TECHNISCHE MECHANIK, Band 28, Heft 3-4, (2008), 219 - 226
Manuskripteingang: 15. Oktober 2007

Simultaneous Resonances Involving Three Mode Shapes of Parametrically-Excited Rectangular Plates

Nguyen Hai

It is known that, for multi-degree-of-freedom systems under time-dependent excitation, the combination of an internal resonance with an external resonance will give rise to simultaneous resonances, and these resonances are characterized by the fact that the system in question can resonate simultaneously in more than one normal mode while only one resonant mode is directly excited by the excitation. The present work deals with the problem of the occurrence of simultaneous resonances in parametrically-excited and simply-supported rectangular plates. The analysis is based on the dynamic analog of von Kármán's large-deflection theory and the governing equations are satisfied using the orthogonality properties of the assumed functions. The nonlinear temporal response of the damped system is determined by the first-order generalized asymptotic method. The solution for simply supported plates indicates the possibility of principal parametric resonances and simultaneous resonances. Simultaneous resonances involving two modes of vibration have been presented in a number of papers by the present author. Numerical results of simultaneous resonances involving three mode shapes are presented in this paper for the first time.

1 Introduction

In structural dynamics, dynamic instability has received considerable attention in the past 50 years, and a typical example in regard to dynamic instability of structures is the case of a rectangular plate acted upon by a periodic in-plane load. When a such plate sustains an in-plane load of the form $n(t) = n_0 + n_t \cos\lambda t$, it may become laterally unstable over certain regions of the (n_0, n_t, λ) parameter space, and this phenomenon is referred to as parametric or dynamic instability. It is known that when the natural frequencies of the system are distinct, and in the absence of internal resonances and combination resonances, the periodic in-plane load can excite only one normal mode at a time; and when the plate executes lateral vibration at half the driving frequency, the corresponding resonance is called *principal* parametric resonance. In contrast with this case of *simple* parametric resonance, *simultaneous* resonances may also occur in multi-degree-of-freedom system subjected to parametric excitation such as a plate. This means that, when a parametric (external) resonance is excited in the presence of an internal resonance, the coincidence of these two types of resonances will give rise to simultaneous resonances. Simultaneous resonances are characterized by the fact that several modes might exist in the response, even though only one mode is directly excited by the excitation. Internal resonance is responsible for this phenomenon and, as a consequence, for a significant transfer of energy from the directly excited mode to other modes of vibration.

Simultaneous resonances involving two modes of vibration of parametrically-excited rectangular plates have been studied theoretically and experimentally by the present author [1-6]. The problem of the occurrence of simultaneous resonances involving three spatial forms of vibration is dealt in this paper and covers an existing gap in our understanding of the dynamic buckling of structures. The simply-supported rectangular plate under investigation is acted upon by periodic in-plane forces uniformly distributed along two opposite edges; the other two edges are stress-free. General rectangular plates are used, the aspect ratio of the plate being regarded as an additional parameter of the system.

2 Statement of the Problem

The continuous system under investigation is a rectangular plate simply supported along its edges, and loaded by periodic in-plane forces uniformly distributed along two opposite edges, as shown in Figure 1. The *x-y* plane is selected in the middle plane of the undeformed plate. The plate is assumed to be thin, initially flat, of uniform thickness, and the plate material is elastic, homogeneous, and isotropic.

