TECHNISCHE MECHANIK, Band 27, Heft 1, (2007), 1 – 17
Manuskripteingang: 13.Oktober 2006

Mechanics Based on an Objective Power Functional

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During the recent years, the interest in non-classical material theories has significantly grown, due to the fact that classical theories can not describe certain effects, in principle. In particular, higher order gradient theories have turned out to become a promising remedy, since they open the way to conceptually new formats wide enough to include, e. g., internal length scales.

The inclusion of higher deformation gradients raises many questions. One expects the parallel existence of higher order stress tensors, for which the balances and boundary conditions have to be generalized. This generalization, however, is by no means trivial nor unique. In the present paper, these concepts are derived by posing invariance requirements upon a general principle of virtual power, as a linear and continuous extension of the balance of work. By such a procedure, a catalogue of different, but essentially equivalent load systems and balances can be obtained, which can be further particularised for specific materials.

Introduction

The principal aim of any mechanical theory is to decide whether a given motion of a body under certain circumstances is dynamically admissible or not. This aim can be achieved in different ways. Usually the admissibility is assured by proving that the laws of motion are fulfilled. For this purpose, most authors (i) introduce the forces and moments as primitive concepts, (ii) assume their objectivity under change of observer, and (iii) assume the Newton-Euler laws of motion to be valid with respect to an inertial observer. Consequently, the laws of motion are only invariant under Galilean transformations, but not under Euclidean ones, whereas for the material theory invariance with respect to the whole Euclidean transformation group is usually applied.

This structure of a dynamical theory is both well-understood and well-established for simple materials. However, some drawbacks remain.

- The key concept of this format, namely the forces and torques are generally invisible and not accessible by direct measurement.
- There are, moreover, substantiated doubts that these concepts are unique.
- There are many different force and stress concepts and, accordingly, different transformation behaviours.
- The triade of *force concept*, *equations of motion*, and *inertial observer* contains a circularity. As a consequence, it cannot be validated or falsified by experiments, in principle (see Bertram 1989).

The generalization of these concepts for non-classical materials such as gradient ones is by no means trivial. The classical Cauchy stress principle has to be enlarged, the same as the balance laws and the boundary conditions. For that purpose, a clear scheme is needed, which leads to a framework for gradient theories of any order.

The present approach does not start with forces and moments, but with the (total) mechanical power as a primitive concept. In contrast to stresses, there is only one power, even for materials with internal or external constraints. This power is assumed to be objective or invariant for all admissible processes, as it does not depend on observers or frames of reference.

Before we investigate the consequences of this invariance requirement, we extend the power functional to the so-called *virtual velocities* in a linear and continuous way. It turns out then that the virtual power under all velocity fields resulting from a rigid body motion, i.e., with a transversal and a rotational part, tells us whether a motion is physically admissible. Because of the linear dependence of the virtual power on the velocities, the lost forces and moments can be deduced as the duals of translations and rotations, respectively. In our setting, they have to be objective, in contrast to the forces and moments, which comprise inertia terms. Consequently, the full Euclidean group applies to the equations of motion, and at this stage there is no need to distinguish between inertial and non-inertial observers at all, which appears to be more reasonable, as an absolute space apparently does not exist in mechanics.

Such a procedure, originally proposed by Clairaut, has been often applied, among others by Noll (1959, 1963). The derivation of the equations of motions from an invariance requirement of the power has been proposed for classical materials by Noll (1963), Green/ Rivilin (1964), Gurin/ Williams (1971), Germain (1972, 1973a), Maugin (1980), Gurtin (1981), Bertram (1983, 1989) and others. It turns out to be an efficient tool to derive balance equations and associated boundary conditions for a large variety of coupled physical-mechanical situations. The mechanical theory of materials with microstructure was handled in this way by Germain (1973b), and the theory of coupled fields in deformable continua including electromagnetism has been derived by Maugin (1980). After postulating the principle of virtual power for a system of forces in a Galilean frame, these authors make use of the postulate of objectivity of the virtual power of internal forces with respect to Euclidean transformations. In contrast, the present work departs from this scenario by stating a principle of objectivity for the total power and derives the dynamical concepts, the laws of motion and the boundary terms from it.

For the virtual power one can apply a representation theorem for continuous and linear functionals on vector and tensor fields, by which dual fields of all orders are introduced, which stand for higher order stress fields. These can partially be turned into surface fields by applying the Gauss transformation. By repeatedly applying this procedure, we can produce a catalogue of different, but essentially equivalent forms of the virtual power functional, which correspond to different force and torque concepts and different boundary values for each of them. Such a catalogue becomes helpful for establishing the frameworks for gradient theories. In all cases, we obtain just two vectorial equations of motion in terms of dynamical quantities, which are equivalent to the various forms of the principle of virtual power.

The partition of forces into internal and external contributions can only be made within material theory which is, however, beyond the scope of this treatise. In fact, such a partition does not result from the form of the dynamic fields in which they appear in the balance equations. It will be shown that there remains some freedom to shift the dynamical field from the interior of a body to its surface, or vice versa. The identification of body forces and of surface tractions becomes feasible only after specifying the dependence of the fields on independent variables. Thereafter one would declare tractions to consist of the short range forces, and body forces of the long range ones.

Notations

Throughout the text a direct tensor notation is preferred. Vectors are denoted by bold small letters, tensors by bold capitals, often with a suffix indicating its order. A vector is considered as a first order tensor. The norm of a tensor of arbitrary order k is denoted by $|\mathbf{Y}_k|$, and the inner product between tensors of arbitrary, but equal order is "$\cdot$". The Rayleigh product of a second order tensor $\mathbf{Y}_2$ and a tensor of arbitrary order $\mathbf{T}$ is defined by

$$\mathbf{Y}_2 * \mathbf{T} \ := \ T^{ik\,...\,l}\,(\mathbf{Y}_2\,\mathbf{g}_i) \otimes (\mathbf{Y}_2\,\mathbf{g}_k) \otimes ... \otimes (\mathbf{Y}_2\,\mathbf{g}_l)$$

with respect to any vector basis $\{\mathbf{g}_i\}$.

The symmetric part $\mathbf{Y}_2^{sym}$ of a second order tensor $\mathbf{Y}_2$ is

$$\mathbf{Y}_2^{sym} \ = \ ½\,(\mathbf{Y}_2 + \mathbf{Y}_2^{T})$$

and its skew part

$$\mathbf{Y}_2^{skw} \ = \ ½\,(\mathbf{Y}_2 - \mathbf{Y}_2^{T})\ .$$

The axial vector $\mathbf{Y}_2^{axi}$ of a second order tensor maps any vector $\mathbf{x}$ by means of the cross product as does its skew part

$$\mathbf{Y}_2^{axi} \times \mathbf{x} \ = \ \mathbf{Y}_2^{skw}\,\mathbf{x}$$

Some useful rules will be needed for field formulations. For any vector field $\mathbf{v}$ we have

$$curl\,\mathbf{v} \ = \ 2\,(grad\,\mathbf{v})^{axi}$$

and

$$grad\; curl\, \mathbf{v} \;=\; curl\,(grad\, \mathbf{v})^T.$$

For any second order tensor field $\mathbf{Y}_2$ we obtain the identity

$$div\, \mathbf{Y}_2^{skw} \;=\; -\, curl\, \mathbf{Y}_2^{axi}$$

and for any vector field $\mathbf{v}$

$$\mathbf{Y}_2^{skw} \cdot grad\, \mathbf{v} \;=\; \mathbf{Y}_2^{axi} \cdot curl\, \mathbf{v}\,.$$

For the position vector $\mathbf{r}_o$ with respect to a point of reference O and any constant vector $\boldsymbol{\omega}$, one finds for the skew field

$$[grad\,(\boldsymbol{\omega} \times \mathbf{r}_o)]^{axi} \;=\; ½\, curl(\boldsymbol{\omega} \times \mathbf{r}_o) \;=\; ½\, \boldsymbol{\omega}\,(div\, \mathbf{r}_o) - ½\,(grad\, \mathbf{r}_o)\, \boldsymbol{\omega} \;=\; \boldsymbol{\omega}$$

and, consequently,

$$\mathbf{Y}_2 \cdot grad(\boldsymbol{\omega} \times \mathbf{r}_o) \;=\; \mathbf{Y}_2^{axi} \cdot curl(\boldsymbol{\omega} \times \mathbf{r}_o) \;=\; 2\, \mathbf{Y}_2^{axi} \cdot \boldsymbol{\omega}.$$

For all other notations, see BERTRAM (2005).

