\ell \pi^{\ell+1} 2^{[1+(n+m)/2](\ell+1)} (2^{\ell+1} - 1)}{\ell+1} \right. \\ \quad \left. - i \mu(h-k) \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(h-k)](2\ell-s+1)/2} \frac{\ell! (i\pi)^{\ell-s+1}}{(\ell-s+1)! |h-k|^s} \right] \\ \quad \times [2^{[1+(n+m)/2](\ell+1)-2s} (2^{\ell+1} - 2^s)] \delta^{nm} \end{array} \right. \quad (25)$$

for $\ell \geq 1$, and $\lambda^{(\ell)}_{kh} = \delta_{kh}$, $\gamma^{(\ell)nm}_{kh} = \delta^{nm} \delta_{kh}$ when $\ell = 0$.

The conjugate connection coefficients (24) $\bar{\lambda}^{(\ell)}_{kh}$ are given by

$$\bar{\lambda}^{(\ell)}_{kh} = \lambda^{(\ell)}_{hk}, \quad \bar{\gamma}^{(\ell)nm}_{kh} = \gamma^{(\ell)nm}_{hk} \quad (26)$$

for $\ell \geq 1$, and $\bar{\lambda}^{(\ell)}_{kh} = \delta_{kh}$, $\bar{\gamma}^{(\ell)nm}_{kh} = \gamma^{(\ell)nm}_{hk}$ when $\ell = 0$.

The mixed connection coefficients (23)_{2,4} are trivially zero:

$$\Lambda^{(\ell)m}_{kh} = 0, \quad \zeta^{(\ell)n}_{kh} = 0, \quad \bar{\Lambda}^{(\ell)m}_{kh} = 0, \quad \bar{\zeta}^{(\ell)n}_{kh} = 0. \quad (27)$$

The fundamental connection coefficients are (25),(26),(27). These coefficients enable us to characterize any order derivative of the basis. In fact, according to (23) it is

$$\frac{d^\ell \varphi_k^0(x)}{dx^\ell} = \sum_{m=0}^{\infty} \sum_{h=-\infty}^{\infty} \lambda^{(\ell)m}_{kh} \varphi_h^m(x). \quad (28)$$

A good approximation is obtained by a finite value of M

$$\frac{d^\ell \varphi_k^0(x)}{dx^\ell} \cong \sum_{h=0}^M \lambda^{(\ell)}_{kh} \varphi_h^0(x). \quad (29)$$

Analogously we have,

$$\frac{d^\ell \psi_k^n(x)}{dx^\ell} = \sum_{m=0}^{\infty} \sum_{k,h=-\infty}^{\infty} \gamma^{(\ell)nm}_{kh} \psi_h^m(x), \quad (30)$$

and a good approximation, which depends only on the dilation N and translational parameter M , is

$$\frac{d^\ell \psi_k^n(x)}{dx^\ell} \cong \sum_{m=0}^N \sum_{h=-M}^M \gamma^{(\ell)nm}_{kh} \psi_h^m(x) \quad (31)$$

with $N \leq n$.

For the corresponding conjugate functions we have

$$\begin{aligned} \frac{d^\ell \bar{\varphi}_k^0(x)}{dx^\ell} &= \sum_{m=0}^{\infty} \sum_{h=-\infty}^{\infty} -\lambda^{(\ell)m}_{kh} \bar{\varphi}_h^m(x), & \frac{d^\ell \bar{\varphi}_k^0(x)}{dx^\ell} &\cong \sum_{h=0}^M \lambda^{(\ell)}_{kh} \bar{\varphi}_h^0(x), \\ \frac{d^\ell \bar{\psi}_k^n(x)}{dx^\ell} &= \sum_{m=0}^{\infty} \sum_{k,h=-\infty}^{\infty} -\gamma^{(\ell)nm}_{kh} \bar{\psi}_h^m(x), & \frac{d^\ell \bar{\psi}_k^n(x)}{dx^\ell} &\cong \sum_{m=0}^N \sum_{h=-M}^M \bar{\gamma}^{(\ell)nm}_{kh} \bar{\psi}_h^m(x). \end{aligned} \quad (32)$$

6 Galerkin Approximation of Differential Operators in Wavelet Spaces

Let us consider the classical problem (derived from (6),(7))

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = N_1 \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \quad (33)$$

with the boundary condition

$$u(0, t) = U \cos(2\pi t). \quad (34)$$

We assume also the asymptotic condition that

$$\lim_{t \rightarrow \infty} u_{nl}(x, t) = 0, \quad \forall x,$$

being $u_{nl}(x, t)$ the contribution to wave propagation due to the nonlinear term on the r.h.s. of (33).