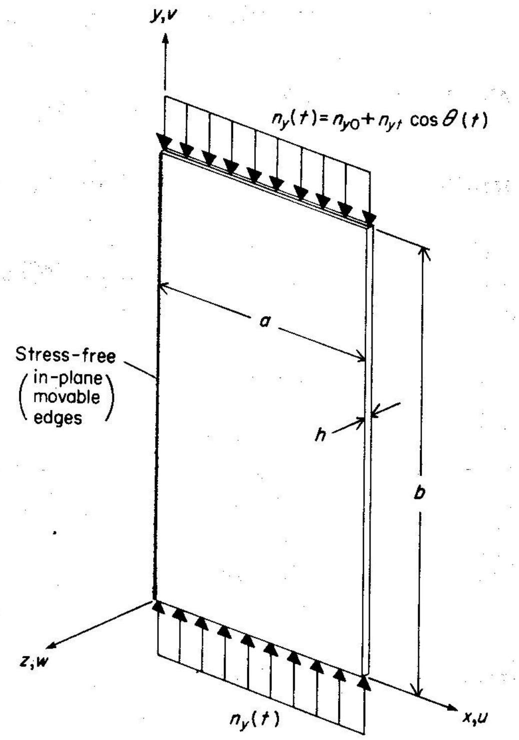

Figure 1. Plate and load configuration

Restricting the problem to the relatively low frequency range where the plate oscillations are predominantly flexural, the effect of transverse shear deformations as well as in-plane and rotatory inertia forces can be neglected. The plate theory used in this analysis may be described as the dynamic analog of von Karman's large-deflection theory. The dimensionless differential equations governing the nonlinear flexural vibrations of the plate can be written as

$$R^4 F_{,XXXX} + 2R^2 F_{,XXYY} + F_{,YYYY} = R^2 \left[W_{,XY}^2 - W_{,XX} W_{,YY} \right] \tag{1}$$

$$R^4 W_{,XXXX} + 2R^2 W_{,XXYY} + W_{,YYYY} = \varsigma \left[R^2 (F_{,YY} W_{,XX} - 2F_{,XY} W_{,XY} + F_{,XX} W_{,YY}) - R^4 W_{,TT} \right] \tag{2}$$

in which a comma denotes partial differentiation with respect to the corresponding coordinates, $R = b/a$ is the plate aspect ratio and $\zeta = 12(1-\nu^2)$, and where

$$X = x/a, Y = y/b, W = w/h, F = f/Eh^2, T = t\left[Eh^2/\rho a^4\right]^{1/2} \tag{3}$$

In equations (3), $w(x, y, t)$ is the lateral displacement and $f(x, y, t)$ the Airy stress function, h denotes the plate thickness, ρ the density, E the Young's modulus, and t the time. The nonlinearity arising in the problem under consideration is due to large amplitudes generating membrane forces.

The boundary conditions are related to both the lateral displacement and the stress function. The stress conditions may be expressed in the dimensionless form as

$$F_{,YY} = F_{,XY} = 0 \quad \text{at } X = 0, 1 \tag{4a}$$

$$F_{,XX} = -N_Y(T), F_{,XY} = 0 \quad \text{at } Y = 0, 1 \tag{4b}$$

in which $N_Y(T) = N_{Y0} + N_{YT}\cos(\Lambda T)$, and $N_Y = (a^2/Eh^3)n_y$. The supporting conditions for a simply-supported rectangular plate are written as

$$W = R^2 W_{,XX} + \nu W_{,YY} = 0 \quad \text{at } X = 0, 1 \tag{5a}$$

$$W = W_{,YY} + \nu R^2 W_{,XX} = 0 \quad \text{at } Y = 0, 1 \tag{5b}$$

The problem consists in determining the functions F and W which satisfy the governing equations, together with the boundary conditions.

3 Method of Solution

The solution for the stress function is represented by double series consisting of appropriate beam functions which satisfy the relevant boundary conditions

$$F(X,Y,T) = \sum_m \sum_n F_{mn}(T)\; X_m(X)\, Y_n(Y) - \tfrac{1}{2} X^2 N_Y(T) \tag{6}$$

where $F_{mn}(T)$ are time-dependent load factors. The solution for the lateral displacement is sought in the form of double series consisting of the eigenfunctions of the freely vibrating system or beam functions which satisfy the relevant boundary conditions

$$W(x,Y,T) = \sum_p \sum_q W_{pq}(T)\, \Phi_p(X)\, \Psi_q(Y) \tag{7}$$

where $W_{pq}(T)$ are time-dependent generalized coordinates of the system.

