

Load Bearing Capacity of Thin Shell Structures Made of Elastoplastic Material by Direct Methods

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A method is introduced to determine the limit load of thin shells using the finite element method. The method is based on an upper bound limit and shakedown analysis with the elastic-perfectly plastic material model. A nonlinear constrained optimisation problem is solved by using Newton's method in conjunction with a penalty method and the Lagrange dual method. The numerical investigation of a pipe bend subjected to bending moments proves the effectiveness of the algorithm.

1 Introduction

In practical engineering, the calculation of the load carrying capacity for structures has been a problem of great interest to many designers. In the early 20th century, it could be relatively easily obtained by imposing the stress intensity at a certain point of the structure equal to the yield stress of the material. This implies that structural failure occurs before yielding. However, many materials, for example the majority of metals, exhibit distinct, plastic properties. Such materials can deform considerably without breaking, even after the stress intensity attains the yield stress. This implies that if the stress intensity reaches the critical (yield) value, the structure does not necessarily fail or deform extensively. To this case, elastic-plastic structural analyses permit higher loads. All design codes for pressure vessels make implicitly use of Limit and Shakedown Analysis (LISA) in the assessment of elastic stresses. This is a kind of extrapolation of linear elastic analyses to the plastic behaviour which is critically discussed in (Taylor et al., 1999). In Europe LISA has been developed as direct plasticity method for the design and the safety analysis of severely loaded engineering structures, such as nuclear power plants and chemical plants, offshore structures etc. (Gokhfeld and Cherniavsky, 1980), (König, 1987), (Staat 2002; Staat and Heitzer, 2003). Annex B of the new European pressure vessel standard EN 13445-3 is based on LISA (European standard, 2005-06), (Taylor et al., 1999), (Zeman, 2006) thus indicating the industrial need for LISA software. All design codes are based on perfect plastic models. The extension of LISA to hardening materials is no problem (Staat and Heitzer, 2002). The direct route calculates the design resistance (limit action) with respect to ultimate limit states of the structure. The following limit states are included in EN 13445-3 Annex B:

- Gross Plastic Deformation (GPD), with excessive local strains and ductile rupture (collapse).
- Progressive plastic Deformation (PD), with incremental collapse (incremental collapse, ratchetting).
- Instability (I), with large displacements to a new stable geometry of the structure under compressive actions (buckling).
- Fatigue (F), with alternating plasticity (AP) or with high cycle fatigue.
- Static Equilibrium (SE), with possible overturning and rigid body movement.

A structure is said to shakedown under a load history if all plastic deformations decay and the plastic dissipation is bounded so that the structural response becomes asymptotically elastic. The PD check and the (non-mandatory) AP check can be made directly by shakedown analysis. Limit analysis is included as the special case of a monotonic load path and is used in a direct DPD check.

Shell structures are used in many engineering applications due to their efficient load carrying capacity relative to material volume. From the engineering point of view, shells often allow to build structures with high strength and stiffness and relatively low weight. From the analysis point of view, shell structures stand for a challenging problem mostly due to the three-dimensional finite rotations. Plasticity in shell structures is accounted either by means of integration of stress over the thickness (layer approach), or stress resultant modelling. By the latter approach the yield surface becomes more complicated than with the former approach, thus requiring a special consideration in algorithm for updating of the stress resultant. Ilyushin (1948) has derived a yield surface which expresses the von Mises yield function in terms of the stress resultants of thin shells and only neglects the effects of the transverse shear forces. In this sense it is an exact yield surface for geometrically linear problems.

Landgraf (1968) has presented a method for determining the yield condition in stress resultants including shear forces axisymmetric plate and shell problems, based solely on Feinberg's static principle and hence not involving any kinematic considerations. The linear approximation of the exact Ilyushin yield surface has been recommended in (Taylor et al., 1999) for the design checks of pressure vessels. Bisbos and Papaioannou (2006) have used it for lower bound shakedown analysis of steel shells. Second-order cone and semidefinite formulations of material yield and failure criteria with application to the linearized Ilyushin yield surface can be found e.g. in (Bisbos, Pardalos, 2007). In the corrections of (Taylor et al., 1999) it is remarked that the linear approximation of the Ilyushin function is not accurate enough for limit analyses and that Ivanov's quadratic approximation is almost as fast but much more accurate.