6.1 Van der Pol Solution

The main feature of the method of successive approximations (MOSA), also called Van der Pol method, consists in the introduction of a small parameter ε . Since the displacements vector $u(x, t)$ (or something similar to it) is assumed to be sufficiently smooth, then it can be expanded for small values of x in the form

$$u(x, t) = u^{(0)}(x, t) + \varepsilon u^{(1)}(x, t) + \varepsilon^2 u^{(2)}(x, t) + \dots, \quad (35)$$

where $u^{(0)}(x, t)$ is the solution of the linear equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (36)$$

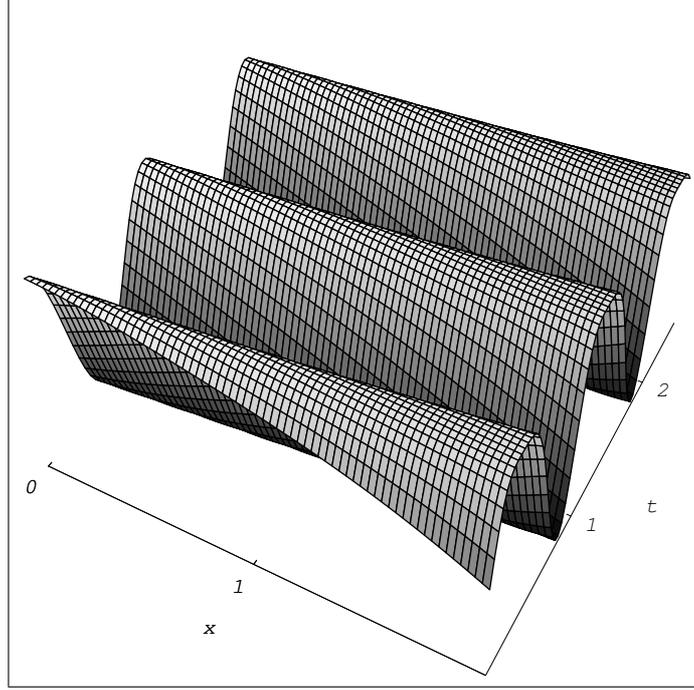


Figure 1: First approximation of the nonlinear wave by MOSA

together with the boundary condition (34). When $u^{(0)}(x, t)$ is known, the second function $u^{(1)}(x, t)$ is computed as the solution of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = N_1 \frac{\partial^2 u^{(0)}}{\partial x^2} \frac{\partial u^{(0)}}{\partial x}, \quad (37)$$

and so on for the successive approximations. Thus the solution of problem (33),(34), up to the second approximation of MOSA leads us to

$$u^{(0)}(x, t) = U \cos(kx - 2\pi t)$$

and $u^{(1)}(x, t)$ as the solution of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{2} N_1 k^3 U^2 \sin 2(kx - 2\pi t), \quad (38)$$

so that it is (see Fig. 1)

$$\begin{aligned} u(x, t) &= u^{(0)}(x) + u^{(1)}(x) \\ &= U \cos(kx - 2\pi t) - \frac{1}{8} N_1 U^2 k^2 x \cos 2(kx - 2\pi t). \end{aligned} \quad (39)$$

6.2 Wavelet Solution

Let us assume that the solution $u(x, t)$ is represented according to (15) by space-dependent amplitudes with time-dependent bases

$$u(x, t) = u_F(x, t) + u_W(x, t)$$

with

$$u_F(x, t) \stackrel{\text{def}}{=} -B_0(x) + \sum_{k=0}^{\infty} A_k(x) \sin 2k\pi t + B_k(x) \cos 2k\pi t \quad (40)$$

and

$$u_W(x, t) \stackrel{\text{def}}{=} \left[\sum_{k=-\infty}^{\infty} \alpha(x) \varphi_k^0(t) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n(x) \psi_k^n(t) \right] + \left[\sum_{k=-\infty}^{\infty} \alpha^*(x) \bar{\varphi}_k^0(t) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^{*n}(x) \bar{\psi}_k^n(t) \right].$$

