TECHNISCHE MECHANIK, Band 28, Heft 3-4, (2008), 310 - 323
Manuskripteingang: 15. Oktober 2007

Parametric Torsional Vibration of Mechanical Drive Systems with non-uniform Transmission Mechanisms

Nguyen Van Khang, Nguyen Phong Dien, Hoang Manh Cuong

This report deals with the torsional vibration of a flexible drive system with non-uniform transmission mechanisms. The dynamic model of this system can be viewed as a parametric torsional vibration system, since the reduced moments of inertia of the rotating parts in the model are time varying. Firstly, the differential equations of motion for the system are established in general by using the Lagrangian equation of the second type. Then, the parametric vibration model of a ship propeller drive will be considered as a typical practical example. The initial conditions will be taken into account to find periodic solutions of the mathematical model derived from the vibration model. Some results of the forced response of the vibration system in time and frequency domain are included.

1 Introduction

Transmission mechanisms are usually used for transformation of the motion from the driver output into motion of the working components in many technical applications, e.g. processing machines, ship propeller drives and agricultural machines.

It is well known that the non-uniform transmission characteristic of mechanisms leads to the phenomena of parametric vibrations of the entire flexible drive system. In general, the dynamic model of this system can be viewed as a parametric torsional vibration system, since the reduced moments of inertia of the rotating parts in the model are time varying. The governing equations of motion for the system in the steady state can be ordinary linear or non-linear differential equations having periodic, time-varying coefficients. An excellent review about the parametric vibration of drive systems with planar mechanisms as well as the theoretical background of the subject may be found in Dresig (2006). This subject ranges from general mathematical principles to applications to specific classes of mechanisms, such as cam, planar four-bar linkages and slider-crank mechanisms.

The purpose of this report is to investigate the parametric torsional vibration of a drive system with non-uniform transmission mechanisms. Firstly, the differential equations of motion for the system are established in general by using the Lagrangian equation of the second type. Then, the parametric vibration model of a ship propeller drive is described in detail. This system includes a four-cylinder engine connected to a flywheel and a gearbox. The initial conditions are taken into account to find periodic solutions of the mathematical model derived from the vibration model. Based on this, the forced response of the vibration system is calculated using numerical methods.

2 Modelling of the Mechanical Drive System as a Parametric Torsional Vibration System

The general model of torsional vibration systems investigated in this report is schematically shown in Figure 1. The mechanical system of the driver shaft, the transmission mechanism and the operating mechanism can be considered as set of n rotating disks connected by elasto-dissipative elements with time-invariant stiffness k_i and constant damping coefficient d_i. Let us introduce into our study the moments of inertia of rotating disks $J_i(\varphi_i)$ as a function of the rotating angle φ_i, $(i = 1, 2..., n)$, and the torques acting on the disks $M_i(\varphi_i, \dot{\varphi}_i)$. This kind of the model is also considered in ref. (Dresig and Holzweissig, 2006).

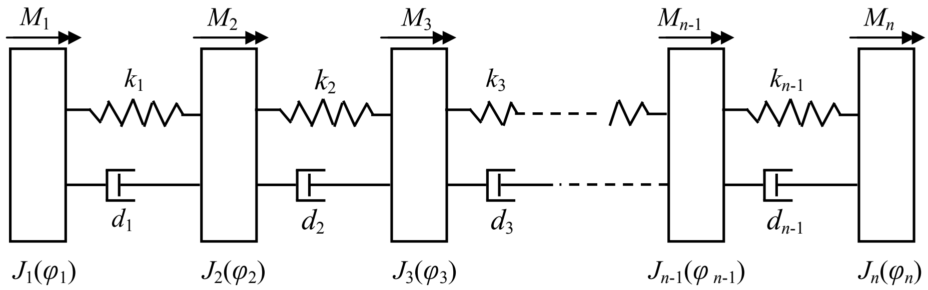

Figure 1. Mechanical model of a parametric torsional vibration system

The kinetic energy T, the potential energy Π and the dissipative function Φ of the system can be expressed in the following forms

$$\mathrm{T} = \frac{1}{2}\sum_{i=1}^{n} J_i(\varphi_i)\dot{\varphi}_i^2 \tag{1}$$

$$\Pi = \frac{1}{2}\sum_{i=1}^{n-1} k_i\left(\varphi_{i+1} - \varphi_i\right)^2 \tag{2}$$

$$\Phi = \frac{1}{2}\sum_{i=1}^{n-1} d_i\left(\dot{\varphi}_{i+1} - \dot{\varphi}_i\right)^2 \tag{3}$$

By using the Lagrangian equations of the second type

$$\frac{d}{dt}\left(\frac{\partial \mathrm{T}}{\partial \dot{\varphi}_i}\right) - \frac{\partial \mathrm{T}}{\partial \varphi_i} = -\frac{\partial \Pi}{\partial \varphi_i} - \frac{\partial \Phi}{\partial \dot{\varphi}_i} + M_i, \tag{4}$$

the differential equations of motion for this system can then be written in the form

$$J_1(\varphi_1)\ddot{\varphi}_1 + \frac{1}{2}J_{1,1}(\varphi_1)\dot{\varphi}_1^2 + d_1\left(\dot{\varphi}_1 - \dot{\varphi}_2\right) + k_1\left(\varphi_1 - \varphi_2\right) = M_1 \tag{5}$$

$$\begin{aligned} J_i(\varphi_i)\ddot{\varphi}_i + \frac{1}{2}J_{i,\,i}(\varphi_i)\dot{\varphi}_i^2 - d_{i-1}\left(\dot{\varphi}_{i-1} - \dot{\varphi}_i\right) + d_i\left(\dot{\varphi}_i - \dot{\varphi}_{i+1}\right) & \\ - k_{i-1}\left(\varphi_{i-1} - \varphi_i\right) + k_i\left(\varphi_i - \varphi_{i+1}\right) &= M_i \end{aligned} \tag{6}$$

(for $i = 2,3,...,n-1$)

$$J_n(\varphi_n)\ddot{\varphi}_n + \frac{1}{2}J_{n,\,n}(\varphi_n)\dot{\varphi}_n^2 + d_{n-1}\left(\dot{\varphi}_{n-1} - \dot{\varphi}_n\right) + k_{n-1}\left(\varphi_{n-1} - \varphi_n\right) = M_n \tag{7}$$

where we use the notation

$$J_{i,\,i} = \frac{d}{d\varphi_i}J_i(\varphi_i) \tag{8}$$

Now we consider the case which often occurs in practical application. When the angular velocity Ω of the driver output is assumed to be constant in the steady state

$$\varphi_1 = \Omega t, \tag{9}$$

it leads to the following relation

$$\varphi_i = \Omega t + q_i,\ \ i = 2,3,...,n \tag{10}$$

$$M_i = M_i(\,\Omega t,\, \varphi_i, \dot{\varphi}_i) = M_i(\,\Omega t + 2\pi,\, \varphi_i, \dot{\varphi}_i), \quad i = 1,2,...,n \tag{11}$$