This paper concerns the application of a kinematic formulation for the finite element limit and shakedown analysis of general thin shells. The technique is based on an upper bound approach using the re-parameterized exact Ilyushin yield surface and a nonlinear optimization procedure. The solution of the problem is obtained by discretizing the shell into finite elements. The exact Ilyushin yield surface has been used by Seitzberger (2000) to solve plastic buckling collapse problems of thin-walled structures with the so-called sequential limit analysis method. It is typical for the direct plasticity methods that the development of algorithms for the structural problem is influenced by the material modelling which is here isotropic, elastic perfectly-plastic von Mises material. In this sense the paper does not follow the design codes which prescribe the Tresca yield function for limit analyses and the von Mises yield function for shakedown analyses. Using different yield surfaces for the same material may be the legal approach of pressure vessel design but not a particularly physical one.

Nomenclature

α^+	upper bound of shakedown load factor
σ_y	yield stress
N_{ij}, M_{ij}	physical in-plane and flexural stress resultant components
\mathbf{n}, \mathbf{m}	normalized in-plane and flexural stress resultant vectors
$\bar{\varepsilon}_{ij}, \kappa_{ij}$	physical mid-plane strain and curvature components
$\dot{\bar{\varepsilon}}, \mathbf{k}$	normalized mid-plane strain rate and curvature vectors
$\dot{\bar{\varepsilon}}, \tilde{\sigma}$	“engineering” strain rate and stress resultant vectors
$N_0, M_0, \varepsilon_0, \kappa_0$	normalized quantities
$P_\varepsilon, P_{\varepsilon\kappa}, P_\kappa$	intensities of quadratic incremental strain resultants
$Q_\varepsilon, Q_{\varepsilon\kappa}, Q_\kappa$	intensities of quadratic stress resultants
ν, β, γ	new parameters of the Burgoyne and Brennan yield surface
$d\bar{\varepsilon}^p$	incremental plastic strain resultant vector
m	number of load vertices ($m = 1$ in limit analysis)
AP	alternating plastic deformation (low cycle fatigue)
GPD	gross plastic deformation (collapse)
PD	progressive plastic deformation (ratchetting)
NG	total number of Gauss' points on the structure

2 Plastic Dissipation Function in Term of Stress Resultants

Consider a convex polyhedral load domain \mathcal{L} and a special load path consisting of all load vertices \hat{P}_k ($k = 1, \dots, m$) of \mathcal{L} . In lower bound shakedown analysis an interior approximation to the shakedown load domain $\alpha\mathcal{L}$ is calculated. Melan's static theorem states that a given structure will shakedown if the yield condition is satisfied and the stresses are statically admissible for all load vertices \hat{P}_k ($k = 1, \dots, m$), (Gokhfeld and Cherniavsky, 1980), (König, 1987). The maximum safe load factor α^- is obtained for which the structure is safe against GPD, PD and AP for all load histories included in $\alpha^- \mathcal{L}$.

In upper bound shakedown analysis an exterior approximation of the shakedown load domain is calculated. Koiter's kinematic theorem states that a given structure will not shakedown over a certain load path if the total

plastic strain rate $\Delta \dot{\tilde{\boldsymbol{\varepsilon}}} = \sum_{k=1}^m \dot{\tilde{\boldsymbol{\varepsilon}}}_k$ over this load path is kinematically compatible and the external power exceeds the plastically dissipated power $D^p(\dot{\tilde{\boldsymbol{\varepsilon}}})$, (Gokhfeld and Cherniavsky, 1980), (König, 1987). The minimum failure load factor α^+ is obtained for which the structure may fail by GPD, PD or AP for some load history included in $\alpha^+ \mathcal{L}$. It is shown in optimization theory that the largest lower bound and the least upper bound are the same (i.e. $\sup\{\alpha^-\} = \inf\{\alpha^+\}$) so that both methods give the same solution. The GPD check is usually the more restrictive one because it involves a safety factor whereas no safety factor is included in the PD check. Therefore the numerical examples in this paper concentrate on limit analyses. Limit analysis is the special case that \mathcal{L} contains only one point. The only load vertex \hat{P}_1 ($m=1$) represents a monotonous loading to the plastic collapse at the limit load $\alpha \hat{P}_1$.