The equation (33) becomes

$$\frac{\partial^2}{\partial t^2}(u_F + u_W) - \frac{\partial^2}{\partial x^2}(u_F + u_W) = N_1 \left[\frac{\partial^2}{\partial x^2}(u_F + u_W) \right] \frac{\partial}{\partial x}(u_F + u_W),$$

and by the projection $\Pi^{N,M}$ we get two separate sets of ordinary differential equations, in the wavelet coefficients

$$\left\{ \begin{array}{l} \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial t^2}, \varphi_k^0(t) \right\rangle - \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial x^2}, \varphi_k^0(t) \right\rangle = \left\langle \Pi^{N,M} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \right], \varphi_k^0(t) \right\rangle, \\ \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial t^2}, \bar{\varphi}_k^0(t) \right\rangle - \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial x^2}, \bar{\varphi}_k^0(t) \right\rangle = \left\langle \Pi^{N,M} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \right], \bar{\varphi}_k^0(t) \right\rangle, \\ \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial t^2}, \bar{\psi}_k^n(t) \right\rangle - \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial x^2}, \bar{\psi}_k^n(t) \right\rangle = \left\langle \Pi^{N,M} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \right], \bar{\psi}_k^n(t) \right\rangle, \\ \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial t^2}, \psi_k^n(t) \right\rangle - \left\langle \Pi^{N,M} \frac{\partial^2 u}{\partial x^2}, \psi_k^n(t) \right\rangle = \left\langle \Pi^{N,M} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \right], \psi_k^n(t) \right\rangle, \end{array} \right. \quad (k = 0, 1, \dots, M; n = 0, \dots, N) \quad (41)$$

and

$$\frac{\partial^2 u_F}{\partial t^2} - \frac{\partial^2 u_F}{\partial x^2} = N_1 \frac{\partial^2 u_F}{\partial x^2} \frac{\partial u_F}{\partial x}.$$

Due to (40),(34), the last one leads us in the linear case to the linear ordinary system

$$\left\{ \begin{array}{l} \frac{d^2 B_1}{dx^2} + 4\pi^2 B_1 = 0 \quad , \quad B_1(0) = 1 \\ \frac{d^2 A_k}{dx^2} + 4\pi^2 A_k = 0 \quad , \quad A_k(0) = 0 \end{array} \right. \quad (42)$$

which gives, unless some inessential constants,

$$u_F(x, t) = U \cos 2\pi(kx - t).$$

This function explains, as expected, the main contribution to the solution, and coincides with $u^{(0)}(x, t)$ obtained by MOSA. Thus comparing with the MOSA model, we can write

$$u_W(t) = xu^{(1)}(x, t) + x^2 u^{(2)}(x, t) + \dots \quad ,$$

so that the projection into the wavelet space coincides with the contribution due to the nonlinearities. In other words, the wavelet component $u_W(t)$ describes the evolution of the nonlinearity effects. From this, there follows the importance of the wavelet analysis. In the linear case the wavelet contribution $u_W(t)$ is absent.

In particular, the projection into the N -scale approximation wavelet space $\Pi^{N,M}$ of the derivatives is according to (17)

$$\left\{ \begin{array}{l} \Pi^{N,M} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [\Pi^{N,M} u(x, t)] = \Pi^{N,M} \frac{\partial u_W}{\partial x} \\ = \left[\sum_{k=0}^M \left(\frac{d}{dx} \alpha_k(x) \right) \varphi_k^0(t) + \sum_{n=0}^N \sum_{k=-M}^M \left(\frac{d}{dx} \beta_k^n(x) \right) \psi_k^n(t) \right] \\ + \left[\sum_{k=0}^M \left(\frac{d}{dx} \alpha_k^*(x) \right) \bar{\varphi}_k^0(t) + \sum_{n=0}^N \sum_{k=-M}^M \left(\frac{d}{dx} \beta_k^n(x) \right) \bar{\psi}_k^n(t) \right] \end{array} \right. \quad (43)$$