Applying the generalized double Fourier approach to the governing equations, using the orthogonality properties of the assumed functions, solving for the time-dependent stress coefficients in terms of time-dependent generalized coordinates of the system, omitting all indices associated with the half-wave spatial form in the unloaded direction, and introducing linear (viscous) damping lead to a system of nonlinear ordinary differential equations for the time functions as follows

$$\ddot{W}_m + 2C_m \dot{W}_m + \Omega_m^2 \left(1 - 2\mu_m \cos\theta\right) W_m + \sum_i \sum_j \sum_k M_m^{ijk} W_i W_j W_k = 0 \tag{8}$$

where m = 1, 2, 3, …, C_m represents the coefficient of viscous damping, $\Omega_m = \omega_m[1 - N_{Y0}/N_m]^{1/2}$ is the free vibration circular frequency of a rectangular plate loaded by the constant component N_{Y0} of the in-plane force while N_m represents the static critical load according to the linear theory, $\mu_m = N_{YT}/2(N_m - N_{Y0})$ is the load (or excitation) parameter, in which N_{YT} is the dimensionless amplitude of the harmonic in-plane loading, and $\theta(T)$ is the total phase angle of harmonic excitation.

The set of equations (8) constitutes an infinite number of simultaneous nonlinear ordinary differential equations. In practice, however, only a finite number of these equations are taken into account for the solution. Moreover, the plate theory used in the present analysis restricts the study to an investigation of the lower flexural modes which are generally the most important in dynamic stability problems. Then, bearing in mind that the actual problem concerns only three spatial forms of vibration, the set of differential equations of motion can be generalized as follows

$$\begin{aligned}
\ddot{W}_i + \Omega_i^2 W_i &= -2C_i \dot{W}_i + 2\mu_i \Omega_i^2 \cos\theta W_i - (\Gamma_{i1} W_i^3 + \Gamma_{i2} W_i^2 W_j + \Gamma_{i3} W_i^2 W_k + \Gamma_{i4} W_i W_j^2 \\
&\quad + \Gamma_{i5} W_i W_k^2 + \Gamma_{i6} W_j^3 + \Gamma_{i7} W_j^2 W_k + \Gamma_{i8} W_j W_k^2 + \Gamma_{i9} W_k^3 + \Gamma_{i10} W_i W_j W_k) \\
\ddot{W}_j + \Omega_j^2 W_j &= -2C_j \dot{W}_j + 2\mu_j \Omega_j^2 \cos\theta W_j - (\cdots \Gamma_j \cdots) \\
\ddot{W}_k + \Omega_k^2 W_k &= -2C_k \dot{W}_k + 2\mu_k \Omega_k^2 \cos\theta W_k - (\cdots \Gamma_k \cdots)
\end{aligned} \tag{9}$$

where Γ_{m1} through Γ_{m10} are coefficients of the nonlinear (cubic) terms, and i, j, k are three generalized spatial forms of vibration appeared in the possible internal resonances. Hence, the number of terms in the expansion for the lateral displacement depends on the highest mode appeared in the internal resonance to be analyzed.

4 Solution of the Temporal Equations of Motion

Mathematical techniques for solving nonlinear problems are relatively limited and approximate methods are generally used. The method of asymptotic expansion in powers of a small parameter, ε, is an effective tool for studying nonlinear vibrating systems with slowly varying parameters. In the present analysis, this method is used to solve the temporal equations of motion.