Let h be the shell thickness and σ_y the uniaxial yield stress. The non-dimensional ‘engineering’ stress and strain resultant vectors are introduced as follows

$$\begin{aligned} \tilde{\boldsymbol{\sigma}} &= [\mathbf{n} \quad \mathbf{m}]^T, \quad \mathbf{n} = \frac{1}{N_0} [N_{11} \quad N_{22} \quad N_{12}]^T, \quad \mathbf{m} = \frac{1}{M_0} [M_{11} \quad M_{22} \quad M_{12}]^T \\ \tilde{\boldsymbol{\varepsilon}} &= [\bar{\mathbf{e}} \quad \mathbf{k}]^T, \quad \bar{\mathbf{e}} = \frac{1}{\varepsilon_0} [\bar{\varepsilon}_{11} \quad \bar{\varepsilon}_{22} \quad 2\bar{\varepsilon}_{12}]^T, \quad \mathbf{k} = \frac{1}{\kappa_0} [\kappa_{11} \quad \kappa_{22} \quad 2\kappa_{12}]^T \end{aligned} \quad (1)$$

in which $N_0 = \sigma_y h$, $M_0 = \sigma_y h^2 / 4$, $\varepsilon_0 = \sigma_y (1 - \nu^2) / E$ and $\kappa_0 = 4\varepsilon_0 / h$ are the normalized quantities, $N_{\alpha\beta}, M_{\alpha\beta}$ ($\alpha, \beta \in \{1, 2\}$) are the physical in-plane force and bending moment components (see figure 1) and $\bar{\varepsilon}_{\alpha\beta}, \kappa_{\alpha\beta}$ ($\alpha, \beta \in \{1, 2\}$) are physical mid-plane strain and curvature components. The quadratic strain intensities are defined in terms of the incremental ‘engineering’ strain resultant by

$$\begin{aligned} P_\varepsilon &= \frac{3}{4} (d\bar{\mathbf{e}}^p)^T \mathbf{P}^{-1} d\bar{\mathbf{e}}^p = \frac{3}{4} (d\tilde{\boldsymbol{\varepsilon}}^p)^T \mathbf{P}_1 d\tilde{\boldsymbol{\varepsilon}}^p \quad (\geq 0), \\ P_{\varepsilon\kappa} &= 3 (d\bar{\mathbf{e}}^p)^T \mathbf{P}^{-1} d\mathbf{k}^p = 3 (d\tilde{\boldsymbol{\varepsilon}}^p)^T \mathbf{P}_2 d\tilde{\boldsymbol{\varepsilon}}^p, \\ P_\kappa &= 12 (d\mathbf{k}^p)^T \mathbf{P}^{-1} d\mathbf{k}^p = 12 (d\tilde{\boldsymbol{\varepsilon}}^p)^T \mathbf{P}_3 d\tilde{\boldsymbol{\varepsilon}}^p \quad (\geq 0). \end{aligned} \quad (2)$$

where $d\tilde{\boldsymbol{\varepsilon}}^p$ is the plastic strain increment resultant vector, \mathbf{P} and its inverse \mathbf{P}^{-1} , \mathbf{P}_i ($i=1, 2, 3$) are

$$\begin{aligned} \mathbf{P} &= \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \\ \mathbf{P}_1 &= \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{P}^{-1}/2 \\ \mathbf{P}^{-1}/2 & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_3 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} \end{pmatrix}. \end{aligned} \quad (3)$$

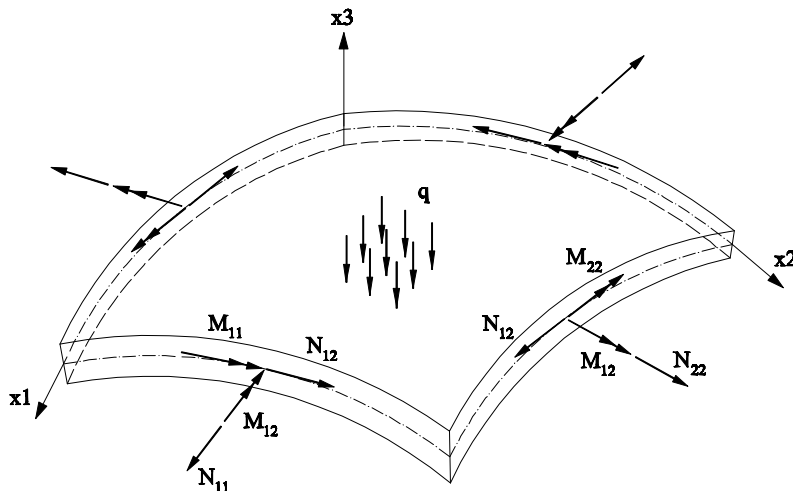


Figure 1. Static shell quantities

Ilyushin (1948) published the derivation of a stress resultant yield surface as presented in figure 2, describing the case where a cross-section of a shell is fully plastified. The derivation of this yield surface is based on the following assumptions: perfectly plastic isotropic material behaviour obeying the von Mises' yield criterion, the validity of the normality rule for the plastic deformations, plane stress conditions in each material point and the validity of the Kirchhoff hypothesis for both total strains and plastic strains. Corresponding to the quadratic strain resultant intensities, quadratic stress resultant intensities can also be defined