Table of frequently used notations

φ, ψ	observers (frames of reference)
$\boldsymbol{\omega}$	angular velocity between two observers
ρ	mass density
$\mathbf{c}$	shift between two observers
$\mathbf{d}_{o\varphi}$	moment of momentum
$\mathbf{f}_\varphi$	resultant force
$\underline{\mathbf{f}}_\varphi$	resultant lost force
$\mathbf{l}_\varphi$	linear momentum
$\mathbf{L}_\varphi$	velocity gradient
$\mathbf{m}_{o\varphi}$	resultant moment with respect to a point of reference O
$\underline{\mathbf{m}}_{o\varphi}$	resultant lost moment with respect to a point of reference O
p_φ	power
$\mathbf{Q}$	relative rotation between two observers
$\mathbf{r}_\varphi$	position vector
$\mathbf{v}_\varphi$	velocity field
$\mathbf{Y}_{\varphi\, i}$	dynamic variables, tensor fields of order i

Global Theory

Let φ be an observer (or a frame of reference), and κ_φ a motion of a material body as it is monitored by φ. Then this motion appears for another observer ψ as κ_ψ, and the mapping between the two motions is induced by the usual Euclidean transformation of the position vectors

$$\mathbf{r}_\varphi\,(P, t) \;=\; \mathbf{Q}(t)\, \mathbf{r}_\psi\,(P\,, t) + \mathbf{c}(t) \tag{1}$$

point-wise applied, i. e., for all material points P and all times t. $\mathbf{Q}(t)$ is a time-dependent rotation (proper orthogonal) and $\mathbf{c}(t)$ a time-dependent translation vector. The action of the Euclidean group on all other kinematical quantities is well known, in particular that on the velocity field $\mathbf{v}_{\varphi,\,\psi} \;=\; \mathbf{r}_{\varphi,\,\psi}\,(P, t)^\bullet$ at a certain instant

$$\mathbf{v}_\varphi = \; \mathbf{Q}\, \mathbf{v}_\psi + \mathbf{Q}^\bullet\, \mathbf{Q}^T (\mathbf{r}_\varphi - \mathbf{c}) + \mathbf{c}^\bullet \;=\; \mathbf{Q}\, \mathbf{v}_\psi + \boldsymbol{\omega} \times (\mathbf{r}_\varphi - \mathbf{c}) + \mathbf{c}^\bullet \tag{2}$$

with $\boldsymbol{\omega}$ being the axial vector of the skew tensor $\mathbf{Q}^\bullet\, \mathbf{Q}^T$. By an appropriate choice of the points of reference it is possible to achieve $\mathbf{c} \equiv \mathbf{o}$ at any particular instant. The action of the Euclidean group on the velocity gradients $\mathbf{L}_{\varphi,\,\psi} := \; grad\, \mathbf{v}_{\varphi,\,\psi}$ is

$$\mathbf{L}_\varphi = \mathbf{Q}\,\mathbf{L}_\psi\,\mathbf{Q}^T + \mathbf{Q}^\bullet\,\mathbf{Q}^T$$

so that their symmetric parts $\mathbf{D}_{\varphi,\,\psi} := (grad\;\mathbf{v}_{\varphi,\,\psi})^{sym}$ are transformed in an objective way

$$\mathbf{D}_\varphi = \mathbf{Q}\,\mathbf{D}_\psi\,\mathbf{Q}^T$$

and their skew parts $\mathbf{W}_{\varphi,\,\psi} := (grad\;\mathbf{v}_{\varphi,\,\psi})^{skw}$ after

$$\mathbf{W}_\varphi = \mathbf{Q}\,\mathbf{W}_\psi\,\mathbf{Q}^T + \mathbf{Q}^\bullet\,\mathbf{Q}^T.$$

Our analysis is based on the notion of the (total) mechanical power as a primitive concept. Roughly speaking, the current power of the body at a certain instant is a real number, which depends not only on the current value, but on the entire motion of the whole body, if the material has a memory. If, however, the body is currently at rest with respect to some observer, then he would expect the power to be zero at such instants. This gives rise for the following assumption.

Principle of Determinism.
For any motion of the body κ_φ for some observer φ there exists a *power functional* $p_\varphi\,(\kappa_\varphi)$ that gives the power which the body currently produces, i.e. at the end of this motion. It is zero, if the current velocity field is zero for all points of the body.

Thus, by the last assumption we have the implication

$$\mathbf{v}_\varphi \equiv \mathbf{o} \text{ everywhere} \qquad \Rightarrow \qquad p_\varphi\,(\kappa_\varphi) = 0\,.$$

Instead of postulating laws of motion, the criterion for distinguishing between dynamically admissible from non-admissible motions is chosen here as an objectivity requirement.

Principle of Objectivity.
A motion of the body is dynamically admissible if and only if the power functional for all (sub)processes and (sub)bodies is objective (or invariant) under all changes of observer

$$p_\varphi\,(\kappa_\varphi) = p_\psi\,(\kappa_\psi)\,. \tag{3}$$

One is tempted to state that the power is a linear functional of the velocity field. As the power is only defined for some particular process, the velocity field at its end is then determined by the process, so that linearity has no meaning, unless we allow for all other fields of a linear space. We will call these fields *virtual velocities*. Their space $\delta\mathscr{V}_\varphi$, called the *space of virtual velocities*, contains the velocity field as a distinguished member. Also, it contains all velocity fields resulting from arbitrary rigid body motions. It has to be endowed with a linear and topological structure, in order to give properties of functions like linearity and continuity a precise meaning.

For the linear operations of such fields we will introduce them point-wise, as usual. The topological structure, however, is non-trivial. We will assume further-on that all velocity fields are p-times piecewise differentiable for some $p \geq 1$, which shall not be specified yet. For introducing the topological structure on $\delta\mathscr{V}_\varphi$, we use the p-norm

$$|\delta\mathbf{v}_\varphi|^p := \sqrt{\int_{\mathscr{B}} [|\delta\mathbf{v}_\varphi\,(x)|^2 + |grad\;\delta\mathbf{v}_\varphi\,(x)|^2 + \ldots + |grad^p\;\delta\mathbf{v}_\varphi\,(x)|^2]\;dV} \tag{4}$$

for all $\delta\mathbf{v}_\varphi \in \delta\mathscr{V}_\varphi$. This makes a topological vector space out of $\delta\mathscr{V}_\varphi$, which is contained in the usual Sobolev-space $\mathscr{W}^{\,p,2}$. We will further-on assume that the current velocity field is always contained in $\delta\mathscr{V}_\varphi$. This assumption restricts the regularity of the velocity fields in a way which is not appropriate for certain purposes and so could be released. E. g., shock waves, shear bands, and other localizations will require weaker regularities in order to allow also for non-smooth fields with singularities. For the present context, however, we will not include such behaviour for the sake of simplicity and clearness.

For another observer ψ we analogously introduce the space of virtual velocities as $\delta\mathscr{V}_\psi$, the elements of which are transformed in the same way as the velocities

$$\delta\mathbf{v}_\varphi = \mathbf{Q}\,\delta\mathbf{v}_\psi + \mathbf{Q}^\bullet\,\mathbf{Q}^T(\mathbf{r}_\varphi - \mathbf{c}) + \mathbf{c}^\bullet \tag{5}$$

between two observers.
We are now able to introduce the virtual power as a continuous extension of the power being linear in the virtual velocities.

Definition. During a motion of a body, at each instant the *virtual power* is a functional

$$\delta p_\varphi(\kappa_\varphi, \bullet): \ \delta\mathscr{V}_\varphi \rightarrow \mathscr{R}$$

with the following properties:
(P1) it is continuous and linear
(P2) it extends p_φ, i.e.,

$$\delta p_\varphi(\kappa_\varphi, \mathbf{v}_\varphi) = p_\varphi(\kappa_\varphi) \qquad \text{for } \mathbf{v}_\varphi \in \delta\mathscr{V}_\varphi \tag{6}$$

(P3) it transforms like p_φ, i.e., for all observers φ and ψ we have

$$\delta p_\varphi(\kappa_\varphi, \delta\mathbf{v}_\varphi) - \delta p_\psi(\kappa_\psi, \delta\mathbf{v}_\psi) = p_\varphi(\kappa_\varphi) - p_\psi(\kappa_\psi) \tag{7}$$

if $\delta\mathbf{v}$ is transformed like $\mathbf{v}$ after (5).

The existence of such an extension is generally assured, but it is by no means unique. However, this non-uniqueness will have no influence on the distinction between dynamically admissible and non-admissible motions.