Assuming that the actual mechanical system is weakly nonlinear, the damping, the excitation and the nonlinearity can be expressed in terms of the above-mentioned small parameter, and that the instantaneous frequency of excitation and the load parameter vary slowly with time. Then, the system of temporal equations of motion (9) can be rewritten in the following asymptotic form

$$\ddot{W}_m + \Omega_m^2 W_m = \varepsilon \left[2\mu_m \Omega_m^2 \cos\theta W_m - 2C_m \dot{W}_m - \sum_i \sum_j \sum_k M_m^{ijk} W_i W_j W_k \right], m = i, j, k \tag{10}$$

Confining ourselves to the first order of approximation in ε, we seek a solution for the system of equation (10) in the following form

$$W_m = a_m(\tau) \cos\psi_m(\tau) \tag{11}$$

where $\tau = \varepsilon T$ represents the *slowing* time, and where a_m and ψ_m are functions of time defined by the system of differential equations

$$da_m / dT = \dot{a}_m = \varepsilon A_1^m \left(\tau, \theta, a_m, \psi_m\right) \tag{12}$$

$$d\psi_m / dT = \dot{\psi}_m = \Omega_m(\tau) + \varepsilon B_1^m \left(\tau, \theta, a_m, \psi_m\right) \tag{13}$$

Functions $A_1^m(\tau, \theta, a_m, \psi_m)$ and $B_1^m(\tau, \theta, a_m, \psi_m)$ are selected in such a way that equation (11) will, after replacing a_m and ψ_m by the functions defined in equations (12) and (13), represent a solution of (10).

Following the general scheme of constructing asymptotic solutions and performing numerous transformations and manipulations, we arrive finally at a system of equations describing the nonstationary response of the discretized system.

5 Stationary Response Related to Simultaneous Resonances

It is known that the governing equations with cubic nonlinearities are associated with many physical systems. The presence of these nonlinear terms has an important influence upon the behavior of the system, especially under a condition of internal resonance. An internal resonance is possible when two or more natural frequencies are commensurable or almost commensurable

$$\sum_i n_i \Omega_i \cong 0 \tag{14}$$

where n_i are positive or negative integers. For convenience, of all possible internal resonances associated with a flat rectangular plate, we will consider in this analysis only an internal resonance of the type $2\Omega_k - \Omega_i - \Omega_j \approx 0$.

When an internal resonance coincides with an external resonance (in this case, it is a principal parametric resonance when the excitation frequency is approximately equal to twice the natural frequency associated with a particular mode of vibration, i.e., $\Lambda \approx 2\Omega_m$,) the combination of the two types gives rise to simultaneous resonances. This kind of resonances is characterized by the fact that the system in question vibrates simultaneously in more than one normal mode and at different frequencies, although only one of the modes is directly excited by the parametric excitation. In this paper, the following case of simultaneous resonances is investigated: $\Lambda \approx 2\Omega_k$ and $2\Omega_k - \Omega_i - \Omega_j \approx 0$.

As mentioned previously, performing numerous transformations and manipulations, stationary values for the specified simultaneous resonances are found to be

$$-C_j a_j - \frac{r}{8\Omega_j} \Gamma_{j6} a_j a_k^2 \sin\psi' = 0 \tag{15a}$$

$$\begin{aligned} &2\Omega_k - \Omega_j - \Omega_i + \left(\frac{3\Gamma_{k13}}{4\Omega_k} - \frac{\Gamma_{j11}}{4\Omega_j} - \frac{\Gamma_{i6}}{4\Omega_i} \right) a_k^2 - \frac{2}{\Lambda} \mu_k \Omega_k^2 \cos\psi + [\left(\frac{\Gamma_{k3}}{2\Omega_k} - \frac{\Gamma_{j2}}{4\Omega_j} - \frac{3\Gamma_{i1}}{8\Omega_i} \right) r^2 + \\ &\frac{\Gamma_{k9}}{2\Omega_k} - \frac{3\Gamma_{j8}}{8\Omega_j} - \frac{\Gamma_{i5}}{4\Omega_i}] a_j^2 + [\frac{r\Gamma_{k17}}{4\Omega_k} a_j^2 - \left(\frac{r\Gamma_{j6}}{8\Omega_j} + \frac{\Gamma_{i11}}{8r\Omega_i} \right) a_k^2] \cos\psi' = 0 \end{aligned} \tag{15b}$$