$$\begin{aligned} Q_t &= \mathbf{n}^T \mathbf{P} \mathbf{n} \quad (\geq 0), \\ Q_m &= \mathbf{n}^T \mathbf{P} \mathbf{m}, \\ Q_n &= \mathbf{m}^T \mathbf{P} \mathbf{m} \quad (\geq 0). \end{aligned} \quad (4)$$

However, this yield surface has not been used because the parametric form in which it was described was not amenable to numerical calculation because it is ill-conditioned and unstable. In order to avoid the difficulties arising with the parameterization of the Ilyushin yield surface and to use the exact yield surface in practical computations, Burgoyne and Brennan (1993) introduced three new parameters

$$\nu = \frac{P_\varepsilon}{P_\kappa}, \quad \beta = -\frac{P_{\varepsilon\kappa}}{P_\kappa} \quad \text{and} \quad \gamma = \nu - \beta^2. \quad (5)$$

With these parameters, the plastic dissipation function per unit area of the mid-plane of the shell may be written in the form (Tran et al., 2007)

$$D^p(\dot{\boldsymbol{\varepsilon}}^p) = N_0 \varepsilon_0 \sqrt{\frac{P_\kappa}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right) \quad (6)$$

where β_1 , β_2 and K_0 are

$$\begin{aligned} \beta_1 &= 0.5 - \beta, \quad \beta_2 = 0.5 + \beta, \\ K_0 &= \ln \left(\frac{\sqrt{(0.5 - \beta)^2 + \gamma} + (0.5 - \beta)}{\sqrt{(0.5 + \beta)^2 + \gamma} - (0.5 + \beta)} \right). \end{aligned} \quad (7)$$

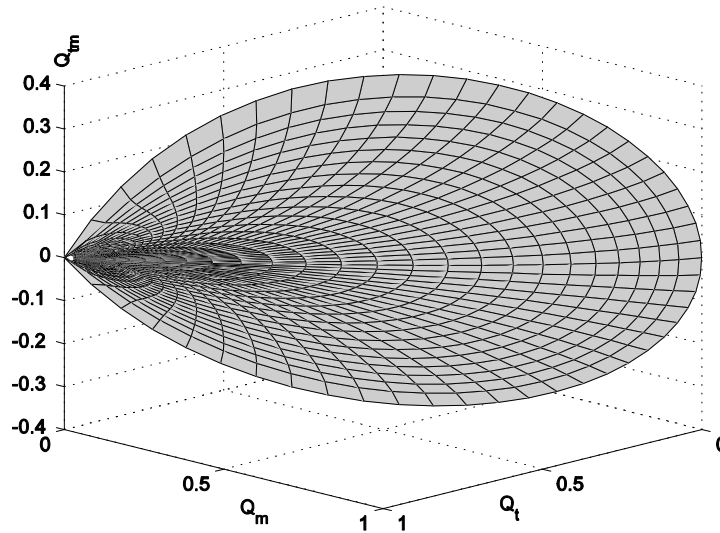


Figure 2. Exact Ilyushin yield surface eq (4) constructed in terms of β and γ ; the edge line is $\gamma = 0$.

It is noted that D^p is convex but not everywhere differentiable (Capsoni and Corradi, 1997). In order to allow a direct nonlinear, non-smooth, constrained optimization problem, a “smooth regularization method” should be used by adding to γ and P_κ a small positive number, namely η^2 . Thus, in this case, equation (7) is amenable to a numerical evaluation for all values of $\dot{\boldsymbol{\varepsilon}}^p$.