If we substitute the transformation of the virtual velocity field in (P3), we obtain for all motions (admissible or not)

$$\delta p_\varphi(\kappa_\varphi, \mathbf{Q}\,\delta\mathbf{v}_\psi + \mathbf{Q}^\bullet\,\mathbf{Q}^T(\mathbf{r}_\varphi - \mathbf{c}) + \mathbf{c}^\bullet) - \delta p_\psi(\kappa_\psi, \delta\mathbf{v}_\psi) = p_\varphi(\kappa_\varphi) - p_\psi(\kappa_\psi)$$

and by the linearity of the virtual power functional

$$\delta p_\varphi(\kappa_\varphi, \mathbf{Q}\,\delta\mathbf{v}_\psi) - \delta p_\psi(\kappa_\psi, \delta\mathbf{v}_\psi) = p_\varphi(\kappa_\varphi) - p_\psi(\kappa_\psi) - \delta p_\varphi(\kappa_\varphi, \boldsymbol{\omega} \times (\mathbf{r}_\varphi - \mathbf{c})) - \delta p_\varphi(\kappa_\varphi, \mathbf{c}^\bullet)$$

for all $\delta\mathbf{v}_\psi \in \delta\mathscr{V}_\psi$. Regarding the dependencies upon the virtual power, the right hand side of this equation is constant. The only linear function which equals a constant, is the zero function. Thus,

$$\delta p_\varphi(\kappa_\varphi, \mathbf{Q}\,\delta\mathbf{v}_\psi) = \delta p_\psi(\kappa_\psi, \delta\mathbf{v}_\psi) \quad \text{for all } \delta\mathbf{v}_\psi \in \delta\mathscr{V}_\psi\,. \tag{8}$$

The remaining parts of the equation are

$$\delta p_\varphi(\kappa_\varphi, \boldsymbol{\omega} \times (\mathbf{r}_\varphi - \mathbf{c})) + \delta p_\varphi(\kappa_\varphi, \mathbf{c}^\bullet) = p_\varphi(\kappa_\varphi) - p_\psi(\kappa_\psi)\,.$$

By the linearity of the virtual power, there exist two time-dependent vectors for every observer which give the virtual power for a translational field

$$\delta p_\varphi(\kappa_\varphi, \mathbf{v}_o) = \underline{\mathbf{f}}_\varphi \cdot \mathbf{v}_o \tag{9}$$

and for a rotational field

$$\delta p_\varphi(\kappa_\varphi, \boldsymbol{\omega} \times \mathbf{r}_\varphi) = \underline{\mathbf{m}}_{o\varphi} \cdot \boldsymbol{\omega} \tag{10}$$

such that

$$\delta p_\varphi(\kappa_\varphi, \boldsymbol{\omega} \times (\mathbf{r}_\varphi - \mathbf{c})) + \delta p_\varphi(\kappa_\varphi, \mathbf{c}^\bullet) = \underline{\mathbf{f}}_\varphi \cdot \mathbf{v}_o + \underline{\mathbf{m}}_{o\varphi} \cdot \boldsymbol{\omega} \tag{11}$$

with the relative velocity of the reference point (which need not be a material point)

$$\mathbf{v}_o = \mathbf{c}^\bullet - \boldsymbol{\omega} \times \mathbf{c}\,.$$

Following Hamel (1949), we will call the vector $\underline{\mathbf{f}}_\varphi$ the *resultant lost (or generalized) force* and the vector $\underline{\mathbf{m}}_{o\varphi}$ the *resultant lost (or generalized) moment* of the body induced by the virtual power functional δp_φ . The *resultant force* $\mathbf{f}_\varphi$ is the lost force completed by the inertia term

$$\mathbf{f}_\varphi \ := \ \underline{\mathbf{f}}_\varphi + \mathbf{l}_\varphi^\bullet \tag{12}$$

and the *resultant moment* $\mathbf{m}_{o\varphi}$ analogously

$$\mathbf{m}_{o\varphi} \ := \ \underline{\mathbf{m}}_{o\varphi} + \mathbf{d}_{o\varphi}^\bullet \tag{13}$$

with the linear momentum

$$\mathbf{l}_\varphi \ := \int_{\mathscr{B}} \mathbf{r}_\varphi^\bullet \,\rho\, dV \qquad \Rightarrow \qquad \mathbf{l}_\varphi^\bullet \ = \int_{\mathscr{B}} \mathbf{r}_\varphi^{\bullet\bullet} \,\rho\, dV \tag{14}$$

and the moment of momentum

$$\mathbf{d}_{o\varphi} \ := \int_{\mathscr{B}} \mathbf{r}_\varphi \times \mathbf{r}_\varphi^\bullet \,\rho\, dV \qquad \Rightarrow \qquad \mathbf{d}_{o\varphi}^\bullet \ = \int_{\mathscr{B}} \mathbf{r}_\varphi \times \mathbf{r}_\varphi^{\bullet\bullet} \,\rho\, dV \tag{15}$$

as usual.
The resultant lost forces and moments are observer-dependent functionals of the motion, like the power itself. The observer-dependence will be clarified by the next theorem.

Theorem 1. (transformations of forces and moments)
The resultant lost force $\underline{\mathbf{f}}_\varphi$ and the resultant lost moment $\underline{\mathbf{m}}_{o\varphi}$ are objective vectors

$$\underline{\mathbf{f}}_\varphi \ = \ \mathbf{Q}\, \underline{\mathbf{f}}_\psi \qquad \text{and} \qquad \underline{\mathbf{m}}_{o\varphi} \ = \ \mathbf{Q}\, \underline{\mathbf{m}}_{o\psi}\,. \tag{16}$$

Proof. We apply (P3) to the field $\delta\mathbf{v}_\psi \equiv \mathbf{a} \times \mathbf{r}_\varphi + \mathbf{b}$ with two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$

$$p_\varphi(\kappa_\varphi) - p_\psi(\kappa_\psi) =$$

$$\delta p_\varphi(\kappa_\varphi, \mathbf{Q}(\mathbf{a} \times \mathbf{r}_\varphi + \mathbf{b}) + \boldsymbol{\omega} \times (\mathbf{r}_\varphi - \mathbf{c}) + \mathbf{c}^\bullet) - \delta p_\psi(\kappa_\psi, \mathbf{a} \times \mathbf{r}_\varphi + \mathbf{b})$$

$$= \ \underline{\mathbf{f}}_\varphi \cdot (\mathbf{Q}\,\mathbf{b}) + \underline{\mathbf{m}}_{o\varphi} \cdot (\mathbf{Q}\,\mathbf{a}) + \underline{\mathbf{f}}_\varphi \cdot \mathbf{v}_o + \underline{\mathbf{m}}_{o\varphi} \cdot \boldsymbol{\omega} - \underline{\mathbf{f}}_\psi \cdot \mathbf{b} - \underline{\mathbf{m}}_{o\psi} \cdot \mathbf{a}$$

$$= \ (\mathbf{Q}^T \underline{\mathbf{f}}_\varphi - \underline{\mathbf{f}}_\psi) \cdot \mathbf{b} + (\mathbf{Q}^T \underline{\mathbf{m}}_{o\varphi} - \underline{\mathbf{m}}_{o\psi}) \cdot \mathbf{a} + \underline{\mathbf{f}}_\varphi \cdot \mathbf{v}_o + \underline{\mathbf{m}}_{o\varphi} \cdot \boldsymbol{\omega}$$

as $\mathbf{Q}(\mathbf{a} \times \mathbf{r}) \ = \ \mathbf{Q}\,\mathbf{a} \times \mathbf{Q}\,\mathbf{r}_\varphi \ = \ \mathbf{Q}\,\mathbf{a} \times \mathbf{r}_\psi$ after an appropriate choice of the point of reference for the second observer. By the arbitrariness of $\mathbf{a}$ and $\mathbf{b}$ we conclude the objectivity of the two tensors; q.e.d.

If the resultant lost force and moment are objective vectors, then the resultant force and the resultant moment can not be objective vectors, in principle. However, this procedure saves us from distinguishing between inertial an non-inertial observers, as is usually done.
While the forces do not depend on a point of reference, the moments do so (through the position vector). This dependence is specified by the following theorem.

Theorem 2. (Varignon´s principle)
The moment depends on the point of reference after

$$\underline{\mathbf{m}}_{o'} \ = \ \underline{\mathbf{m}}_o + \overrightarrow{o'o} \times \underline{\mathbf{f}} \tag{17}$$

with $\overrightarrow{o'o}$ being the position vector of the second point of reference with respect to the first.