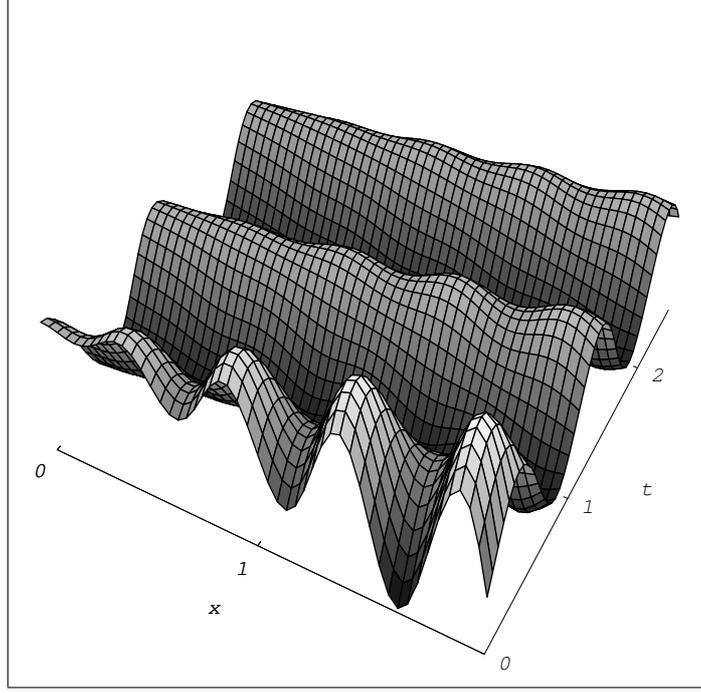


Figure 2: *Lowest scale wavelet approximation of the nonlinear wave .*

and

$$\left\{ \begin{aligned} \Pi^{N,M} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2} [\Pi^{N,M} u(x, t)] = \frac{\partial^2}{\partial x^2} [\Pi^{N,M} u_W(x, t)] \\ &= \left[\sum_{k=0}^M \left(\frac{d^2}{dx^2} \alpha_k(x) \right) \varphi_k^0(t) + \sum_{n=0}^N \sum_{k=-M}^M \left(\frac{d^2}{dx^2} \beta_k^n(x) \right) \psi_k^n(t) \right] \\ &+ \left[\sum_{k=0}^M \left(\frac{d^2}{dx^2} \alpha_k^*(x) \right) \bar{\varphi}_k^0(t) + \sum_{n=0}^N \sum_{k=-M}^M \left(\frac{d^2}{dx^2} \beta_k^n(x) \right) \bar{\psi}_k^n(t) \right] \end{aligned} \right. \quad (44)$$

The projection of the evolution operator gives

$$\begin{aligned} \Pi^{N,M} \left[\frac{\partial^2}{\partial t^2} u(x, t) \right] &= \left[\sum_{k=0}^M \alpha_k(x) \left(\frac{d^2}{dt^2} \varphi_k^0(t) \right) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^n(x) \left(\frac{d^2}{dt^2} \psi_k^n(t) \right) \right] \\ &+ \left[\sum_{k=0}^M \alpha_k^*(x) \left(\frac{d^2}{dt^2} \bar{\varphi}_k^0(t) \right) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^{*n}(x) \left(\frac{d^2}{dt^2} \bar{\psi}_k^n(t) \right) \right] \end{aligned}$$

and, according to (29),(31),

$$\begin{aligned} \Pi^{N,M} \left[\frac{\partial^2}{\partial t^2} u(x, t) \right] &= \left[\sum_{k,h=0}^M \alpha_k(x) \lambda_{kh}^{(2)} \varphi_h^0(t) + \sum_{n,m=0}^N \sum_{k,h=-M}^M \beta_k^n(x) \gamma_{kh}^{(2)nm} \psi_h^m(t) \right] \\ &+ \left[\sum_{k,h=0}^M \alpha_k^*(x) \bar{\lambda}_{kh}^{(\ell)} \bar{\varphi}_h^0(t) + \sum_{n,m=0}^N \sum_{k,h=-M}^M \beta_k^{*n}(x) \bar{\gamma}_{kh}^{(\ell)nm} \bar{\psi}_h^m(t) \right] \end{aligned} \quad (45)$$

By putting (43),(44),(45) into (41) and by the scalar product with the basis functions $\varphi_i^0, \bar{\varphi}_i^0, \psi_i^r(x), \bar{\psi}_i^r(x)$ we