$$-C_k a_k + \frac{1}{\Lambda}\mu_k \Omega_k^2 a_k \sin\psi + \frac{r\Gamma_{k17}}{8\Omega_k} a_j^2 a_k \sin\psi' = 0 \tag{15c}$$

$$\Lambda - 2\Omega_k - [\frac{r^2\Gamma_{k3}}{2\Omega_k} + \frac{\Gamma_{k9}}{2\Omega_k}]a_j^2 - \frac{3\Gamma_{k13}}{4\Omega_k} a_k^2 + \frac{2}{\Lambda}\mu_k \Omega_k^2 \cos\psi - \frac{r\Gamma_{k17}}{4\Omega_k} a_j^2 \cos\psi' = 0 \tag{15d}$$

in which

$$r = \frac{a_i}{a_j} = \left[\frac{\Gamma_{i11}\Omega_j C_j}{\Gamma_{j6}\Omega_i C_i}\right]^{1/2} \tag{16}$$

and where $\psi = \theta - 2\psi_k$ is the phase angle associated with the principal parametric resonance involving mode k, and $\psi' = 2\psi_k - \psi_j - \psi_i$ represents the phase angle corresponding to the above-mentioned internal resonance. The steady-state amplitudes, a_i, a_j and a_k, and the phase angles, ψ and ψ', can be obtained by solving equations (15) by a numerical technique. Nevertheless, from equation (16), it follows that one of the resonance conditions is

$$\operatorname{sgn}\Gamma_{i11} = \operatorname{sgn}\Gamma_{j6} \tag{17}$$

It appears from equations (15) that there are two possibilities for a nontrivial solution Either a_k is nonzero and both a_i and a_j are zero, or all three are nonzero. The first possibility indicates that the specified internal resonance has no effect on the system response and only the principal parametric resonance involving mode k may occur. For the latter possibility, as mode k is the only mode excited by the parametric excitation, the presence of two other modes i and j in the response is possible only by the transfer of energy from the excited mode to these two modes through internal mechanism.

6 Numerical Results

In order to gain further insight into the occurrence of simultaneous resonances involving three spatial forms of vibration, numerical evaluation was performed for two thin rectangular plates of different aspect ratios. The various values of the plate parameters and material constants used for the numerical calculations are presented in Table 1. The specific material properties are selected in accordance with rectangular plates cut from commercially available sheets.

Specimen	Dimensions (mm) $a \times b \times h$	Aspect ratio R
P-1	440 × 762 × 1.016	1.7318
Material: Polycarbonate Modulus of elasticity, E = 2.385586 Gpa Poisson's ratio, ν = 0.45 Density, ρ = 1205.48 kg/m^3		
P-2	174.6 × 508 × 0.914	2.9095
Material: Plexiglas Modulus of elasticity, E = 4.412645 Gpa Poisson's ratio, ν = 0.38 Density, ρ = 1187.85 kg/m^3		

Table 1. Specifications of plate parameters

For the numerical evaluation, both specimens have been reduced to seven-degree-of-freedom systems. Loaded and unloaded (free) natural frequencies, as well as possible internal resonances, of each specimen are given in Table 2. In this table, $P_{cr} = N_{Y0}/N^*$ designates the ratio of static critical loading, in which N^* is the lowest critical load, and is used as a parameter in the calculations.

Plate	Mode m	Free natural frequency (Hz) ω_m	Loaded natural frequency (Hz) Ω_m ($P_{cr} = 0.5$)	Internal resonances
P-1	1	4.998473	3.926654	$\Omega_3 - 3\Omega_1 \cong 0$
	2	8.748084	6.185829	$2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$
	3	14.997434	11.782527	$2\Omega_3 - \Omega_7 + \Omega_5 \cong 0$
	4	23.746525	20.269177	$\Omega_1 + \Omega_2 + \Omega_5 - \Omega_6 \cong 0$
	5	34.995357	31.393024	
	6	48.743928	45.073165	
	7	64.992240	61.280105	
P-2	1	32.456921	28.536560	$2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$
	2	42.744122	29.505831	$2\Omega_2 - \Omega_7 + \Omega_5 \cong 0$
	3	59.889458	37.877419	$2\Omega_3 - \Omega_6 + \Omega_1 \cong 0$
	4	83.892927	56.675918	
	5	114.754531	84.797939	
	6	152.474269	120.996734	
	7	197.052142	164.659768	