If we assume that φ_i varies little from its mean value during the steady state motion, i.e. there is only small torsional vibration q_i in the system, then the moment of inertia $J_i(\varphi_i)$ and torque $M_i(\Omega t, \varphi_i, \dot{\varphi}_i)$ depend essentially on the rotating angular $\varphi_1 = \Omega t$. Using the Taylor series expansion around $q_i = 0$, we get

$$J_i(\Omega t + q_i) = \bar{J}_i(\Omega t) + \bar{J}_{i,i}(\Omega t)q_i + \frac{1}{2}\bar{J}_{i,ii}(\Omega t)q_i^2 + ... \quad (12)$$

$$M_i(\Omega t, \Omega t + q_i, \Omega + \dot{q}_i) = \bar{M}_i(\Omega t) + \bar{M}_{i,i}(\Omega t)q_i + \bar{M}_{i,ip}(\Omega t)\dot{q}_i + ... \quad (13)$$

where $M_{i,i} = \dfrac{\partial M_i}{\partial \varphi_i}$, $M_{i,ip} = \dfrac{\partial M_i}{\partial \dot{\varphi}_i}$, $J_{i,ii} = \dfrac{d^2}{d\varphi_i^2} J_i(\varphi_i)$.

Substituting Eqs. (12) and (13) into Eqs. (6) and (7) and neglecting nonlinear terms, we obtain the linear differential equations of the torsional vibration for the mechanical drive system with non-uniform transmission mechanisms as follows

$$\begin{aligned} &\bar{J}_2(\Omega t)\ddot{q}_2 + \left[d_1 + d_2 + \Omega \bar{J}_{2,2}(\Omega t) - \bar{M}_{2,2p}(\Omega t)\right]\dot{q}_2 - d_2\dot{q}_3 \\ &+\left[k_1 + k_2 + \frac{1}{2}\Omega^2 \bar{J}_{2,2}(\Omega t) - \bar{M}_{2,2}(\Omega t)\right]q_2 - k_2 q_3 = -\frac{1}{2}\Omega^2 \bar{J}_{2,2}(\Omega t) + \bar{M}_2(\Omega t) \end{aligned} \quad (14)$$

$$\begin{aligned} &\bar{J}_i(\Omega t)\ddot{q}_i - d_{i-1}\dot{q}_{i-1} + \left[d_{i-1} + d_i + \Omega \bar{J}_{i,i}(\Omega t) - \bar{M}_{i,ip}(\Omega t)\right]\dot{q}_i - d_i\dot{q}_{i+1} \\ &-k_{i-1}q_{i-1} + \left[k_{i-1} + k_i + \frac{1}{2}\Omega^2 \bar{J}_{i,ii}(\Omega t) - \bar{M}_{i,i}(\Omega t)\right]q_i - k_i q_{i+1} = -\frac{1}{2}\Omega^2 \bar{J}_{i,i}(\Omega t) + \bar{M}_i(\Omega t) \end{aligned} \quad (15)$$

(for $i = 3, ..., n-1$)

$$\begin{aligned} &\bar{J}_n(\Omega t)\ddot{q}_n - d_{n-1}\dot{q}_{n-1} + \left[d_{n-1} + \Omega \bar{J}_{n,n}(\Omega t) - \bar{M}_{n,np}(\Omega t)\right]\dot{q}_n \\ &-k_{n-1}q_{n-1} + \left[k_{n-1} + \frac{1}{2}\Omega^2 \bar{J}_{n,nn}(\Omega t) - \bar{M}_{n,n}(\Omega t)\right]q_n = -\frac{1}{2}\Omega^2 \bar{J}_{n,n}(\Omega t) + \bar{M}_n(\Omega t) \end{aligned} \quad (16)$$

Eqs. (14) - (16) can be rewritten in the compact matrix form as

$$\mathbf{M}(\Omega t)\ddot{\mathbf{q}} + \mathbf{D}(\Omega t)\dot{\mathbf{q}} + \mathbf{K}(\Omega t)\mathbf{q} = \mathbf{h}(\Omega t) \quad (17)$$

Where

$$\mathbf{q} = \begin{bmatrix} q_2 & q_3 & ... & q_n \end{bmatrix}^T \quad (18)$$

$$\mathbf{M}(\Omega t) = \begin{bmatrix} \bar{J}_2(\Omega t) & 0 & ... & 0 \\ 0 & \bar{J}_3(\Omega t) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \bar{J}_n(\Omega t) \end{bmatrix} \quad (19)$$

$$\mathbf{D}(\Omega t) = \begin{bmatrix} d_1 + d_2 + \Omega \bar{J}_{2,2} - \bar{M}_{2,2p} & -d_2 & 0 & ... \\ -d_2 & d_2 + d_3 + \Omega \bar{J}_{3,3} - \bar{M}_{3,3p} & -d_3 & ... \\ 0 & -d_3 & d_3 + d_4 + \Omega \bar{J}_{4,4} - \bar{M}_{4,4p} & ... \\ 0 & 0 & -d_4 & ... \\ ... & ... & ... & ... \end{bmatrix} \quad (20)$$

$$\mathbf{K}(\Omega t)=\begin{bmatrix} k_1+k_2+\frac{1}{2}\Omega^2\bar{J}_{2,22}-\bar{M}_{2,2} & -k_2 & 0 & \dots \\ -k_2 & k_2+k_3+\frac{1}{2}\Omega^2\bar{J}_{3,33}-\bar{M}_{3,3} & -k_3 & \dots \\ 0 & -k_3 & k_3+k_4+\frac{1}{2}\Omega^2\bar{J}_{4,44}-\bar{M}_{4,4} & \dots \\ 0 & 0 & -k_4 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \tag{21}$$

$$\mathbf{h}(\Omega t)=\begin{bmatrix} -\frac{1}{2}\Omega^2\bar{J}_{2,2}+\bar{M}_2 \\ -\frac{1}{2}\Omega^2\bar{J}_{3,3}+\bar{M}_3 \\ \dots \\ -\frac{1}{2}\Omega^2\bar{J}_{n,n}+\bar{M}_n \end{bmatrix}. \tag{22}$$

The methods for finding periodic solutions of Eq. (17) under considering the initial conditions have already been investigated in some studies, e.g. references Nguyen Van Khang (1982, 1986), Nguyen Van Khang (1996), Nguyen Van Khang and Vu Van Khiem (1996). The above described theory on parametric torsional vibration is illustrated by the following example with numerical simulations.