Proof. We use the equation of the position vectors $\mathbf{r}_{\varphi\, o'} = \mathbf{r}_{\varphi\, o} + \overrightarrow{o'o}$, so that

$$\mathbf{a} \times \mathbf{r}_{\varphi\, o} + \mathbf{b} \;=\; \mathbf{a} \times (\mathbf{r}_{\varphi\, o'} - \overrightarrow{o'o}\,) + \mathbf{b} \;=\; \mathbf{a} \times \mathbf{r}_{\varphi\, o'} + \mathbf{b} - \mathbf{a} \times \overrightarrow{o'o}$$

holds for arbitrary vectors **a** and **b** , and

$$\begin{aligned} &\delta p_\varphi\, (\kappa_\varphi\, , \mathbf{a} \times \mathbf{r}_{\varphi\, o} + \mathbf{b}) \;=\; \underline{\mathbf{f}}_\varphi \cdot \mathbf{b} + \underline{\mathbf{m}}_{o\varphi} \cdot \mathbf{a} \\ &= \delta p_\varphi\, (\kappa_\varphi\, , \mathbf{a} \times \mathbf{r}_{\varphi\, o'} + \mathbf{b} - \mathbf{a} \times \overrightarrow{o'o}\,) \;=\; \underline{\mathbf{f}}_\varphi \cdot (\mathbf{b} - \mathbf{a} \times \overrightarrow{o'o}\,) + \underline{\mathbf{m}}_{o'\varphi} \cdot \mathbf{a} \\ &= \; \underline{\mathbf{f}}_\varphi \cdot \mathbf{b} - \overrightarrow{o'o} \times \underline{\mathbf{f}}_\varphi \cdot \mathbf{a} + \underline{\mathbf{m}}_{o'\varphi} \cdot \mathbf{a}\, . \end{aligned}$$

A comparison in **a** and **b** leads to Varignon´s formula; q.e.d.

By the definition of the moments, Varignon´s principle holds analogously for the resultant moments

$$\mathbf{m}_{o'} \;=\; \mathbf{m}_o + \overrightarrow{o'o} \times \mathbf{f}\, .$$

The next results are direct consequences of equation (11).

Theorem 3. (Principle of d´Alembert)
A motion of the body is dynamically admissible if and only if

$$\underline{\mathbf{f}}_\varphi \;=\; \mathbf{o} \qquad\qquad \underline{\mathbf{m}}_{o\varphi} \;=\; \mathbf{o} \tag{18}$$

hold for one observer (and hence for all).

Theorem 4. (Newton-Euler laws of motion)
A motion of the body is dynamically admissible if and only if the laws of motion

$$\mathbf{f}_\varphi \;=\; \mathbf{l}_\varphi^{\bullet} \qquad\qquad \mathbf{m}_{o\varphi} \;=\; \mathbf{d}_{o\varphi}^{\bullet} \tag{19}$$

hold for one observer (and hence for all).

The following statement is a direct consequence of the foregoing theorem.

Theorem 5. (Principle of virtual power, global version)
A motion of the body is dynamically admissible if and only if the balance of virtual power

$$\underline{\mathbf{f}}_\varphi \cdot \delta\mathbf{v}_o + \underline{\mathbf{m}}_{o\varphi} \cdot \delta\boldsymbol{\omega} \;=\; 0 \tag{20}$$

holds for all vectors $\delta\mathbf{v}_o$ and $\delta\boldsymbol{\omega}$ for one observer (and hence for all).

Note that the virtual power of this theorem is not identical to the virtual power functional δp_φ , but only contains its essential parts for the distinction between admissible and non-admissible processes. With these laws we are already able to completely describe the dynamics of rigid bodies. For deformable bodies, however, a field formulation of these concepts is desirable, which will be given in the next Section.

Field equations

The key for the localization of the foregoing global concepts is the Riesz representation of a linear continuous functional on topological vector spaces (see, e.g., Adams, 1975 p. 48).

Theorem 6. (field formulation of the virtual power)
For each observer φ there exist $p{+}1$ tensor fields $\mathbf{Y}_{\varphi\, i}$ of order $i = 1, \ldots , p{+}1$ such that

$$\delta p_\varphi\, (\kappa_\varphi\, , \delta\mathbf{v}_\varphi) \;=\; \int_{\mathscr{B}} [\mathbf{Y}_{\varphi\, 1}(x) \cdot \delta\mathbf{v}_\varphi\, (x) + \mathbf{Y}_{\varphi\, 2}(x) \cdot grad\, \delta\mathbf{v}_\varphi\, (x) + \ldots + \mathbf{Y}_{\varphi\, p+1}(x) \cdot grad^{\,p}\, \delta\mathbf{v}_\varphi\, (x)]\; dV$$

for all $\delta\mathbf{v}_\varphi \in \delta\mathscr{V}_\varphi$. (21)

As almost all of our variables are observer-dependent, we will suppress the observer suffix, whenever no distinction between different observers is intended.
The dynamical variables $\mathbf{Y}_1 , \mathbf{Y}_2 , \mathbf{Y}_3 , \ldots , \mathbf{Y}_{p+1}$ in each material point are still functionals of the motion, but do not depend on the virtual velocity. These functionals must be further specified by material laws, which is, however, beyond the scope of this paper.
By (P2) we obtain the same representation for the power

$$p_\varphi (\kappa_\varphi) = \int_{\mathscr{B}} [\mathbf{Y}_{\varphi\,1}(x) \cdot \mathbf{v}_\varphi (x) + \mathbf{Y}_{\varphi\,2}(x) \cdot grad\, \mathbf{v}_\varphi (x) + \ldots + \mathbf{Y}_{\varphi\,p+1}(x) \cdot grad^p\, \mathbf{v}_\varphi (x)]\, dV. \tag{22}$$

Theorem 7. (transformations of dynamical fields)
The fields of the dynamical variables $\mathbf{Y}_{\varphi\,i}$, $i = 1, \ldots , p+1$, are objective under change of observer

$$\mathbf{Y}_{\varphi\,i} = \mathbf{Q} * \mathbf{Y}_{\psi\,i} . \tag{23}$$

Proof. By (8) we have for all motions

$$\delta p_\varphi (\kappa_\varphi , \mathbf{Q}\, \delta \mathbf{v}_\psi) = \delta p_\psi (\kappa_\psi , \delta \mathbf{v}_\psi)$$

$$= \int_{\mathscr{B}} [\mathbf{Y}_{\varphi\,1} \cdot (\mathbf{Q}\, \delta\mathbf{v}_\psi) + \mathbf{Y}_{\varphi\,2} \cdot (\mathbf{Q}\, grad_\varphi\, \delta\mathbf{v}_\psi\, \mathbf{Q}^T) + \ldots + \mathbf{Y}_{\varphi\,p+1} \cdot (\mathbf{Q} * grad_\varphi{}^p\, \delta\mathbf{v}_\psi)]\, dV$$

$$= \int_{\mathscr{B}} [(\mathbf{Q}^T\, \mathbf{Y}_{\varphi\,1}) \cdot \delta\mathbf{v}_\psi + (\mathbf{Q}^T\, \mathbf{Y}_{\varphi\,2}\, \mathbf{Q}) \cdot grad_\varphi\, \delta\mathbf{v}_\psi + \ldots + (\mathbf{Q}^T * \mathbf{Y}_{\varphi\,p+1}) \cdot grad_\varphi{}^p\, \delta\mathbf{v}_\psi]\, dV$$

$$= \int_{\mathscr{B}} [\mathbf{Y}_{\psi\,1} \cdot \delta \mathbf{v}_\psi + \mathbf{Y}_{\psi\,2} \cdot grad_\psi\, \delta\mathbf{v}_\psi + \ldots + \mathbf{Y}_{\psi\,p+1} \cdot grad_\psi{}^p\, \delta\mathbf{v}_\psi]\, dV \qquad \text{for all } \delta\mathbf{v}_\psi \in \delta\mathscr{V}_\psi$$

as

$$grad_\varphi (\mathbf{Q}\, \delta\mathbf{v}_\psi) = \mathbf{Q}\, grad_\varphi\, \delta\mathbf{v}_\psi = \mathbf{Q}\, grad_\psi\, \delta\mathbf{v}_\psi\, \mathbf{Q}^T$$

and, more generally for higher gradients, by the use of the Rayleigh product

$$grad_\varphi{}^p (\mathbf{Q}\, \delta\mathbf{v}_\psi) = \mathbf{Q} * grad_\psi{}^p\, \delta\mathbf{v}_\psi \qquad \text{for all } \delta\mathbf{v}_\psi \in \delta\mathscr{V}_\psi .$$

A comparison in the arbitrary fields $\delta\mathbf{v}_\psi \in \delta\mathscr{V}_\psi$ leads to the desired result; q.e.d.