3 Upper Bound Limit and Shakedown Algorithm for General Shell Structures

The kinematic shakedown theorem (Koiter's theorem) states that a given structure will not shakedown over a certain load path contained within the load domain \mathcal{L} if the total plastic strain rate over this load path (cycle) is kinematically compatible and the external power exceeds the internally dissipated power. It means that the upper bound shakedown load factor α^+ which is the smaller one of the low cycle fatigue limit, and the ratchetting limit is the ratio between the dissipated power and the external power. By discretizing the whole structure by finite elements and forcing the external power equal to one, the shakedown limit, may be found by the following minimization

$$\alpha^+ = \min \sum_{k=1}^m \sum_{i=1}^{NG} w_i N_0 \varepsilon_0 \sqrt{\frac{P_K}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^m \dot{\tilde{\mathbf{e}}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = 1, \dots, NG \\ \sum_{k=1}^m \sum_{i=1}^{NG} w_i N_0 \varepsilon_0 \dot{\tilde{\mathbf{e}}}_{ik}^T \tilde{\boldsymbol{\sigma}}_{ik}^E = 1. \end{cases} \quad (8)$$

The first constraint means that the total generalized strain rate over a load cycle $\dot{\tilde{\mathbf{e}}}$ must be kinematically compatible. The second one is the normalization condition implying that the total external power is equal to one. \mathbf{B}_i denotes the deformation matrix, $\dot{\mathbf{u}}$ is the displacement rate vector, w_i is the weighting factor of the Gauss point i^{th} and NG is the total number of Gauss points in the structure. $\tilde{\boldsymbol{\sigma}}^E$ denotes the fictitious elastic generalized stress vector which would appear in an infinitely elastic material for the same loading. By introducing some new notations

$$\dot{\mathbf{e}}_{ik} = w_i \dot{\tilde{\mathbf{e}}}_{ik}, \quad \mathbf{t}_{ik} = N_0 \varepsilon_0 \tilde{\boldsymbol{\sigma}}_{ik}^E, \quad \hat{\mathbf{B}}_i = w_i \mathbf{B}_i \quad (9)$$

where $\dot{\mathbf{e}}_{ik}$, \mathbf{t}_{ik} , $\hat{\mathbf{B}}_i$ are the new strain rate vector, new fictitious elastic stress vector, and new deformation matrix, respectively, we obtain a simplified version for the upper bound shakedown analysis

$$\alpha^+ = \min \sum_{k=1}^m \sum_{i=1}^{NG} N_0 \varepsilon_0 \sqrt{\frac{P_K}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^m \dot{\mathbf{e}}_{ik} = \hat{\mathbf{B}}_i \dot{\mathbf{u}} & \forall i = 1, \dots, NG \\ \sum_{k=1}^m \sum_{i=1}^{NG} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} = 1. \end{cases} \quad (10)$$

This is a nonlinear constrained optimization problem. By applying Newton's method in conjunction with a penalty method and the Lagrange dual method to solve the Karush-Kuhn-Tucker optimality conditions of the system (10) we obtain the Newton directions $d\dot{\mathbf{u}}$ and $d\mathbf{e}_{ik}$, which assure that a suitable step along them will lead to a decrease of the objective function α^+ . If the relative improvement between two steps is smaller than a given constant, the algorithm stops and leads to the shakedown limit factor (Tran et al., 2007). It is noted, that in limit analysis there is only one load and this load does not vary. Then the load domain reduces to one point ($m = 1$). This fact means that the above upper bound of the shakedown load factor reduces to the upper bound of the limit load factor.

4 Numerical Example

We present in this section the numerical calculation of plastic collapse limit load for a well-known shell problem in order to evaluate our algorithm. Consider an 90° elbow with mean radius r , bend radius of curvature R and thickness h . One of its ends is supposed clamped and the other one is subjected to a constant in-plane closing moment M_I or a constant out-of-plane bending moment M_{II} as shown in figure 3b. The curvature factor is defined as follow

$$\lambda = \frac{Rh}{r^2}. \quad (11)$$

Generally, $\lambda \leq 0.5$ corresponds to a highly-curved pipe, while $\lambda \rightarrow \infty$ corresponds to a straight pipe. In order to evaluate the model, different values of λ within the range $[0.1, 1.2]$ are examined. Our model that is used for elastic-plastic analysis is meshed by 700 quadrangular flat 4-node shell elements as shown in figure 3a. The elastic-perfectly plastic material model is used with $E = 208000$ MPa, $\nu = 0.3$, $\sigma_y = 250$ MPa. For each test case, some existing analytical and numerical solutions found in literature including large displacement analysis are briefly represented and compared for the sake of completeness.