3 Parametric Torsional Vibration of a Ship Propeller Drive

3.1 Vibration Model

A ship propeller drive is schematically shown in Figure 2. This system includes a four-cylinder engine connected to a flywheel and a gearbox. Four crankshafts are the part of the engine which translate reciprocating linear motion of pistons into the rotating motion of the driving shaft. The drive system can be considered as a typical torsional vibration system due to the torsional elasticity of the driving shaft, gears and the propeller.

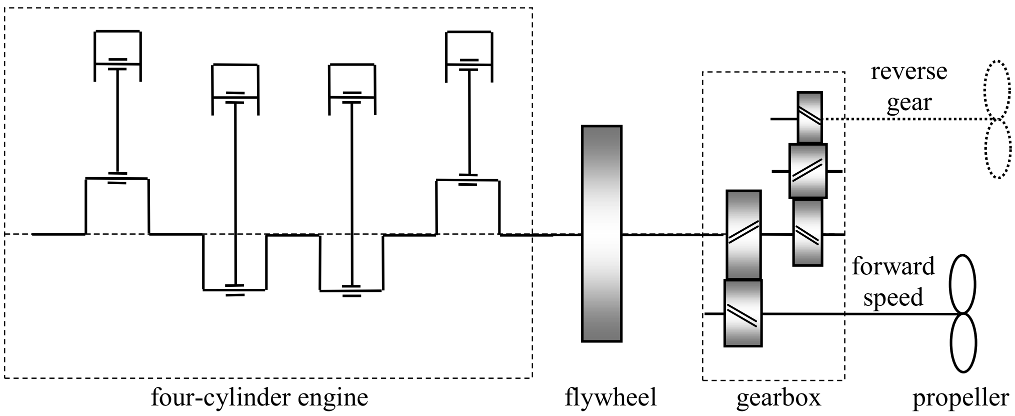

Figure 2. A ship propeller drive

Figure 3 shows the torsional vibration model of the ship propeller drive. The crankshafts are modelled by four rotating disks with the reduced moments of inertia $J_i(\varphi_i)$ $(i=1,2,3,4)$. Moments of inertia J_5, J_6 and J_7 for the flywheel, gearbox and propeller are assumed to be constant. M_1, M_2, M_3 and M_4 denote the torques load applied on the system and M_7 represents the resistance torque acting on the propeller.

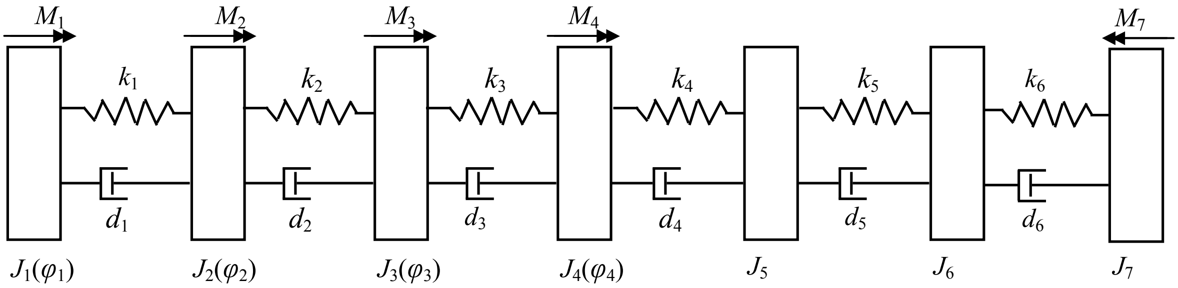

Figure 3. Torsional vibration model of the ship propeller drive

3.2 Reduced Moment of Inertia of the Crankshaft

Let us consider the motion of piston B connected to crank AC through connecting rod CB as shown in Figure 4 (as would be found in internal combustion engines). The well-known kinematic relationship of the mechanism is given by

$$x_B = r\cos\varphi + l\cos\psi \tag{23}$$

$$\sin\psi = \lambda\sin\varphi \tag{24}$$

where l and r are rod length and crank radius respectively, φ is crank angle and x_B the piston pin position along cylinder bore centerline x, $\lambda = r/l$.

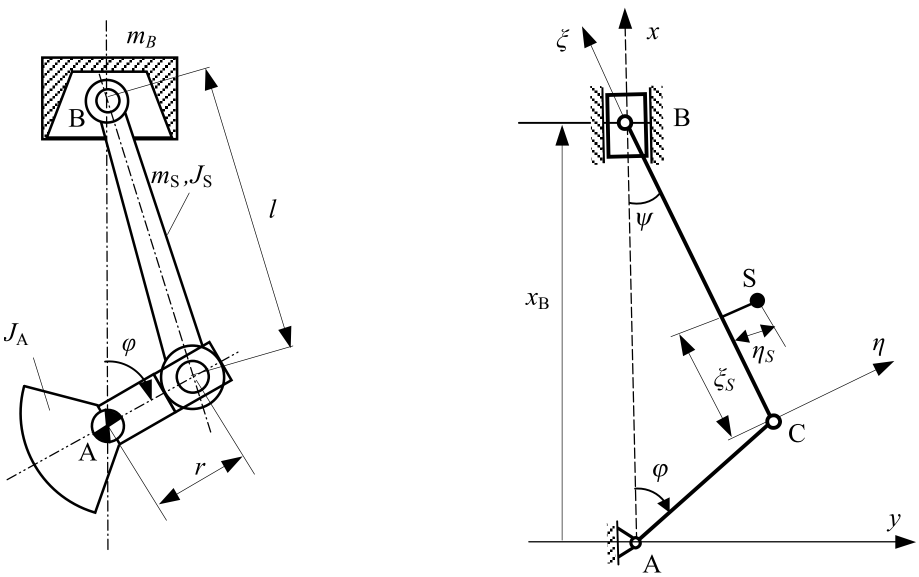

Figure 4. Geometric layout and kinematic schema of a crankshaft

The position of the center of mass S of the connecting rod in the fixed coordinate frame $\{x,y\}$ is given by

$$x_S = r\cos\varphi + \xi_S\cos\psi + \eta_S\sin\psi \tag{25}$$

$$y_S = r\sin\varphi - \xi_S\sin\psi + \eta_S\cos\psi \tag{26}$$

Where ξ_S, η_S are coordinates of S in the body-fixed coordinate frame $\{\xi,\eta\}$ as shown in figure 4. By differentiating Eqs. (23) - (26) with respect to time, we obtain velocity relationships

$$\dot{x}_B = -r\dot{\varphi}\sin\varphi - l\dot{\psi}\sin\psi \tag{27}$$

$$\dot{\psi} = \frac{\lambda\dot{\varphi}\cos\varphi}{\cos\psi} \tag{28}$$

$$\dot{x}_S = -r\dot{\varphi}\sin\varphi - \dot{\psi}\left(\xi_S \sin\psi - \eta_S \cos\psi\right) \tag{29}$$

$$\dot{y}_S = r\dot{\varphi}\cos\varphi - \dot{\psi}\left(\xi_S \cos\psi + \eta_S \sin\psi\right) \tag{30}$$