Table 2. Natural frequencies and possible internal resonances of both specimens

Of all internal resonances mentioned, only the internal resonance $2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$, together with the principal parametric resonance $\Lambda = 2\Omega_5$, will be considered in this paper. Typical results associated with the specified simultaneous resonances are shown in Figures 2 to 4. In the figures, D_{cr} (= N_{YT}/N^*) denotes the dynamic component N_{YT} of the periodic in-plane force normalized to the lowest critical load N^*, and is called the ratio of dynamic critical loading; Δ (= $2\pi C_m/\Omega_m$) is the decrement of viscous damping and has a value of 0.01.

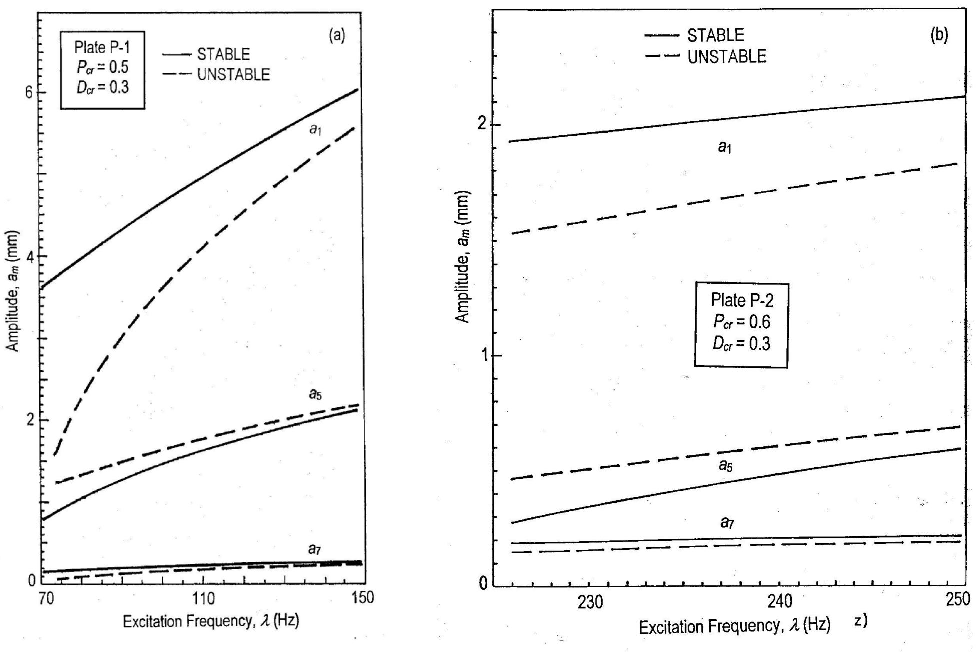

Figure 2. Frequency-response curves associated with the simultaneous resonances $\Lambda = 2\Omega_5$ (principal parametric) and $2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$ (internal)

In Figures 2 and 3, the results show the response amplitudes as functions of the excitation frequency (λ, in Hz). In Figure 2, the frequency-response curves are associated with the simultaneously occurring resonances $\Lambda = 2\Omega_5$ (principal parametric resonance involving the fifth spatial form) and $2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$ (internal resonance involving the first, fifth and seventh mode shapes). Hence the fifth mode is excited parametrically and the presence of the first and seventh spatial forms is possible only by the transfer of energy from the directly-excited fifth mode to two other modes through internal mechanism. It is interesting to note that the lowest mode (first mode) dominates the response, and the unstable branch of the parametrically-excited fifth mode is over its stable solution. This study confirms again that the lowest mode always dominates the deflection, no matter what parametrically-excited mode is.