An external moment applied to one end of the pipe bend, tends to deform the annular cross section significantly both in and out of its own plane, i.e. it is subjected to warping and ovalization. The moment-end rotation curves show a defined limit load behaviour for the closing mode of in-plane bending but not for the opening mode (Shalaby, Younan, 1998). This difference can be explained as an effect of large displacements. We do not consider the opening mode here. The direct plasticity methods have been used to treat stochastic uncertainties with great advantages in (Tran et al. 2007). The direct limit and shakedown analysis is not only very effective but it also provides a well defined limit state function and the sensitivities that are required for first and second order reliability analysis. This approach could be extended to stochastic fields to consider the effect of imperfections such as thickness variations on the collapse of shells.

Elbow under in-plane closing Bending Moment

We define the limit load factor $\alpha_l = M_l / M_l^s$, where M_l is the limit moment of the elbow and M_l^s is the limit moment of the straight pipe which has the same radius as the elbow. Calladine (1974) proposed a lower bound solution for an infinite, strongly-curved pipe ($\lambda \leq 0.5$)

$$\alpha_l^c = 0.9346\lambda^{2/3}. \quad (12)$$

This solution is considered in the literature to come close to the experimental limit load factor (Bolt and Greenstreet, 1972; Goodall, 1978; Griffiths, 1979). According to Yan (1997), it is a good approximation when $\lambda < 0.7$. For a slightly-curved pipe ($\lambda \geq 0.7$), he proposed an approximate solution which is validated by numerical analysis

$$\alpha_l^y = \cos\left(\frac{\pi}{6\lambda}\right). \quad (13)$$

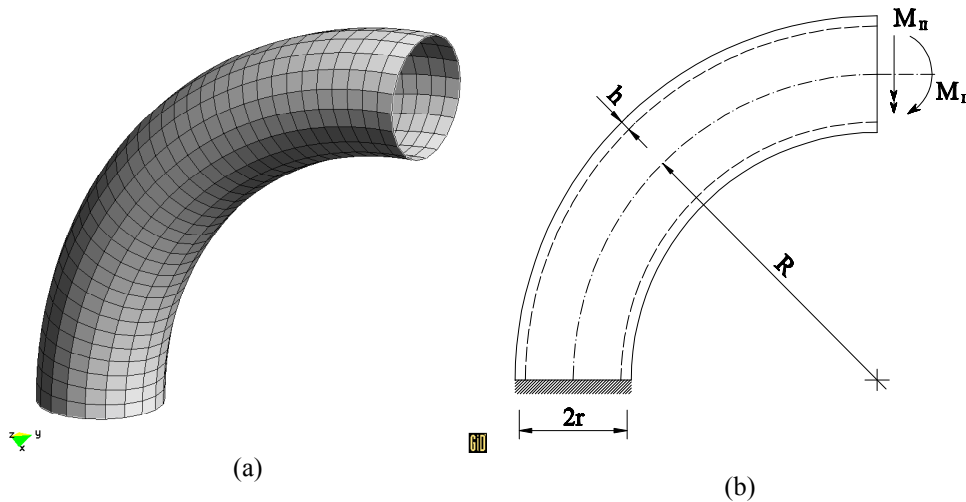


Figure 3. FE-mesh and geometrical dimensions

Desquines (1997) proposed a more general analytical solution as a lower bound, which can be applied for any value of λ

$$\alpha_I^{De} = \frac{1}{\sqrt{1 + 0.3015\lambda^{-2}}}. \quad (14)$$

Spence and Findlay (1973) also expressed an analytical solution for the limit load of an elbow

$$\alpha_I^{SF} = 0.8\lambda^{0.6}, \quad \lambda < 1.45. \quad (15)$$

All the foregoing expressions are based on small displacement analysis and assume perfectly plastic material behavior. Based on large displacement analysis, Goodall (1978) proposed the maximum load-carrying capacity of the elbow subjected to closing bending moment as

$$\alpha_I^G = \frac{1.04\lambda^{2/3}}{1 + \beta}. \quad (16)$$

where

$$\beta = \frac{4\sqrt{3(1-\nu^2)}r\sigma_y}{\pi Eh} \left[2 + \frac{(3\lambda)^{2/3}}{3} \right]$$

Based on the experimental study at CEA DENT, Touboul et al. (1989) proposed the following equations of closing collapse moments of elbows

$$\alpha_I^T = 0.715\lambda^{2/3}. \quad (17)$$

Drubay et al. (1995) expressed another closing mode collapse moments of elbows as

$$\alpha_I^{Dr} = 0.769\lambda^{2/3}. \quad (18)$$

Our numerical results are introduced in table 1 and figure 4, compared with these above analytical solutions and a numerical solution of Yan (1997). It is shown that our solutions compare well with the other analytical solutions, which are based on small displacement theory, but bigger than those which are based on large displacement theory. They converge as an upper bound of Calladine's solution and lower bound of Desquines's solution.