For any observer (and dropping the observer-index), by the definition of the resultant lost force (9)

$$\delta p(\kappa , \mathbf{v}_o) = \underline{\mathbf{f}} \cdot \mathbf{v}_o = \int_{\mathscr{B}} \mathbf{Y}_1(x) \cdot \mathbf{v}_o\, dV = \int_{\mathscr{B}} \mathbf{Y}_1(x)\, dV \cdot \mathbf{v}_o$$

we obtain the representation

$$\underline{\mathbf{f}} = \int_{\mathscr{B}} \mathbf{Y}_1(x)\, dV.$$

Analogously, the definition of the resultant lost moment (10)

$$\delta p(\kappa , \boldsymbol{\omega} \times \mathbf{r}) = \underline{\mathbf{m}}_o \cdot \boldsymbol{\omega} = \int_{\mathscr{B}} [\mathbf{Y}_1(x) \cdot \boldsymbol{\omega} \times \mathbf{r} + \mathbf{Y}_2(x) \cdot grad\, (\boldsymbol{\omega} \times \mathbf{r})]\, dV$$

$$= \int_{\mathscr{B}} [\mathbf{r} \times \mathbf{Y}_1(x) + 2\, \mathbf{Y}_2^{axi}(x)]\, dV \cdot \boldsymbol{\omega}$$

leads to the representation

$$\underline{\mathbf{m}}_o = \int_{\mathscr{B}} [\mathbf{r} \times \mathbf{Y}_1(x) + 2\, \mathbf{Y}_2^{axi}(x)]\, dV.$$

The resultant force has the representation

$$\mathbf{f} = \int_{\mathscr{B}} [\mathbf{Y}_1(x) + \rho\, \mathbf{r}(x)^{\bullet\bullet}]\, dV$$

and the resultant moment

$$\mathbf{m}_o = \int_{\mathscr{B}} \{\mathbf{r}(x) \times [\mathbf{Y}_1(x) + \rho(x)\, \mathbf{r}(x)^{\bullet\bullet}] + 2\, \mathbf{Y}_2^{axi}(x)\}\, dV\,.$$

By Theorem 7, we know that $\mathbf{Y}_1$ and $\mathbf{Y}_2^{axi}$ are objective vector fields.
If we substitute these representations into the Principle of d´Alembert, we obtain local forms of the laws of motion.

Theorem 8. (local form of the Newton-Euler laws of motion)
A motion of the body is dynamically admissible if and only if the local form of the laws of motion hold

$$\begin{aligned} &\mathbf{Y}_1 = \mathbf{0} \\ &\mathbf{Y}_2^{axi} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{Y}_2 = \mathbf{Y}_2^T \end{aligned} \tag{24}$$

almost everywhere in the body.

Note that the other fields $\mathbf{Y}_2^{sym}$, $\mathbf{Y}_3$, ... , $\mathbf{Y}_{p+1}$ do not directly enter the laws of motion. This, however, does not mean that such quantities cannot play a useful role in mechanical theories.
The next theorem is a stronger version of the Principle of virtual power of the last Section.

Theorem 9. (Principle of virtual power, integral version)
A motion of the body is dynamically admissible if and only if the balance of virtual power holds in the form

$$\int_{\mathscr{B}} (\mathbf{Y}_1 \cdot \delta\mathbf{v} + \mathbf{Y}_2^{axi} \cdot curl\, \delta\mathbf{v})\, dV = 0 \tag{25}$$

for all vector fields $\delta\mathbf{v} \in \delta\mathscr{V}$ for one observer (and hence for all).

Proof. We multiply the local laws of motion (24) by arbitrary vectors $\delta\mathbf{v}$ and $2\delta\boldsymbol{\omega}$. Then

$$\mathbf{Y}_1 \cdot \delta\mathbf{v} + 2\, \mathbf{Y}_2^{axi} \cdot \delta\boldsymbol{\omega} = 0$$

if and only if (24) holds, i.e. if the motion is dynamically admissible. If we interpret $\delta\mathbf{v}$ as the local value of some virtual velocity field $\delta\mathbf{v} \in \delta\mathscr{V}$, and $2\, \delta\boldsymbol{\omega}$ as the local value of its curl, then we obtain the above form of the balance of virtual power as a necessary and sufficient condition for a motion to be dynamically admissible; q.e.d.

In the sequel we will derive a number of alternative forms of the forces and moments which are altogether equivalent to those of (24). By applying the Gauss transformation in the form

$$\int_{\mathscr{B}} \mathbf{Y}_{p+1} \cdot grad^p\, \delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}} (\mathbf{Y}_{p+1}\, \mathbf{n}) \cdot grad^{p-1}\, \delta\mathbf{v}\, dA - \int_{\mathscr{B}} div\, \mathbf{Y}_{p+1} \cdot grad^{p-1}\, \delta\mathbf{v}\, dV$$

for any $p > 1$, we can introduce the surface integrals

$$\delta p(\kappa\,,\delta\mathbf{v}) \;=\; \int_{\mathscr{B}} \;[(\mathbf{Y}_{1} - div\,\mathbf{Y}_{2})\cdot\delta\mathbf{v} - div\,\mathbf{Y}_{3}\cdot grad\,\delta\mathbf{v} - \ldots - div\,\mathbf{Y}_{p+1}\cdot grad^{\,p-1}\,\delta\mathbf{v}]\;dV$$

$$+\int_{\partial\mathscr{B}} \;[(\mathbf{Y}_{2}\,\mathbf{n})\cdot\delta\mathbf{v} + (\mathbf{Y}_{3}\,\mathbf{n})\cdot grad\,\delta\mathbf{v} + \ldots + (\mathbf{Y}_{p+1}\,\mathbf{n})\cdot grad^{\,p-1}\,\delta\mathbf{v}]\;dA\,. \tag{26}$$

For a constant field $\delta\mathbf{v}$ this gives the resultant force

$$\mathbf{f} \;=\int_{\mathscr{B}} \;(\mathbf{Y}_{1} - div\,\mathbf{Y}_{2} + \rho\,\mathbf{r}^{\bullet\bullet})\;dV + \int_{\partial\mathscr{B}} \;\mathbf{Y}_{2}\,\mathbf{n}\;dA \tag{27}$$

and for $\delta\mathbf{v} \equiv \boldsymbol{\omega}\times\mathbf{r}$ the resultant moment

$$\mathbf{m}_{\mathrm{o}} = \int_{\mathscr{B}} \;[\mathbf{r}\times(\mathbf{Y}_{1} - div\,\mathbf{Y}_{2} + \rho\,\mathbf{r}^{\bullet\bullet}) - 2\,(div\,\mathbf{Y}_{3})^{axi}]\;dV + \int_{\partial\mathscr{B}} \;[\mathbf{r}\times\mathbf{Y}_{2}\,\mathbf{n} + 2\,(\mathbf{Y}_{3}\,\mathbf{n})^{axi}]\;dA\,. \tag{28}$$

By applying the Gauss transformation repeatedly, we obtain for the virtual power more surface terms

$$\delta p(\kappa,\delta\mathbf{v}) \;=\; \int_{\mathscr{B}} \;[\mathbf{Y}_{1} - div\,\mathbf{Y}_{2} + div^{2}\,\mathbf{Y}_{3}\ldots + (-1)^{p}\,div^{p}\,\mathbf{Y}_{p+1}]\cdot\delta\mathbf{v}\;dV$$

$$\begin{aligned} &+\int_{\partial\mathscr{B}} \;[\{\mathbf{Y}_{2} - div\,\mathbf{Y}_{3}\ldots - (-1)^{p}\,div^{p-1}\,\mathbf{Y}_{p+1}\}\;\mathbf{n}\cdot\delta\mathbf{v} \\ &+\{\mathbf{Y}_{3} - div\,\mathbf{Y}_{4}\ldots - (-1)^{p-1}\,div^{p-2}\,\mathbf{Y}_{p+1}\}\;\mathbf{n}\cdot grad\,\delta\mathbf{v} \\ &+\ldots + \mathbf{Y}_{p+1}\,\mathbf{n}\cdot grad^{p-1}\,\delta\mathbf{v}]\;dA\,. \end{aligned} \tag{29}$$

