TECHNISCHE MECHANIK, Band 27, Heft 1, (2007), 28-36
Manuskripteingang: 29. November 2006

Load-Transfer to an Orthotropic Fibre-Reinforced Composite Strip via an Elastic Element

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An asymptotic approach is proposed to describe the load-transfer to an orthotropic fibre-reinforced composite strip via an elastic element. To start with we simplified the input boundary value problem (BVP) using ratios of the elastic constants as small parameters. The simplified BVP can be solved exactly. Then we homogenized the obtained solution and estimate the asymptotic error of the homogenization method. We also analysed the influence of the weak interface between fibres and matrix.

1 Introduction

The wide-spread use of composites with increasingly complex internal structures raises the question how the load-transfer from one to another part of the composite can be properly described. Melan (1932) was the first to study the problem of the stress transmission between an infinite fibre and an infinite linear elastic matrix. His paper was reconsidered and extended by many authors; see for the references papers by Lenci (2000), Muki and Sternberg (1968), Sternberg (1970), and books by Christensen (1979), Grigolyuk and Tolkachev (1987), Manevitch and Pavlenko (1991). Exact analytical solutions can be obtained only for the infinite matrix (Lenci, 2000; Muki and Sternberg, 1968). For the solution of the plane problem for a strip one obtains a singular integral equation or a system of equations that can be solved only numerically (Grigolyuk and Tolkachev, 1987; Sternberg, 1970). The method of boundary layers, i.e., the asymptotic approach for a highly anysotropic matrix (Christensen, 1979; Manevitch and Pavlenko, 1991) must be used carefully for the isotropic case. A lot of papers are devoted to a half-plane or a strip isotropic matrix reinforced along the boundaries (Alexandrov and Arutyunyan, 1983; Antipov, 1993; Antipov and Arutyunyan, 1993; Bardzokas and Sfyris, 2005; Bufler, 1964; Melan, 1932). On the other hand, many papers deal with fibre-reinforced composites without boundary reinforce element (Grigolyuk and Tolkachev, 1987; Lenci, 2000; Manevitch and Pavlenko, 1991). For example, the problem of interfaces between fibres and a matrix with friction has been analysed by Antipov and Arutyunyan (1993), who considered the simultaneous presence of Coulomb friction and the perfect adherence. In a paper by Lenci (2000), Melan's problem has been reconsidered under the hypothesis that the adherence between the fibre and the matrix is not perfect. Authors do not know any result for the combined problem – transfer of load to a fibre-reinforced composite via the boundary reinforce element.

In the present paper we address a special problem of load transfer between an orthotropic fibre-reinforced composite strip and the boundary elastic element. This problem is of practical importance for layered, in particular sandwich composites with at least one layer being fibre-reinforced. It can also be interesting for foundation engineering for the calculation of structures of the "wall in the soil" type. The following strategy is applied: firstly, an asymptotic simplification of boundary conditions allows obtaining the exact solution of the governing problem. Then we analyse the area of applicability of the homogenized solution. In the last section of the paper we report a solution for the case of weak interfaces between the fibres and the matrix.

2 Statement of the Problem

We study a composite strip ($0 \le x \le H$, $-\infty < y < \infty$) reinforced in the x-direction by a regular system of fibres (Fig 1). The proposed analysis is based on the following approximate assumptions: the fibre is a 1D elastic continuum; bending of the fibre is neglected; the matrix is of the fibre a 2D elastic continuum according to the conventional theory of the generalized plane stress; the attachment of the fibre is modelled as an ideal line-contact. We suppose firstly that the bond between the matrix and the fibre is perfect and continuous

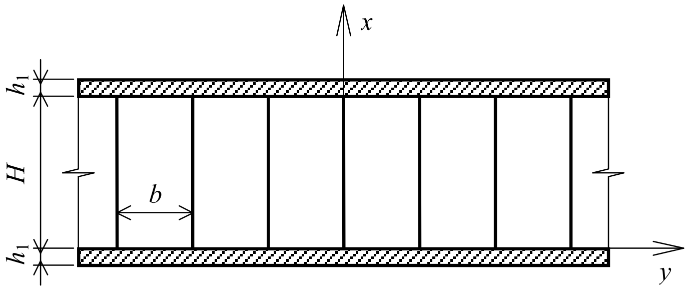

Figure 1. Fibre-reinforced composite strip with boundary beams.

$$u_f = u, \tag{1}$$

where u_f and u are, respectively, displacements of the fibre and of the matrix at the line of interface.