λ	Calladine (12)	Yan (13)	Desquines (14)	Spence & Findlay (15)	Goodall (16)	Touboul (17)	Drubay (18)	Present paper
0.100	0.2013	-	0.1791	0.2001	0.1489	0.154	0.1657	0.2155
0.200	0.3196	-	0.3422	0.3046	0.2817	0.2445	0.263	0.3279
0.250	0.3709	-	0.4144	0.3482	0.3401	0.2838	0.3052	0.3900
0.300	0.4188	-	0.4794	0.3885	0.3947	0.3204	0.3446	0.4614
0.363	0.4756	-	0.5515	0.4355	0.4593	0.3638	0.3913	0.5200
0.400	0.5074	-	0.5888	0.4617	0.4955	0.3882	0.4175	0.5589
0.500	0.5887	-	0.6732	0.5278	0.5879	0.4504	0.4844	0.6260
0.600	0.6648	-	0.7377	0.5888	0.6741	0.5087	0.5471	0.6930
0.650	0.7013	-	0.7639	0.6178	0.7153	0.5365	0.577	0.7227
0.700	0.7368	0.7330	0.7868	0.6458	0.7554	0.5637	0.6063	0.7494
0.750	-	0.7660	0.8069	0.6732	0.7945	0.5902	0.6348	0.7773
0.800	-	0.7933	0.8244	0.6998	0.8327	0.6162	0.6627	0.7974
0.903	-	0.8365	0.8544	0.7525	0.909	0.668	0.7184	0.8341
1.000	-	0.8660	0.8765	0.8000	0.9782	0.715	0.769	0.8617
1.200	-	0.9063	0.9093	0.8925	1.1138	0.8074	0.8684	0.9006