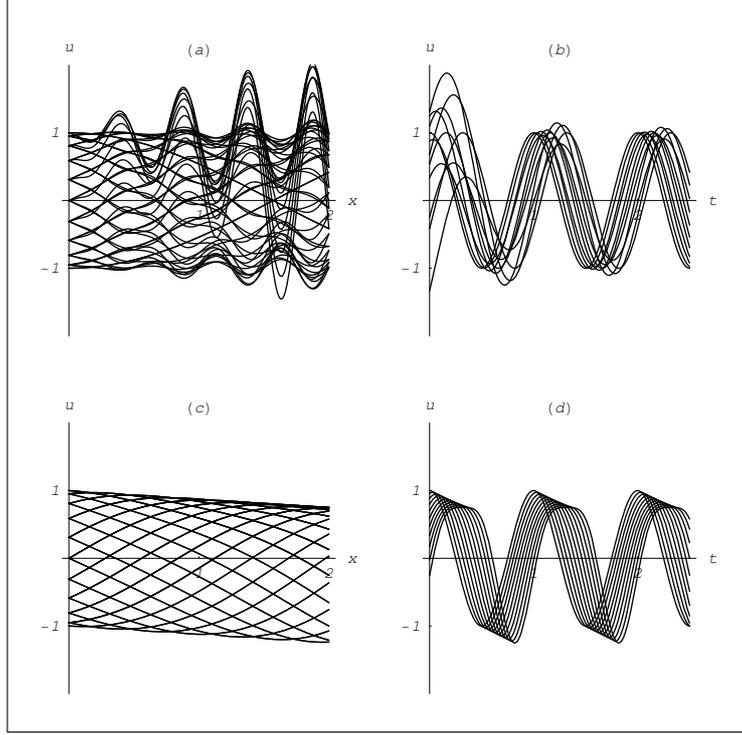


Figure 3: Comparison of the first approximation by MOSA with the wavelet approximation. Space [time] evolution of a discrete set of points in time [space]: (a) $u_F(x, 0.005j) + u_w(x, 0.005j)$, $j = 0, \dots, 499$; (b) $u_F(0.2j, t) + u_w(0.2j, t)$, $j = 0, \dots, 9$; (c) $u^{(0)}(x, 0.005j) + u^{(1)}(x, 0.005j)$, $j = 0, \dots, 499$; (d) $u^{(0)}(0.2j, t) + u^{(1)}(0.2j, t)$, $j = 0, \dots, 9$.

obtain the algebraic system

$$\left\{ \begin{array}{l} \sum_{k=0}^M \alpha_k \lambda^{(2)}_{ki} - \frac{d^2 \alpha_i}{dx^2} = a_i(x) \quad , \quad (i = 0, \dots, M) \\ \sum_{k=0}^M \alpha_k^* \lambda^{(2)}_{ik} - \frac{d^2 \alpha_i^*}{dx^2} = a_i^*(x) \quad , \quad (i = 0, \dots, M) \\ \sum_{n=0}^N \sum_{k=-M}^M \beta_k^n \gamma^{(2)}_{ki} - \frac{d^2 \beta_i^r}{dx^2} = b_i^r(x) \quad , \quad (r = 0, \dots, N; i = -M, \dots, M) \\ \sum_{n=0}^N \sum_{k=-M}^M \beta_k^{*n} \gamma^{(2)}_{ki} - \frac{d^2 \beta_i^{*r}}{dx^2} = b_i^{*r}(x) \quad , \quad (r = 0, \dots, N; i = -M, \dots, M) . \end{array} \right. \quad (46)$$

The r.h.s. terms a_i , a_i^* , b_i^r , b_i^{*r} are functions both of the wavelet coefficients and of the nonlinear connection coefficients. In order to compare with the MOSA method, let us assume that the r.h.s. is given as it is the case for the equation (38). In this case the coarse wavelet approximation of the r.h.s. of (38) is

$$\sin 2\pi(kx - 2\pi t) = \frac{1}{4} \sin 4\pi x [\varphi_0^0(t) + \bar{\varphi}_0^0(t)] = \frac{1}{2} \sin 4\pi x \frac{\sin 2\pi t}{\pi t}$$

so that the corresponding wavelet coefficients are

$$a_0(x) = \frac{1}{2} \sin 4\pi x \quad , \quad a_0^*(x) = \frac{1}{2} \sin 4\pi x \quad , \quad b_i^r(x) = b_i^{*r}(x) = 0 .$$

If $\lambda^{(2)}_{00} \stackrel{(25)}{=} -4\pi^2/3$, the only nontrivial equation is