The interaction between the specified internal resonance and the mentioned principal parametric resonance on the frequency-response curves is illustrated in Figure 3. As can be seen, the parametric response of the fifth mode occurs when $\Lambda \cong 2\Omega_5$. At a certain frequency, however, the specified internal resonace occurs simultaneously; this causes the amplitude of the first mode, which is directly excited by the parametric excitation, drops drastically and becomes less than the amplitude of the first mode, which is due to internal resonance, but is still higher than the amplitude of the seventh mode. This implies that there is a significant transfer of energy from the fifth mode to the first mode, and only a small amount of energy from the fifth mode to the seventh mode.

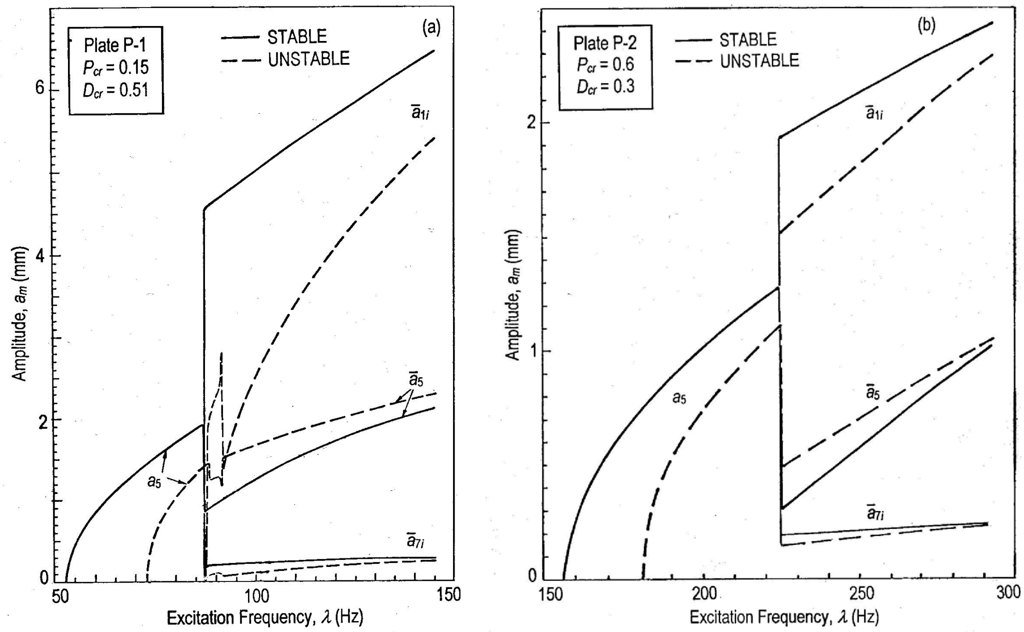

Figure 3. Effect of the internal resonance $2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$ on the frequency-response curves corresponding to the principal parametric resonance $\Lambda = 2\Omega_5$

Figure 4 depicts the load amplitude-response curves associated with the specified simultaneous resonances. The results illustrate again the domination of the first mode to the deflection of the motion and show once more the significance of energy transfer when these kinds of resonances occur.

7 Concluding Remarks

The present work covers an existing gap in our understanding of the parametric excitation of continuous systems and presents a rational analysis of the occurrence of simultaneous resonances involving three spatial forms of vibration in parametrically-excited rectangular plates.

The results of this investigation indicate that the coincidence between a principal parametric resonance and an internal resonance will give rise to simultaneous resonances. Simultaneous resonances are characterized by the fact that the system in question resonates simultaneously in more than one normal mode although only one of the modes is parametrically excited by a single harmonic excitation. Internal resonance is responsible for strong modal coupling and, consequently, for a significant transfer of energy from the excited mode into other nonexcited modes. Because of this modal interaction, modes other than the one excited can dominate the response and, generally, the lowest mode dominates higher modes.