Taking into account that $\lambda < 1$ and

$$\cos\psi = \sqrt{1-\lambda^2 \sin^2\varphi} \approx 1 - \frac{\lambda^2}{2}\sin^2\varphi, \tag{31}$$

then Eq.(28) takes the form

$$\dot{\psi} \approx \lambda\dot{\varphi}\cos\varphi\left(1+\frac{1}{2}\lambda^2\sin^2\varphi\right) \tag{32}$$

The kinetic energy of the crankshaft can be expressed in the following form

$$T = \frac{1}{2}J_A\dot{\varphi}^2 + \frac{1}{2}J_S\dot{\psi}^2 + \frac{1}{2}m_S\left(\dot{x}_S^2 + \dot{y}_S^2\right) + \frac{1}{2}m_B\dot{x}_B^2 \tag{33}$$

where J_A is mass moment of the inertia of the crank referred to the *z*-axis through fixed point A, m_S denotes the mass of the connecting rod and m_B is the mass of the piston, J_S is the moment of the inertia of the connecting rod referred to the center of mass *S*.

By substituting Eqs. (27), (29), (30), (32) into Eq. (33) and using some trigonometric relations, for instance

$$\cos\varphi\sin^2\varphi = \frac{1}{4}\left(\cos\varphi - \cos 3\varphi\right),$$

$$\cos\varphi\sin^3\varphi = \frac{1}{4}\sin 2\varphi - \frac{1}{8}\sin 4\varphi,$$

$$\cos\varphi\sin^4\varphi = \frac{1}{8}\cos\varphi - \frac{3}{16}\cos 3\varphi + \frac{1}{16}\cos 5\varphi,$$

$$\cos\varphi\sin^5\varphi = \frac{5}{32}\sin 2\varphi - \frac{1}{8}\sin 4\varphi + \frac{1}{32}\sin 6\varphi,$$

we can rewrite kinetic energy *T* of the crankshaft in term of φ and $\dot{\varphi}$ as

$$T = \frac{1}{2}J(\varphi)\dot{\varphi}^2. \tag{34}$$

The reduced moment of inertia $J(\varphi)$ has the form of a periodic function

$$J(\varphi) = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\varphi + b_k^{(1)}\sin k\varphi\right] \tag{35}$$

where the constant term

$$\begin{aligned} J_0 = J_A + J_S\left(\frac{\lambda^2}{2}+\frac{\lambda^4}{8}\right) + m_S\Bigg[r^2 + \left(\xi_S^2+\eta_S^2\right)\left(\frac{\lambda^2}{2}+\frac{\lambda^4}{8}\right) \\ -2r\xi_S\left(\frac{1}{2}-\frac{\lambda^4}{64}\right)\lambda\Bigg] + m_B\left[\frac{r^2}{2} - \frac{l^2\lambda^4}{8} + \frac{l^2\lambda^6}{16}\right], \end{aligned} \tag{36}$$

and constant coefficients $a_k^{(1)}$ and $b_k^{(1)}$ are

$$a_1^{(1)} = m_S r\xi_S\left(\frac{\lambda^2}{2}+\frac{\lambda^4}{8}\right) + m_B rl\left(\frac{\lambda^2}{2}+\frac{\lambda^4}{8}\right) \tag{37}$$

$$a_2^{(1)} = J_S\frac{\lambda^2}{2} + \frac{m_S\left(\xi_S^2+\eta_S^2\right)\lambda^2}{2} - m_S r\xi_S\lambda - \frac{1}{64}m_S r\xi_S\lambda^5 - m_B\left(\frac{r^2}{2}+\frac{l^2\lambda^6}{32}\right) \tag{38}$$

$$a_3^{(1)} = m_S r\xi_S\left(\frac{\lambda^2}{2}+\frac{3\lambda^4}{16}\right) + m_B rl\left(\frac{\lambda^2}{2}+\frac{3\lambda^4}{16}\right) \tag{39}$$

$$a_4^{(1)} = J_S\frac{\lambda^4}{8} + \frac{m_S\left(\xi_S^2+\eta_S^2\right)\lambda^4}{8} + \frac{1}{32}m_S r\xi_S\lambda^5 + m_B l^2\left(\frac{\lambda^4}{8}+\frac{\lambda^6}{16}\right) \tag{40}$$

$$a_5^{(1)} = \frac{1}{16}m_S r\xi_S\lambda^4 + \frac{m_B r l\lambda^4}{16}, \quad a_6^{(1)} = \frac{1}{64}m_S r\xi_S\lambda^5 + \frac{m_B l^2\lambda^6}{32} \tag{41}$$

$$b_1^{(1)} = m_S r\eta_S\left(\frac{\lambda^2}{2}+\frac{\lambda^4}{8}\right), \; b_2^{(1)} = m_S r\eta_S\left(\lambda+\frac{5\lambda^5}{64}\right) \tag{42}$$

$$b_3^{(1)} = m_S r\xi_S\left(\frac{\lambda^2}{2}+\frac{3\lambda^4}{16}\right) + m_B r l\left(\frac{\lambda^2}{2}+\frac{3\lambda^4}{16}\right) \tag{43}$$

$$b_4^{(1)} = m_S r\eta_S\frac{\lambda^5}{16}, \; b_5^{(1)} = m_S r\eta_S\frac{\lambda^4}{16}, \; b_6^{(1)} = m_S r\eta_S\frac{\lambda^5}{64} \tag{44}$$

According to the flat configuration, the ship engine uses four cranks with throws spaced 180 degrees apart. Using Eq. (35) we obtain

$$J_1(\varphi_1) = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\varphi_1 + b_k^{(1)}\sin k\varphi_1\right] \tag{45}$$

$$J_2(\varphi_2) = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\varphi_2-3\pi\right) + b_k^{(1)}\sin k\left(\varphi_2-3\pi\right)\right] \tag{46}$$

$$J_3(\varphi_3) = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\varphi_3-\pi\right) + b_k^{(1)}\sin k\left(\varphi_3-\pi\right)\right] \tag{47}$$

$$J_4(\varphi_4) = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\varphi_4-2\pi\right) + b_k^{(1)}\sin k\left(\varphi_4-2\pi\right)\right] \tag{48}$$

where coefficients $a_k^{(1)}$ and $b_k^{(1)}$ for k = 1, 2, .., 6 are determined by Eqs. (37)-(44).