$$-4\pi^2/3\alpha - \frac{d^2 \alpha}{dx^2} = \frac{1}{2} \sin 4\pi x \quad ,$$

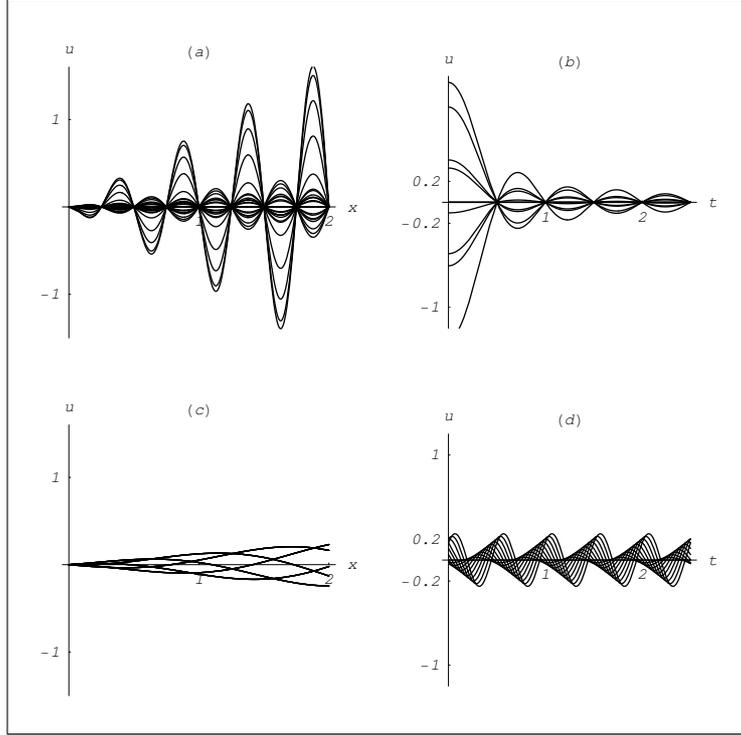


Figure 4: Comparison of the the only nonlinear contribution to the first approximation by MOSA and the wavelet approximation. Space [time] evolution of a discrete set of points in time [space]: (a) $u_w(x, 0.005j)$, $j = 0, \dots, 499$; (b) $u_w(0.2j, t)$, $j = 0, \dots, 9$; (c) $u^{(1)}(x, 0.005j)$, $j = 0, \dots, 499$; (d) $u^{(1)}(0.2j, t)$, $j = 0, \dots, 9$.

which, apart from some inessential constants, is solved by

$$\alpha(x) = \frac{3}{88\pi^2} \sin 4\pi x .$$

Therefore the lowest level of $u_W(x, t)$ is

$$u_W(x, t) = x \frac{3}{44\pi^2} \sin 4\pi x \frac{\sin 2\pi t}{\pi t}$$

and the wavelet approximation at the coarsest level is

$$u(x, t) = U \cos(kx - 2\pi t)x + \frac{3}{44\pi^2} \sin 4\pi x \frac{\sin 2\pi t}{\pi t} . \quad (47)$$

Comparing (47) with (39) (see also Figs. 1,2,3, 4, 5) we can see that

1. the wavelet solution even at the coarsest level can easily give a qualitatively good representation of the solution
2. the nonlinearities are singled out by the wavelet coefficients
3. the MOSA method gives a too smooth approximation of the solution which instead, due to the nonlinearities, should be more like the wavelet approximation (with higher frequency oscillations, see e.g. Fig. 5).

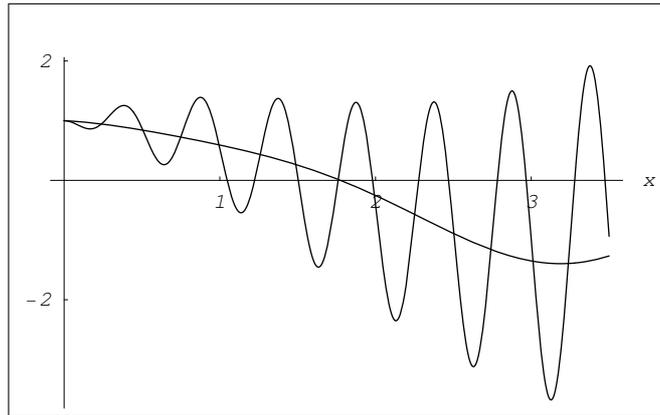


Figure 5: Comparison of the the boundary profiles of the first approximation by MOSA and the wavelet approximation: $u_F(x, 0) + u_w(x, 0)$ against $u^{(0)}(x, 0) + u^{(1)}(x, 0)$.

Conclusion

It has been shown that by using

1. a suitable complex wavelet basis
2. the connection coefficients
3. the Galerkin method

the wavelet solution might be an expedient tool for the analysis of nonlinear wave propagation. Due to their localization properties wavelets are not only a good tools for capturing nonlinearities, but this can be performed with a minimum number of coefficients. Moreover, the behavior at different scales can be further investigated in order to focus on the relevant scales of the phenomenon. The comparison with the MOSA also shows that the wavelet approach seems to be more expedient for describing the oscillations during the wave evolution.

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