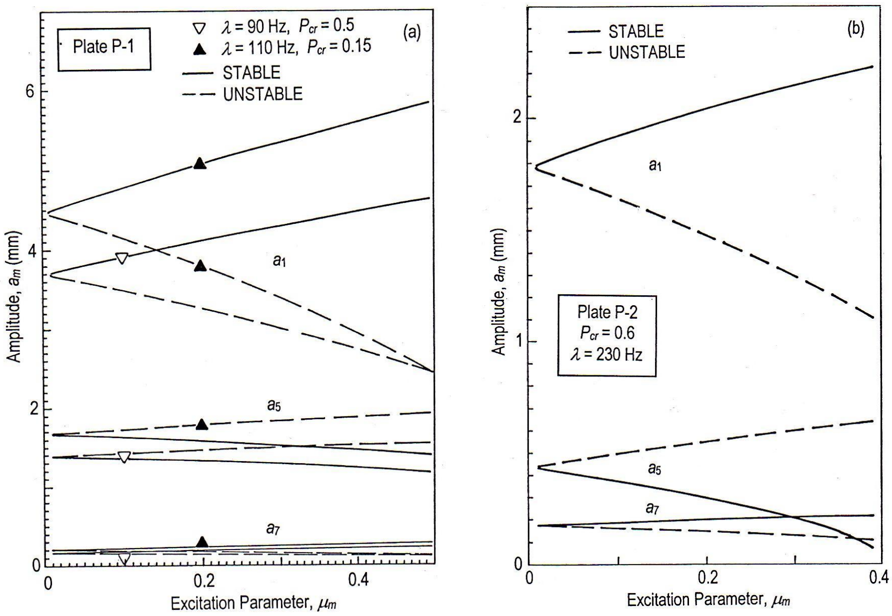

Figure 4. Load amplitude-response curves associated with the simultaneous resonances $\Lambda = 2\Omega_5$ and $2\Omega_5 - \Omega_7 - \Omega_1 \cong 0$

References

[1] H. Nguyen and G.L. Ostiguy: Effect of boundary conditions on the dynamic instability and non-linear response of rectangular plates, Part I: Theory, *J. Sound and Vibration* (1989) **133**(3), 381-400.

[2] H. Nguyen, G.L. Ostiguy, and L.P. Samson: Effect of boundary conditions on the dynamic instability and non-linear response of rectangular plates, Part II: Experiment, *J. Sound and Vibration* (1989) **133**(3), 401-422.

[3] G.L. Ostiguy, L.P. Samson, and H. Nguyen: On the occurrence of simultaneous resonances in parametrically-excited rectangular plates, *Trans. ASME, J. Vibration and Acoustics* (1993) **115**, 344-352.

[4] H. Nguyen and G.L. Ostiguy: Recent developments on the dynamic stability and response of parametrically-excited rectangular plates, *JSAEM Studies in Applied Electromagnetics and Mechanics*, K. Miya, M. Yamamoto, and X.H. Nguyen (eds.), (1998) **6**, 102-111.

[5] H. Nguyen and G.L. Ostiguy: On the application of two perturbation methods to nonlinear systems, *IUTAM Symposium on Recent Developments in Non-linear Oscillations of Mechanical Systems (Hanoi, 1999)*, Kluwer Academic Publishers, Dordrecht, 2000, 165-174.

[6] Nguyen Hai: Simultaneous resonances of parametrically-excited rectangular plates, *Proceedings of the International Symposium on Dynamics and Control (Hanoi, 2003)*, E.J. Kreuzer and V.K. Nguyen (eds.), Vietnam National University Publishers, Hanoi, 2004, 45-56.

Address: Nguyen Hai, Hochiminh City University of Technology, Hochiminh City, Vietnam
email: haiha96@yahoo.com