3.3 Differential Equations of Motion of the System in the Steady State

As already mentioned in the previous section, the differential equations of forced torsional vibration of the system in the steady state can be written in the compact form

$$\mathbf{M}(\Omega t)\ddot{\mathbf{q}} + \mathbf{D}(\Omega t)\dot{\mathbf{q}} + \mathbf{K}(\Omega t)\mathbf{q} = \mathbf{h}(\Omega t) \tag{49}$$

where angular velocity $\dot{\varphi}_7$ of the propeller is assumed to be constant, $\dot{\varphi}_7 = \Omega$, and the generalized coordinates

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \cdots & q_6 \end{bmatrix}^{\mathrm{T}}, \tag{50}$$

are defined by the following relation (see also Figure 3)

$$\varphi_i = \Omega t + q_i, \; i = 1,2,...,6 \tag{51}$$

According to Eqs. (19)-(21) we can find

$$\mathbf{M}(\Omega t) = \begin{bmatrix} \overline{J}_1(\Omega t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{J}_2(\Omega t) & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{J}_3(\Omega t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{J}_4(\Omega t) & 0 & 0 \\ 0 & 0 & 0 & 0 & J_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_6 \end{bmatrix} \tag{52}$$

$$\mathbf{D}=\begin{bmatrix} d_1+\Omega\bar{J}_{1,1}-\bar{M}_{1,1p} & \text{-}d_1 & 0 & 0 & 0 & 0 \\ \text{-}d_1 & d_1+d_2+\Omega\bar{J}_{2,2}-\bar{M}_{2,2p} & \text{-}d_2 & 0 & 0 & 0 \\ 0 & \text{-}d_2 & d_2+d_3+\Omega\bar{J}_{3,3}-\bar{M}_{3,3p} & \text{-}d_3 & 0 & 0 \\ 0 & 0 & \text{-}d_3 & d_3+d_4+\Omega\bar{J}_{4,4}-\bar{M}_{4,4p} & \text{-}d_4 & 0 \\ 0 & 0 & 0 & \text{-}d_4 & d_4+d_5 & \text{-}d_5 \\ 0 & 0 & 0 & 0 & \text{-}d_5 & d_5+d_6 \end{bmatrix} \quad (53)$$

$$\mathbf{K}=\begin{bmatrix} k_1+\frac{\Omega^2}{2}\bar{J}_{1,11}-\bar{M}_{1,1} & \text{-}k_1 & 0 & 0 & 0 & 0 \\ \text{-}k_1 & k_1+k_2+\frac{\Omega^2}{2}\bar{J}_{2,22}-\bar{M}_{2,2} & \text{-}k_2 & 0 & 0 & 0 \\ 0 & \text{-}k_2 & k_2+k_3+\frac{\Omega^2}{2}\bar{J}_{3,33}-\bar{M}_{3,3} & \text{-}k_3 & 0 & 0 \\ 0 & 0 & \text{-}k_3 & k_3+k_4+\frac{\Omega^2}{2}\bar{J}_{4,44}-\bar{M}_{4,4} & \text{-}k_4 & 0 \\ 0 & 0 & 0 & \text{-}k_4 & k_4+k_5 & \text{-}k_5 \\ 0 & 0 & 0 & 0 & \text{-}k_5 & k_5+k_6 \end{bmatrix} \quad (54)$$

where

$$\bar{J}_1 = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\Omega t + b_k^{(1)}\sin k\Omega t\right]$$

$$\bar{J}_2 = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\Omega t - 3\pi\right) + b_k^{(1)}\sin k\left(\Omega t - 3\pi\right)\right]$$

$$\bar{J}_3 = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\Omega t - \pi\right) + b_k^{(1)}\sin k\left(\Omega t - \pi\right)\right]$$

$$\bar{J}_4 = J_0 + \sum_{k=1}^{6}\left[a_k^{(1)}\cos k\left(\Omega t - 2\pi\right) + b_k^{(1)}\sin k\left(\Omega t - 2\pi\right)\right]$$

$$\bar{J}_{1,1} = \sum_{k=1}^{6}\left[-ka_k^{(1)}\sin k\Omega t + k\, b_k^{(1)}\cos k\Omega t\right]$$

$$\bar{J}_{2,2} = \sum_{k=1}^{6}\left[-ka_k^{(1)}\sin k\left(\Omega t - 3\pi\right) + kb_k^{(1)}\cos k\left(\Omega t - 3\pi\right)\right]$$

$$\bar{J}_{3,3} = \sum_{k=1}^{6}\left[-ka_k^{(1)}\sin k\left(\Omega t - \pi\right) + kb_k^{(1)}\cos k\left(\Omega t - \pi\right)\right]$$

$$\bar{J}_{4,4} = \sum_{k=1}^{6}\left[-ka_k^{(1)}\sin k\left(\Omega t - 2\pi\right) + kb_k^{(1)}\cos k\left(\Omega t - 2\pi\right)\right]$$

$$\bar{J}_{1,11} = -\sum_{k=1}^{6}\left[k^2 a_k^{(1)}\cos k\Omega t + k^2 b_k^{(1)}\ \sin k\Omega t\right]$$

$$\bar{J}_{2,22} = -\sum_{k=1}^{6}\left[k^2 a_k^{(1)}\cos k\left(\Omega t - 3\pi\right) + k^2 b_k^{(1)}\ \sin k\left(\Omega t - 3\pi\right)\right]$$

$$\bar{J}_{3,33} = -\sum_{k=1}^{6}\left[k^2 a_k^{(1)}\cos k\left(\Omega t - \pi\right) + k^2 b_k^{(1)}\ \sin k\left(\Omega t - \pi\right)\right]$$

$$\bar{J}_{4,44} = -\sum_{k=1}^{6}\left[k^2 a_k^{(1)}\cos k\left(\Omega t - 2\pi\right) + k^2 b_k^{(1)}\sin k\left(\Omega t - 2\pi\right)\right]$$

Where the torques acting on four cranks can be approximately represented by a truncated Fourier series

$$
\begin{aligned}
\bar{M}_1(\Omega t) &= M_0 + \sum_{k=1}^{m}\left[a_k^{(2)} \cos\frac{k}{2}\Omega t + b_k^{(2)}\sin\frac{k}{2}\Omega t \right] \\
\bar{M}_2(\Omega t) &= M_0 + \sum_{k=1}^{m}\left[a_k^{(2)} \cos\frac{k}{2}\left(\Omega t - 3\pi\right) + b_k^{(2)}\sin\frac{k}{2}\left(\Omega t - 3\pi\right) \right] \\
\bar{M}_3(\Omega t) &= M_0 + \sum_{k=1}^{m}\left[a_k^{(2)} \cos\frac{k}{2}\left(\Omega t - \pi\right) + b_k^{(2)}\sin\frac{k}{2}\left(\Omega t - \pi\right) \right] \\
\bar{M}_4(\Omega t) &= M_0 + \sum_{k=1}^{m}\left[a_k^{(2)} \cos\frac{k}{2}\left(\Omega t - 2\pi\right) + b_k^{(2)}\sin\frac{k}{2}\left(\Omega t - 2\pi\right) \right]
\end{aligned}
\tag{55}
$$