For a constant $\delta\mathbf{v}$ this gives the resultant force in the form

$$\mathbf{f} \;=\int_{\mathscr{B}} \;[\mathbf{Y}_{1} + \rho\,\mathbf{r}^{\bullet\bullet} - div\,\mathbf{Y}_{2} + div^{2}\,\mathbf{Y}_{3}\ldots + (-1)^{p}\,div^{p}\,\mathbf{Y}_{p+1}]\;dV$$

$$+\int_{\partial\mathscr{B}} \;[\mathbf{Y}_{2} - div\,\mathbf{Y}_{3}\ldots - (-1)^{p}\,div^{p-1}\,\mathbf{Y}_{p+1}]\;\mathbf{n}\;dA \tag{30}$$

and for $\delta\mathbf{v} \equiv \boldsymbol{\omega}\times\mathbf{r}$ this gives the resultant moment

$$\mathbf{m}_{\mathrm{o}} = \int_{\mathscr{B}} \;\mathbf{r}\times[\mathbf{Y}_{1} + \rho\,\mathbf{r}^{\bullet\bullet} - div\,\mathbf{Y}_{2} + div^{2}\,\mathbf{Y}_{3} - \ldots + (-1)^{p}\,div^{p}\,\mathbf{Y}_{p+1}]\;dV$$

$$\begin{aligned} &+\int_{\partial\mathscr{B}} \;(\mathbf{r}\times[\mathbf{Y}_{2} - div\,\mathbf{Y}_{3} + \ldots - (-1)^{p}\,div^{p-1}\,\mathbf{Y}_{p+1}]\;\mathbf{n} \\ &+ 2\;\{[\mathbf{Y}_{3} - div\,\mathbf{Y}_{4}\ldots - (-1)^{p-1}\,div^{p-2}\,\mathbf{Y}_{p+1}]\;\mathbf{n}\}^{axi})\;dA\,. \end{aligned} \tag{31}$$

Again, the terms with the fields $\mathbf{Y}_{2}^{sym}$, $\mathbf{Y}_{3}$, ... , $\mathbf{Y}_{p+1}$ in the interior and on the surface of the body can be cancelled out in the laws of motion (24).

By further specifying these fields, we will now deduce different classes of materials. We will start with the most simple one.

Example 1: *polar materials* ($\mathbf{Y}_{2}$ skew, and $\mathbf{Y}_{3,4,..} \equiv \mathbf{0}$)

By assuming that the only non-zero dynamical variables are $\mathbf{Y}_{1}$ and the skew part of $\mathbf{Y}_{2}$, we obtain the following virtual power functional

$$\delta p(\kappa, \delta\mathbf{v}) = \int_{\mathscr{B}} (\mathbf{Y}_1 \cdot \delta\mathbf{v} + \mathbf{Y}_2^{skw} \cdot grad\,\delta\mathbf{v})\, dV$$

$$= \int_{\mathscr{B}} (\mathbf{Y}_1 \cdot \delta\mathbf{v} + \mathbf{Y}_2^{axi} \cdot curl\,\delta\mathbf{v})\, dV \tag{32}$$

$$= \int_{\mathscr{B}} (\mathbf{Y}_1 + curl\,\mathbf{Y}_2^{axi}) \cdot \delta\mathbf{v}\, dV + \int_{\partial\mathscr{B}} (\mathbf{Y}_2^{axi} \times \mathbf{n}) \cdot \delta\mathbf{v}\, dA$$

by use of the Gauss transformation in the form

$$\int_{\mathscr{B}} \mathbf{Y}_2^{axi} \cdot curl\,\delta\mathbf{v}\, dV = \int_{\partial\mathscr{B}} [\mathbf{Y}_2^{axi} \times \mathbf{n}] \cdot \delta\mathbf{v}\, dA + \int_{\mathscr{B}} curl\,\mathbf{Y}_2^{axi} \cdot \delta\mathbf{v}\, dV\,.$$

For a constant $\delta\mathbf{v}$ this gives the resultant force

$$\mathbf{f} = \int_{\mathscr{B}} (\mathbf{Y}_1 + \rho\,\mathbf{r}^{\bullet\bullet})\, dV = \int_{\mathscr{B}} (\mathbf{Y}_1 + \rho\,\mathbf{r}^{\bullet\bullet} + curl\,\mathbf{Y}_2^{axi})\, dV + \int_{\partial\mathscr{B}} \mathbf{Y}_2^{axi} \times \mathbf{n}\, dA \tag{33}$$

and for $\delta\mathbf{v} \equiv \boldsymbol{\omega} \times \mathbf{r}$ the resultant moment

$$\mathbf{m}_o = \int_{\mathscr{B}} [\mathbf{r} \times (\mathbf{Y}_1 + \rho\,\mathbf{r}^{\bullet\bullet}) + 2\,\mathbf{Y}_2^{axi}]\, dV \tag{34}$$

$$= \int_{\mathscr{B}} [\mathbf{r} \times (\mathbf{Y}_1 + \rho\,\mathbf{r}^{\bullet\bullet} + curl\,\mathbf{Y}_2^{axi})]\, dV + \int_{\partial\mathscr{B}} \mathbf{r} \times (\mathbf{Y}_2^{axi} \times \mathbf{n})\, dA\,.$$

Usually one considers for the two vector fields the following decomposition into a divergence field and a rest in the form

$$\mathbf{Y}_1 = \rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet}) + div\,\mathbf{T} \tag{35}$$
$$\mathbf{Y}_2^{axi} = \tfrac{1}{2}\,\rho\,\underline{\mathbf{m}} + \tfrac{1}{2}\,div\,\mathbf{M}\,.$$

These two vector fields are objective after theorem (7). Because of the inertia term in $\mathbf{Y}_1$, however, it is not possible that both, $\mathbf{b}$ and $\mathbf{T}$ are objective. It may be tempting to assume the objectivity of the stress tensor $\mathbf{T}$, while the body force density $\mathbf{b}$ is made to contain all the inertia terms. However, one should keep in mind that the distinction between body forces and surface forces induced by $\mathbf{T}$ is at this stage by no means unique. This issue would become clearer by specifying the dependencies of these two variables on kinematical quantities through constitutive equations, which is, however, beyond the scope of this treatise.

For the virtual power functional we obtain the following versions

$$\delta p(\kappa, \delta\mathbf{v}) = \int_{\mathscr{B}} [(\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) \cdot \delta\mathbf{v} + (\rho\,\underline{\mathbf{m}} + div\,\mathbf{M}) \cdot \tfrac{1}{2}\,curl\,\delta\mathbf{v}]\, dV$$

$$= \int_{\mathscr{B}} [\rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet}) \cdot \delta\mathbf{v} + \rho\,\underline{\mathbf{m}} \cdot \tfrac{1}{2}\,curl\,\delta\mathbf{v} - \mathbf{T} \cdot grad\,\delta\mathbf{v} - \mathbf{M} \cdot \tfrac{1}{2}\,grad\,curl\,\delta\mathbf{v}]\, dV$$

$$+ \int_{\partial\mathscr{B}} [(\mathbf{T}\,\mathbf{n}) \cdot \delta\mathbf{v} + (\mathbf{M}\,\mathbf{n}) \cdot \tfrac{1}{2}\,curl\,\delta\mathbf{v}]\, dA \tag{36}$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} + \tfrac{1}{2}\,curl\,(\rho\,\underline{\mathbf{m}} + div\,\mathbf{M})] \cdot \delta\mathbf{v}\, dV$$

$$+ \int_{\partial\mathscr{B}} [\tfrac{1}{2}\,(\rho\,\underline{\mathbf{m}} + div\,\mathbf{M}) \times \mathbf{n}] \cdot \delta\mathbf{v}\; dA$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}^{sym} + \tfrac{1}{2}\,curl\,(\rho\,\mathbf{m} + div\,\mathbf{M})] \cdot \delta\mathbf{v}\; dV$$

$$+ \int_{\partial\mathscr{B}} (\mathbf{T}^{axi} + \tfrac{1}{2}\,\rho\,\mathbf{m} + \tfrac{1}{2}\,div\,\mathbf{M}) \times \mathbf{n} \cdot \delta\mathbf{v}\; dA$$

with the vector field $\mathbf{m} := \underline{\mathbf{m}} - 2\,\mathbf{T}^{axi}/\rho$.