Table 1. Limit load factors of elbow under in-plane closing bending moment

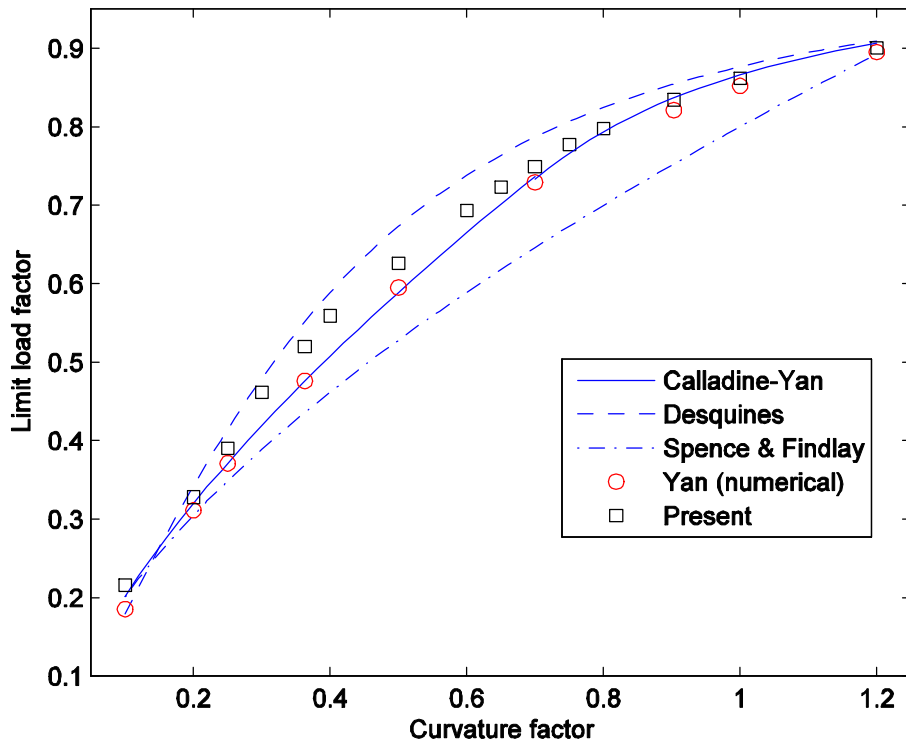


Figure 4. Limit load factors of elbow under in-plane closing bending moment

λ	Yan (19a)	Yan (19b)	Present paper
0.100	0.2763	-	0.2476
0.200	0.4188	-	0.4047
0.250	0.4788	-	0.4709
0.300	0.5341	-	0.5244
0.363	0.5989	-	0.5675
0.400	0.6348	-	0.6063
0.450	0.6813	-	0.6337
0.500	0.7257	0.7143	0.6575
0.550	-	0.7374	0.6924
0.600	-	0.7591	0.7245
0.650	-	0.7796	0.7539
0.700	-	0.7991	0.7808
0.750	-	0.8177	0.8053
0.800	-	0.8355	0.8276
0.903	-	0.8699	0.8674
1.000	-	0.9000	0.8984
1.200	-	0.9564	0.9467

Table 2. Limit load factors of elbow under out-of-plane bending moment

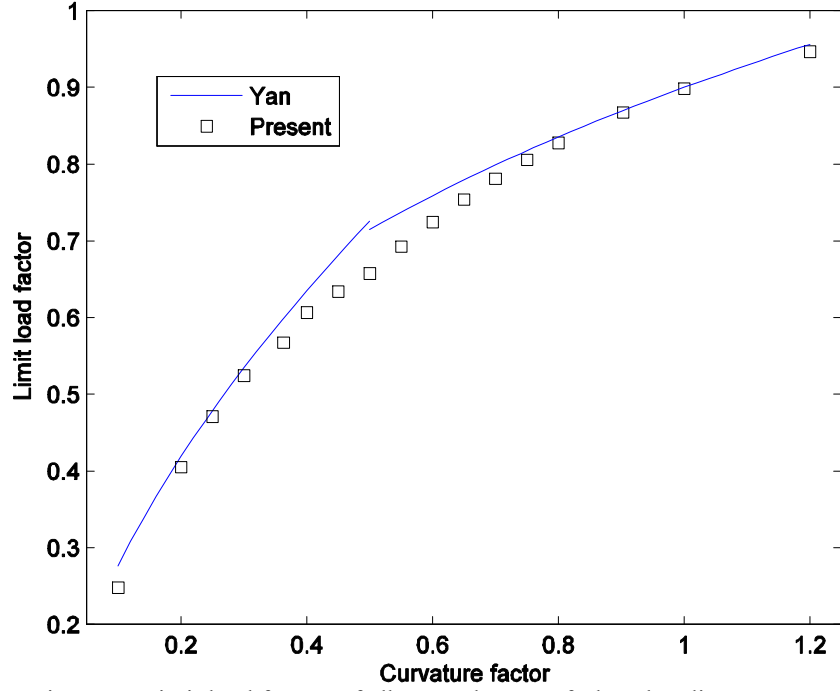


Figure 5. Limit load factors of elbow under out-of-plane bending moment

Elbow under out-of-plane Bending Moment

We define the limit load factor $\alpha_{II} = M_{II} / M_{II}^t$, where M_{II} is out-of-plane limit moment of the elbow, M_{II}^t is the torsion limit moment of the axle which has the same radius as the elbow.

By this definition, Yan (1997) proposed an analytical solution for the elbow subjected to out-of-plane bending moment

$$\alpha_{II}^Y = 1.1\lambda^{0.6}, \quad \lambda < 0.5, \quad (19a)$$

$$\alpha_{II}^Y = 0.9\lambda^{1/3}, \quad 0.5 \leq \lambda \leq 1.4. \quad (19b)$$

Numerical results are introduced in table 2 and figure 5, compared with the analytical solution of Yan (1997). It is shown that our solutions compare well with Yan's solution outside the range $0.4 \leq \lambda \leq 0.7$.

5 Conclusions

The numerical solutions demonstrate that the proposed method is capable of identifying reasonable estimates of the limit load factor for a wide range of thin shell problems. It has been tested against several limit loads which have been calculated in literature. A numerically very effective method is achieved from the lesser computational cost by using shell elements compared with volume elements and by direct plasticity methods which achieve plastic solutions in the computing time of only several linear elastic steps. This method seems to be particularly suited to comparatively large problems or to the application in structural optimization and structural reliability.

Acknowledgment

The first author is the recipient of a Deutscher Akademischer Austausch Dienst (DAAD) research fellowship award under the grant reference A/04/20207. We would like to thank the reviewer for helpful suggestions.

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