Then, using Eq. (54) we have

$$
\begin{aligned}
\bar{M}_{1,1}(\Omega t) &= \sum_{k=1}^{m}\frac{k}{2}\left[-a_k^{(2)} \sin\frac{k}{2}\Omega t + b_k^{(2)}\cos\frac{k}{2}\Omega t \right] \\
\bar{M}_{2,2}(\Omega t) &= \sum_{k=1}^{m}\frac{k}{2}\left[-a_k^{(2)} \sin\frac{k}{2}\left(\Omega t - 3\pi\right) + b_k^{(2)}\cos\frac{k}{2}\left(\Omega t - 3\pi\right) \right] \\
\bar{M}_{3,3}(\Omega t) &= \sum_{k=1}^{m}\frac{k}{2}\left[-a_k^{(2)} \sin\frac{k}{2}\left(\Omega t - \pi\right) + b_k^{(2)}\cos\frac{k}{2}\left(\Omega t - \pi\right) \right] \\
\bar{M}_{4,4}(\Omega t) &= \sum_{k=1}^{m}\frac{k}{2}\left[-a_k^{(2)} \sin\frac{k}{2}\left(\Omega t - 2\pi\right) + b_k^{(2)}\cos\frac{k}{2}\left(\Omega t - 2\pi\right) \right]
\end{aligned}
\tag{56}
$$

Finally, the vector of generalized forces in Eq. (49) takes the form

$$
\mathbf{h}(\Omega t) = \begin{bmatrix} \bar{M}_1(\Omega t) - \frac{1}{2}\Omega^2 \bar{J}_{1,1}(\Omega t) \\ \bar{M}_2(\Omega t) - \frac{1}{2}\Omega^2 \bar{J}_{2,2}(\Omega t) \\ \bar{M}_3(\Omega t) - \frac{1}{2}\Omega^2 \bar{J}_{3,3}(\Omega t) \\ \bar{M}_4(\Omega t) - \frac{1}{2}\Omega^2 \bar{J}_{4,4}(\Omega t) \\ 0 \\ 0 \end{bmatrix}. \tag{57}
$$

3.4 Numerical Calculations

In this work, we consider only periodic vibrations which are a commonly observed phenomenon in mechanical drive systems. The periodic solutions of Eq. (49) can be obtained by choosing appropriate initial conditions for the vector of variables **q**.

The following parameters are used for numerical calculations: Rotating speed of the propeller $n = 1700$ (rpm) corresponding to $\Omega = 178\ (1/s)$, stiffness $k_1 = k_2 = k_3 = 1{,}2.10^6$ Nm; $k_4 = 1{,}8.10^6$ Nm; $k_5 = 0{,}1166.10^6$ and $k_6 = 4{,}555.10^3$ Nm for the forward speed, damping coefficients $d_1 = d_2 = d_3 = d_4 = 1{,}484$ (kg.m^2/s); $d_5 = d_6 = 0$, moments of inertia $J_0 = 0{,}03825$ kg·m^2; $J_5 = 1{,}122$ and $J_6 = 0{,}01355$ (kg.m^2). The Fourier coefficients of $J_i(\varphi_i)$ are $a_1^{(1)} = 1{,}757.10^{-3}$; $a_2^{(1)} = -6{,}52.10^{-3}$; $a_3^{(1)} = -1{,}772.10^{-3}$; $a_4^{(1)} = -119{,}47.10^{-6}$; $a_5^{(1)} = 15{,}91.10^{-6}$; $a_6^{(1)} = 1{,}82.10^{-6}$ kg·m^2 and $b_k^{(1)} = 0$. The mean value of the torques $\bar{M}_i$ (i = 1,...,4) used for the calculation is $M_0 = 60.38$ (Nm). The Fourier coefficients $a_k^{(2)}$ and $b_k^{(2)}$ (see Eq. 55) can be estimated by experimental works, e.g. Haug (1952), Hafner and Maass (1984) and Nguyen Thuong Hien (1998). These values are given in Table 1 for $m = 48$.

k	$a_k^{(2)}$ (Nm)	$b_k^{(2)}$ (Nm)	k	$a_k^{(2)}$ (Nm)	$b_k^{(2)}$ (Nm)
1	-103,591	-90,719	25	1,570	1,540
2	46,335	142,002	26	-0,836	-1,512
3	-0,886	-132,584	27	0,457	1,365
4	-11,925	104,299	28	-0,14	-1,295
5	11,551	-80,856	29	-0,131	1,230
6	-8,926	69,324	30	0,412	-1,126
7	13,933	-60,712	31	-0,601	1,023
8	-17,566	49,307	32	0,847	-0,865
9	17,606	-37,897	33	-0,970	0,601
10	-15,038	30,010	34	0,961	-0,359
11	14,096	-24,808	35	-0,858	0,261
12	-13,684	20,019	36	0,894	-0,205
13	13,109	-15,307	37	-0,926	0,039
14	-11,705	11,388	38	0,865	0,185
15	10,490	-8,356	39	-0,661	-0,324
16	-9,055	6,009	40	0,501	0,361
17	7,870	-4,345	41	-0,375	-0,38
18	-6,936	3,079	42	0,289	0,425
19	6,367	-1,721	43	-0,145	-0,486
20	-5,316	0,396	44	-0,017	0,527
21	4,095	0,333	45	0,215	-0,508
22	-3,146	-0,496	46	-0,351	0,431
23	2,785	0,717	47	0,452	-0,355
24	-2,286	-1,202	48	-0,519	0,315

Table 1. The Fourier coefficients of torques $\bar{M}_i$

Some calculating results are shown in Figures 5-8. The solutions obtained by numeric integration of Eq. (49) are shown in Figures. 5 and 8. These curves can be useful to estimate the dynamic errors within the considered drive system. From the frequency spectrums in Figures 6 and 7, it can be clearly seen that the torsional vibration of the drive shaft are dominant at the second harmonic of the rotational frequency Ω. In addition, the spectrums show different frequency components, such as $\Omega/2, 3\Omega/2, 5\Omega/2$. This is a commonly observed phenomenon in parametric vibration systems. The dynamic loads acting on the transmission system are then obtained as modelling results (see Figure 7).