This gives the resultant lost force in the forms

$$\underline{\mathbf{f}} = \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T})\; dV = \int_{\mathscr{B}} \rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet})\; dV + \int_{\partial\mathscr{B}} \mathbf{T}\,\mathbf{n}\; dA$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} + \tfrac{1}{2}\,curl\,(\rho\,\underline{\mathbf{m}} + div\,\mathbf{M})]\; dV$$

$$+ \int_{\partial\mathscr{B}} [\tfrac{1}{2}\,(\rho\,\underline{\mathbf{m}} + div\,\mathbf{M}) \times \mathbf{n}]\; dA \tag{37}$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}^{sym} + \tfrac{1}{2}\,curl\,(\rho\,\mathbf{m} + div\,\mathbf{M})]\; dV$$

$$+ \int_{\partial\mathscr{B}} (\mathbf{T}^{axi} + \tfrac{1}{2}\rho\,\mathbf{m} + \tfrac{1}{2}\,div\,\mathbf{M}) \times \mathbf{n}\; dA$$

and the resultant lost moment

$$\underline{\mathbf{m}}_o = \int_{\mathscr{B}} [\mathbf{r} \times (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) + 2\,\mathbf{T}^{axi} + \rho\,\mathbf{m} + div\,\mathbf{M}]\; dV$$

$$= \int_{\mathscr{B}} [\mathbf{r} \times (\mathbf{b} - \mathbf{r}^{\bullet\bullet}) + \mathbf{m}]\,\rho\; dV + \int_{\partial\mathscr{B}} [\mathbf{r} \times (\mathbf{T}\,\mathbf{n}) + \mathbf{M}\,\mathbf{n}]\; dA \tag{38}$$

$$= \int_{\mathscr{B}} \mathbf{r} \times [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} + curl\,(\mathbf{T}^{axi} + \tfrac{1}{2}\,\rho\,\mathbf{m} + \tfrac{1}{2}\,div\,\mathbf{M})]\; dV$$

$$+ \int_{\partial\mathscr{B}} \mathbf{r} \times [(\mathbf{T}^{axi} + \tfrac{1}{2}\rho\,\mathbf{m} + \tfrac{1}{2}\,div\,\mathbf{M}) \times \mathbf{n}]\; dA\,.$$

In the absence of distributed moments ($\mathbf{m} \equiv \mathbf{0}$ and $\mathbf{M} \equiv \mathbf{0}$) (non-polar material) we obtain in particular

$$\mathbf{Y}_1 = \rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet}) + div\,\mathbf{T} \tag{39}$$

$$\mathbf{Y}_2^{axi} = \mathbf{T}^{axi}.$$

Note that the two vector fields $\mathbf{b}$ and $\mathbf{T}^{axi}$ can be determined independently, the same as $\mathbf{Y}_1$ and $\mathbf{Y}_2^{axi}$.

This gives the following forms of the virtual power functional

$$\delta p(\kappa, \delta\mathbf{v}) = \int_{\mathscr{B}} [(\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) \cdot \delta\mathbf{v} + \mathbf{T}^{axi} \cdot curl\,\delta\mathbf{v}]\, dV$$

$$= \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} + curl\,\mathbf{T}^{axi}) \cdot \delta\mathbf{v}\, dV + \int_{\partial\mathscr{B}} (\mathbf{T}^{axi} \times \mathbf{n}) \cdot \delta\mathbf{v}\, dA$$

$$= \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}^{sym}) \cdot \delta\mathbf{v}\, dV + \int_{\partial\mathscr{B}} (\mathbf{T}^{axi} \times \mathbf{n}) \cdot \delta\mathbf{v}\, dA \tag{40}$$

$$= \int_{\mathscr{B}} [\rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet}) \cdot \delta\mathbf{v} - \mathbf{T}^{sym} \cdot grad\,\delta\mathbf{v}]\, dV + \int_{\partial\mathscr{B}} (\mathbf{T}\,\mathbf{n}) \cdot \delta\mathbf{v}\, dA.$$

and, thus, for the resultant lost force

$$\underline{\mathbf{f}} = \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T})\, dV = \int_{\mathscr{B}} \rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet})\, dV + \int_{\partial\mathscr{B}} \mathbf{T}\,\mathbf{n}\, dA$$

$$= \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}^{sym})\, dV + \int_{\partial\mathscr{B}} \mathbf{T}^{axi} \times \mathbf{n}\, dA \tag{41}$$

and the resultant lost moment

$$\underline{\mathbf{m}}_o = \int_{\mathscr{B}} [\mathbf{r} \times (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) + 2\,\mathbf{T}^{axi}]\, dV$$

$$= \int_{\mathscr{B}} \mathbf{r} \times \rho\,(\mathbf{b} - \mathbf{r}^{\bullet\bullet})\, dV + \int_{\partial\mathscr{B}} \mathbf{r} \times (\mathbf{T}\,\mathbf{n})\, dA \tag{42}$$

$$= \int_{\mathscr{B}} [\mathbf{r} \times (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) + curl\,\mathbf{T}^{axi}]\, dV + \int_{\partial\mathscr{B}} \mathbf{r} \times (\mathbf{T}^{axi} \times \mathbf{n})\, dA.$$

We therefore obtain by (24) Cauchy´s equations of motion in the usual form

$$div\,\mathbf{T} + \rho\,\mathbf{b} = \rho\,\mathbf{r}^{\bullet\bullet}$$

$$\mathbf{T}^{axi} = \mathbf{0} \iff \mathbf{T} = \mathbf{T}^T. \tag{43}$$

Comparison with Germain's first gradient theory.

The method of virtual power has been used in the past to derive the fundamental equations of continuum mechanics in a systematic way, for instance in a series of papers by Germain (1972, 1973). There are two main differences between Germain's work and the present proposal. Firstly, Germain introduces the virtual power functional as a primitive concept, whereas we have constructed it here as an extension of the actual power. Secondly, in Germain's approach and in contrast to the present work, the various forces which act on the body are first divided into two classes: *external forces* representing the dynamical effects on the body $\mathscr{B}$ due to the interaction with other bodies which have no common part with $\mathscr{B}$, and *internal forces* representing the mutual dynamical effects of subsystems of $\mathscr{B}$. These forces are introduced via the value of the corresponding virtual power.

In the present work, the partition of forces into internal and external ones has not been made from the outset. Accordingly, the principle of objectivity is stated here for the (total) power functional and not only for the power of internal forces, as done in (Germain, 1973). Both theories can be shown to lead to the same field equations for first and second gradient media. The objective of this paragraph is to identify the dynamical variables $\mathbf{Y}_i$ of the present framework in Germain's first gradient theory.

The derivation of the first gradient theory following the method of virtual power by Germain is not given here. Instead, parts of the results obtained in (Germain, 1973) are recalled and compared to the present formulation. According to Germain, the virtual power functional is the sum of the following contributions

$$\delta p(\kappa, \delta \mathbf{v}) = \delta p_i(\kappa, \delta \mathbf{v}) + \delta p_d(\kappa, \delta \mathbf{v}) + \delta p_c(\kappa, \delta \mathbf{v}) + \delta p_a(\kappa, \delta \mathbf{v}) \tag{44}$$

which are specified in the sequel. The virtual power functional of the internal forces δp_i is a linear form with respect to the symmetric part of the virtual velocity gradient

$$\delta p_i(\kappa, \delta \mathbf{v}) = -\int_{\mathscr{B}} \mathbf{S} \cdot \delta \mathbf{D}\, dV \qquad \text{with } \delta \mathbf{D} := (grad\, \delta \mathbf{v})^{sym} \tag{45}$$

where the stress tensor $\mathbf{S}$ is symmetric. The virtual power of external forces is split into the contributions of long range forces and contact forces. The virtual power functional of the long range forces δp_d has the following general form

$$\delta p_d(\kappa, \delta \mathbf{v}) = \int_{\mathscr{B}} (\rho\, \mathbf{b} \cdot \delta \mathbf{v} + \mathbf{C} \cdot \delta \mathbf{W})\, dV \quad \text{with } \delta \mathbf{W} := (grad\, \delta \mathbf{v})^{skw}. \tag{46}$$

The specific body force $\rho\, \mathbf{b}$ and the skew body couple $\mathbf{C}$ (and, possibly, a density of body symmetric double forces) must be introduced in general in the first gradient theory. The virtual power functional of contact forces δp_c is

$$\delta p_c(\kappa, \delta \mathbf{v}) = \int_{\partial \mathscr{B}} \mathbf{t} \cdot \delta \mathbf{v}\, dA\,. \tag{47}$$

The stress vector is found to have the following form

$$\mathbf{t} = \mathbf{T}\, \mathbf{n} \qquad \text{with} \qquad \mathbf{T} = \mathbf{S} + \mathbf{T}^{skw} \tag{48}$$

The tensor $\mathbf{T}$ is the *effective* or *Cauchy stress tensor*. The Gauss transformation can be applied to the previous surface integral to get the alternative expression

$$\delta p_c(\kappa, \delta \mathbf{v}) = \int_{\mathscr{B}} [(div\, \mathbf{T}) \cdot \delta \mathbf{v} + \mathbf{T} \cdot grad\, \delta \mathbf{v}]\, dV. \tag{49}$$

Finally, the virtual power of inertial forces is

$$\delta p_a(\kappa, \delta \mathbf{v} = -\int_{\mathscr{B}} \rho\, \mathbf{r}^{\bullet\bullet} \cdot \delta \mathbf{v}\, dV. \tag{50}$$

The expressions above are indeed the most general forms of contact and long range forces within a first gradient framework. More general contributions would not be balanced by the present form of neither the internal nor the acceleration forces.