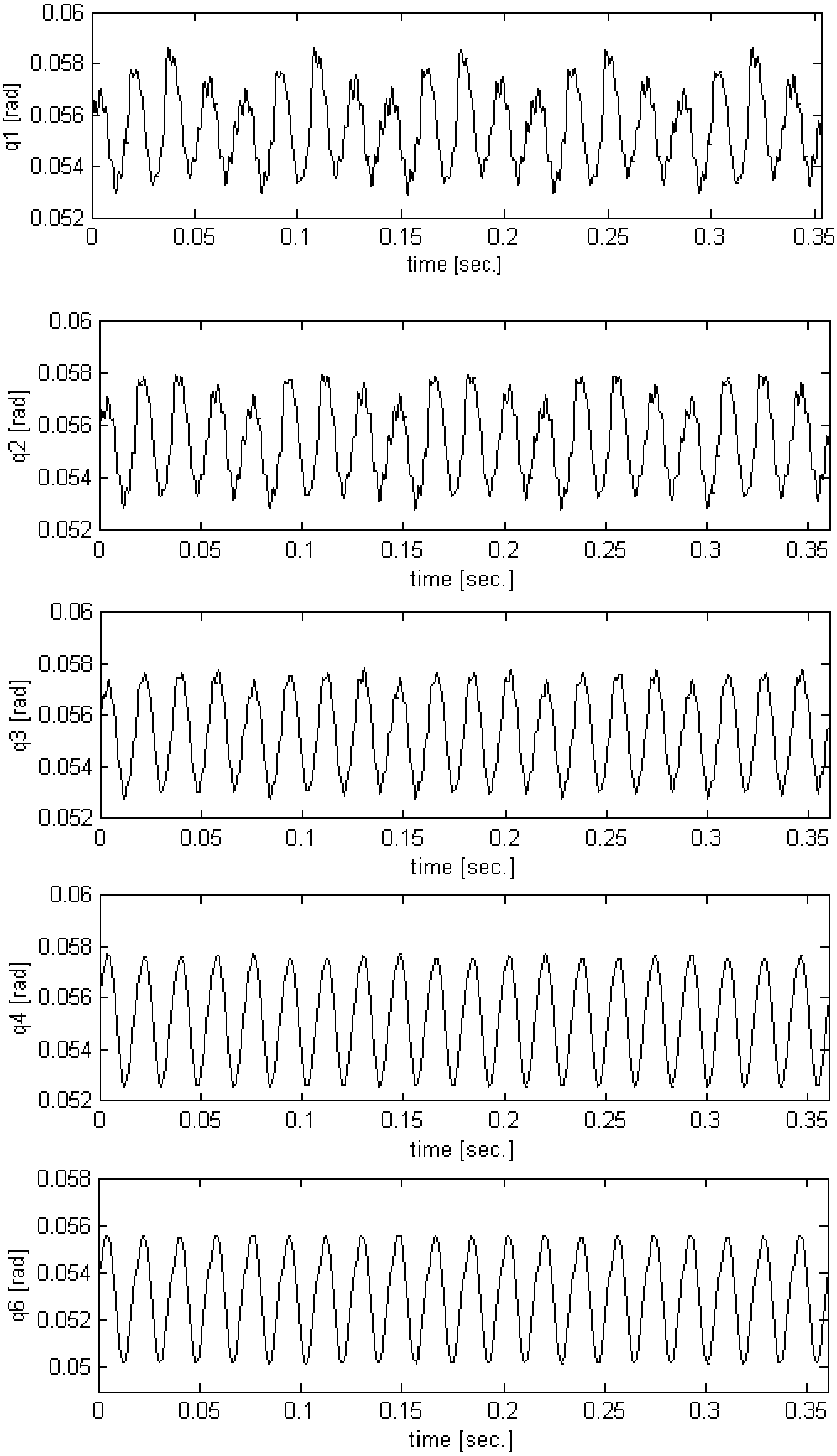

Figure 5. Calculating results for q_i (i = 1,…,6)

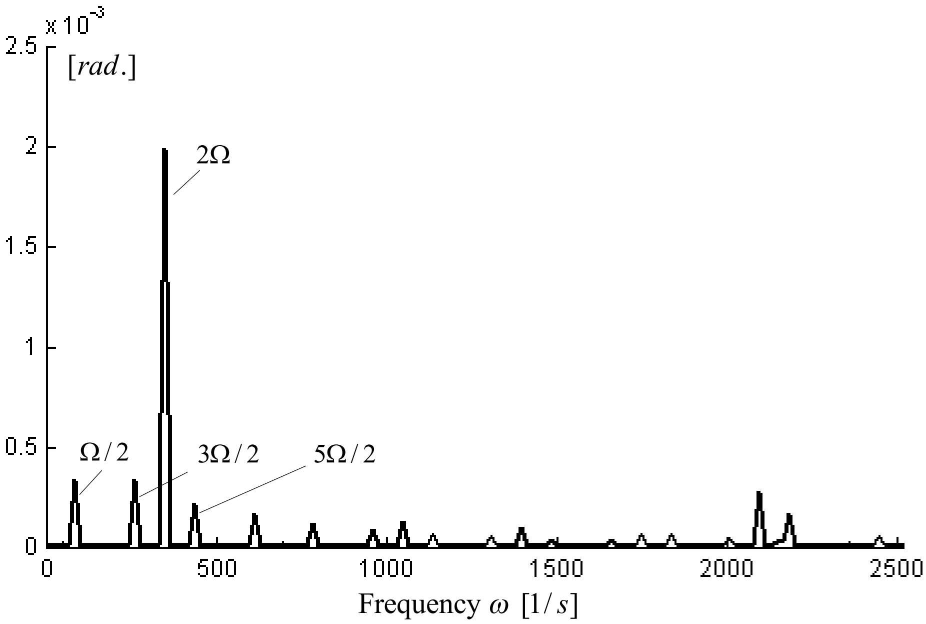

Figure 6. Frequency spectrum of q_1

a)

b)

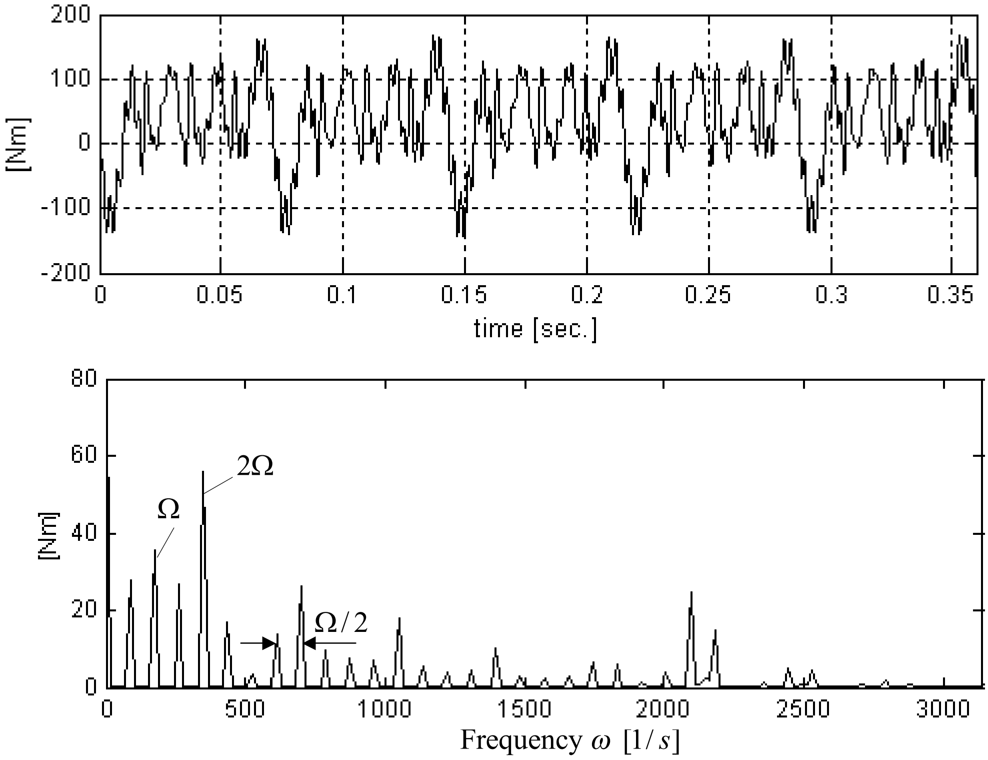

Figure 7. Dynamic load $k_3(q_4 - q_3)$ acting on the shaft segment between disk 3 and disk 4
a) time curve, b) frequency spectrum

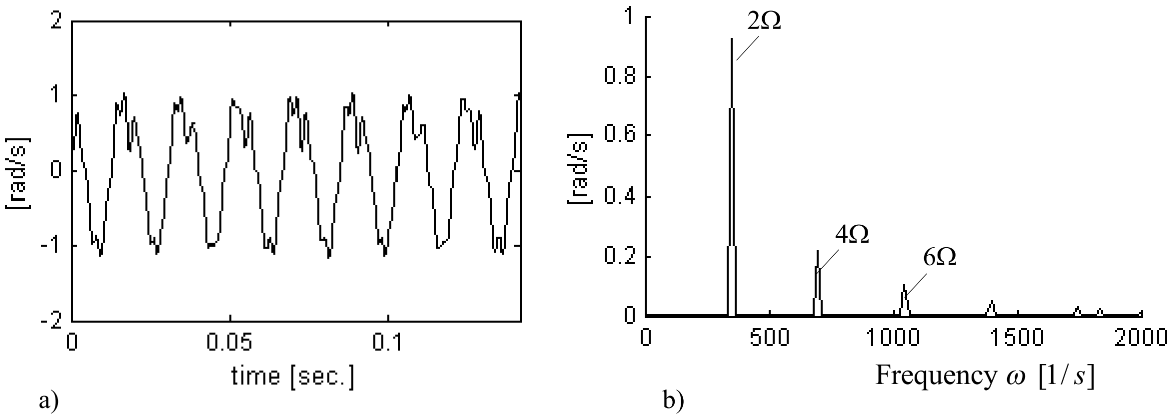

Figure 8. Calculating results of $\dot{q}_6$ a) in time domain, b) in frequency domain

4 Conclusions

In this study, the parametric torsional vibration of flexible drive systems with non-uniform transmission mechanisms was generally investigated. This problem is addressed through the incorporation a time-varying vibration model of non-uniform transmission mechanisms into the time-invariant torsional vibration model of a uniform transmission system. The torsional vibration model of a ship propeller drive with slider-crank mechanisms was presented as an application example. The mathematical model is derived from the vibration model, and the governing equations of motion for the system in the steady state are ordinary equations having periodic, time-varying coefficients.

By using numerical methods, the forced responses of the torsional vibration system are obtained. We consider only periodic solutions of the governing equations of motion. The modelling results can be useful to estimate dynamic errors and dynamic loads within the drive system in the early stage of machine design.

Acknowledgment

This paper was completed with the financial support of the Vietnam Basic Research Program in Natural Science

References

Dresig, H. and Holzweissig, F.: *Maschinendynamik* (7. Auflage). Springer -Verlag, Berlin (2006).

Dresig, H.: *Schwingungen mechanischer Antriebssysteme* (2. Auflage), Springer-Verlag, Berlin (2006).

Schiehlen, W. O. and Müller, P. C.: *Lineare Schwingungen*. Akademische Verlagsgesellschaft, Wiesbaden (1976).

Nguyen Van Khang: *Dynamische Stabilität und periodische Schwingungen in Mechanismen*. Dissertation B, TH Karl-Marx-Stadt (1986).

Nguyen Van Khang: On the dynamic stability and periodic vibration of cam mechanisms with elastic drive. *Machine Vibration*, 5, (1996), 127-130.

Nguyen Van Khang: Numerischer Bestimmung der dynamischen Stabilitätsparameter und periodischen Schwingungen ebener Mechanismen. *Rev. Roum. Sci. Tech. - Mech. Appl.* 27 (1982), 495-507.

Nguyen Van Khang and Vu Van Khiem: Numerischer Berechnung der dynamischen Stabiliätsbedingungen und der periodischen Schwingungen in Kurvengetrieben mit elastischer Stösselstange. *Technische Mechanik*, 16, Heft 4, (1996), 317-325.

Haug, K.: *Die Drehschwingungen in Kolbenmaschinen*. Springer -Verlag, Berlin (1952).

Hafner, K. E. and Maass, H.: *Theorie der Triebwerksschwingunggen der Verbrennungskraftmaschine.* Springer -Verlag, Wien (1984).

Nguyen Thuong Hien: *An Investigation on the torsional vibration of a ship drive system with DTSC50 engine*. Master thesis, Hanoi University of Technology (1998).

Address: Prof. Dr.-Ing. habil. Nguyen Van Khang, Assoc. Prof. Dr.-Ing. Nguyen Phong Dien,
Department of Applied Mechanics, Hanoi University of Technology, 1. Dai Co Viet, Hanoi.
Dipl.-Ing. Hoang Manh Cuong, Haiphong Marine University, Vietnam.

email: nvankhang@mail.hut.edu.vn ; dien@mail.hut.edu.vn