As a result of all previous contributions, the following expression of the virtual power functional is obtained

$$\delta p(\kappa, \delta \mathbf{v}) = \int_{\mathscr{B}} [(\rho\, \mathbf{b} - \rho\, \mathbf{r}^{\bullet\bullet} + div\, \mathbf{T}) \cdot \delta \mathbf{v} + (\mathbf{T}^{skw} + \mathbf{C}) \cdot \delta \mathbf{W}]\, dV \tag{51}$$

from which the dynamical variables $\mathbf{Y}_i$ in the Riesz representation of the power functional (21) can be identified as

$$\mathbf{Y}_1 \;=\; \rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} \tag{52}$$
$$\mathbf{Y}_2 \;=\; \mathbf{Y}_2^{skew} \;=\; \mathbf{T}^{skw} + \mathbf{C}$$

or, equivalently,

$$\mathbf{Y}_2^{axi} \;=\; \mathbf{T}^{axi} + \tfrac{1}{2}\rho\,\mathbf{m} \qquad\qquad \text{with } \rho\,\mathbf{m} := 2\,\mathbf{C}^{axi}.$$

The second law of motion $\mathbf{Y}_2{}^{axi} = \mathbf{0}$ indicates that the skew part of the Cauchy stress tensor is equal to the negative body couple density $\mathbf{C}$. The first gradient theory can be compared to the theory of polar media introduced in the previous Example 1. Unlike general polar media, first gradient continua are not sensitive to surface couple stresses $\mathbf{M}$. Like polar media, first gradient continua can be loaded by body couples $\rho\,\mathbf{m} := 2\,\mathbf{C}^{axi}$.
Symmetric double body forces can also exist in both first gradient and polar media.

Example 2
We consider the case of a second gradient material with $\mathbf{Y}_2$ symmetric, and $\mathbf{Y}_{3,4,..} \equiv \mathbf{0}$. Here we obtain

$$\delta p(\kappa\,,\,\delta\mathbf{v}) \;=\; \int_{\mathscr{B}} (\mathbf{Y}_1 \cdot \delta\mathbf{v} + \mathbf{Y}_2^{sym} \cdot \delta\mathbf{D})\,dV \qquad\qquad \text{with } \delta\mathbf{D} := (grad\;\delta\mathbf{v})^{sym}$$

$$= \int_{\mathscr{B}} (\mathbf{Y}_1 - div\,\mathbf{Y}_2^{sym}) \cdot \delta\mathbf{v}\;dV + \int_{\partial\mathscr{B}} (\mathbf{Y}_2^{sym}\,\mathbf{n}) \cdot \delta\mathbf{v}\;dA\,. \tag{53}$$

The resulting lost force is

$$\underline{\mathbf{f}} = \int_{\mathscr{B}} \mathbf{Y}_1\,dV = \int_{\mathscr{B}} (\mathbf{Y}_1 - div\,\mathbf{Y}_2^{sym})\,dV + \int_{\partial\mathscr{B}} \mathbf{Y}_2^{sym}\,\mathbf{n}\;dA \tag{54}$$

and the resulting lost moment

$$\underline{\mathbf{m}}_{\mathrm{o}} = \int_{\mathscr{B}} \mathbf{r} \times \mathbf{Y}_1\,dV \tag{55}$$

$$= \int_{\mathscr{B}} \mathbf{r} \times (\mathbf{Y}_1 - div\,\mathbf{Y}_2^{sym})\,dV + \int_{\partial\mathscr{B}} \mathbf{r} \times (\mathbf{Y}_2^{sym}\,\mathbf{n})\;dA\,.$$

For the two fields we again make the decomposition into a divergence and a rest

$$\mathbf{Y}_1 \;=\; \rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T} \tag{56}$$

$$\mathbf{Y}_2^{sym} \;=\; \mathbf{S}^{sym} + div\,\mathbf{M}_3 \qquad\qquad \text{with the symmetry } \mathrm{M}_3^{ijk} = \mathrm{M}_3^{jik}\,.$$

This gives

$$\delta p(\kappa\,,\,\delta\mathbf{v}) \;=\; \int_{\mathscr{B}} [(\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T}) \cdot \delta\mathbf{v} + (\mathbf{S}^{sym} + div\,\mathbf{M}_3) \cdot \delta\mathbf{D}]\,dV \tag{57}$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div(\mathbf{T} - \mathbf{S}^{sym} - div\,\mathbf{M}_3)] \cdot \delta\mathbf{v}\;dV + \int_{\partial\mathscr{B}} [(\mathbf{S}^{sym} + div\,\mathbf{M}_3)\,\mathbf{n}] \cdot \delta\mathbf{v}\;dA\,.$$

The resultant lost force becomes

$$\mathbf{f} = \int_{\mathscr{B}} (\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div\,\mathbf{T})\,dV \tag{58}$$

$$= \int_{\mathscr{B}} [\rho\,\mathbf{b} - \rho\,\mathbf{r}^{\bullet\bullet} + div(\mathbf{T} - \mathbf{S}^{sym} - div\,\mathbf{M}_3)]\,dV + \int_{\partial\mathscr{B}} (\mathbf{S}^{sym} + div\,\mathbf{M}_3)\,\mathbf{n}\;dA$$

and the resultant lost moment

$$\mathbf{m}_o = \int_{\mathscr{B}} \mathbf{r} \times (\rho\, \mathbf{b} - \rho\, \mathbf{r}^{\bullet\bullet} + div\, \mathbf{T})\, dV \tag{59}$$

$$= \int_{\mathscr{B}} \mathbf{r} \times [\rho\, \mathbf{b} - \rho\, \mathbf{r}^{\bullet\bullet} + div(\mathbf{T} - \mathbf{S}^{sym} - div\, \mathbf{M}_3)]\, dV + \int_{\partial\mathscr{B}} \mathbf{r} \times (\mathbf{S}^{sym} + div\, \mathbf{M}_3)\, \mathbf{n}\, dA\,.$$

The method of virtual power is the best-suited tool for the derivation of boundary conditions in gradient theories. The simple and double force surface densities must be expressed as in terms of the simple and hyperstress tensors and of the characteristics of the surface (normal vector and curvature). This has been done by Germain (1973) and Trostel (1985, 1993).

Further theories of higher grades can also be specified in the same manner by the general forms in (29).

Conclusions

In the foregoing framework we introduced the fundamental concepts of mechanics on the basis of the mechanical power of a moving body, and its objectivity under Euclidean transformations. After an extension of the power functional to a virtual power functional, we could introduce the forces and moments as the power-conjugate vectors, and derive the laws of motion, which hold for every observer, regardless if inertial or not.

The application of the Riesz theorem for linear continuous functionals leads to field equations for the (virtual) power and for the forces and moments. By applying the Gauss theorem, a catalogue of different, but still equivalent forms can be derived.

The new features of the proposed theory with respect to existing formulations based on power functionals are the following.

- The virtual power is defined as an extension of the actual power.
- The theory is based on the construction of the *total* virtual power functional. There is, at the beginning, no formal partition into internal and external forces. This partition is however necessary as a next step for the formulation of the boundary value problem on the body, especially in order to derive the proper boundary conditions.
- A synthetic framework is presented applying to grade-p materials. The presented balance equations are valid for any $p > 0$. The required generalized stress tensors clearly appear in the Riesz formulation of the power functional. They intervene only indirectly in the local form of the balance equations, as shown in (Germain 1973) for the second grade theory.

By this procedure we have produced a catalogue of different, but essentially equivalent forms of the power and of the balance equations. As a next step, it would be the task of material theories to further specify such theories.

The method of virtual power can also be used to derive the field and boundary equations for other generalized continua like the Cosserat and micromophic theories. However, these models require the introduction of additional degrees of freedom, i.e. enriched kinematics of the continuum, which is not considered in the present work (see Germain, 1973b, and Forest and Sievert, 2003, 2006).

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