Then the problem is reduced to a system of PDEs with respect to the components u and v of the displacement vector

$$\begin{aligned} &\left[B_{11} + E_f F \Phi(y)\right] u_{xx} + B_{33} u_{yy} + \left(B_{12} + B_{33}\right) v_{xy} = 0, \\ &B_{22} v_{yy} + B_{33} v_{xx} + \left(B_{21} + B_{33}\right) u_{xy} = 0, \end{aligned} \tag{2}$$

where B_{ij} are the rigidities of the orthotropic strip, $\Phi(y) = \sum_{k=-\infty}^{\infty} \delta(y - kb)$, b is the distance between the fibres, $\delta(\cdot)$ is the Dirac delta-function, E_f is the Young modulus of the fibres, F is the area of their cross-section, and indexes x and y denote the corresponding partial derivatives.

The expressions for stresses can be written as follows

$$N_1 = B_{11} u_x + B_{12} v_y, \quad N_2 = B_{12} u_x + B_{22} v_y, \quad N_{12} = B_{33}\left(u_y + v_x\right). \tag{3}$$

Then the transmission stress $\tau^{(k)}$ between the k-th fibre and the matrix is:

$$\tau^{(k)} = 0.5\left(N_{12}^{(-k)} - N_{12}^{(+k)}\right), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{4}$$

where $N_{12}^{(\pm k)} = \lim_{y \to bk \pm 0} N_{12}$.

The fibre-force $T^{(k)}$ satisfies the equilibrium equation

$$T^{(k)} = E_f F \left.\frac{d^2 u}{dx^2}\right|_{y=bk} = 2\tau^{(k)}. \tag{5}$$

We assume that the composite strip is reinforced at $x = 0, H$ by elastic beams symmetrically located with respect to the middle plane of the strip. Then the boundary conditions for system (2) is

$$B_{33}\left(u_y + v_x\right) = \pm E_1 F_1 v_{yy} \quad \text{at } x = 0, H, \tag{6}$$

$$\left[B_{11} + E_f F \Phi(y)\right] u_x + B_{12} v_y = T_{10} \cos\frac{2\pi y}{L} \pm E_1 I_1 u_{yyyy} \quad \text{at } x = 0, H, \tag{7}$$

where E_1 is Young's modulus and F_1 is the cross-section's area of the boundary beams, $F_1 = h_1 h_2$, h_1 is the height of the beams (Fig. 1), h_2 is the thickness of the strip, I is the first moment of the cross-section of the beams, $I = h_1^3 h_2/12$, T_{10} is the intensity and L is the period of the applied load.

The boundary conditions (6), (7) are very important from the practical standpoint, because usually external loads are transferred to composite structures via an elastic element. Unfortunately, the mixed BVP (2), (6), (7) does not allow for an exact analytical solution. On the other hand, as a limiting case one can obtain from (6), (7) the following boundary conditions at $x = 0, H$

$$v = 0\,, \tag{8}$$

$$\left[B_{11} + E_f F \Phi(y)\right] u_x + B_{12} v_y = T_{10} \cos\frac{2\pi y}{L}\,. \tag{9}$$

The BVP (2), (8), (9) can be solved exactly. Below we shall show that the boundary conditions (6), (7) can be approximately reduced to a simplified form.

3 Analysis of Boundary Conditions

Let us introduce the non-dimensional co-ordinates $\xi = 2\pi x/L$, $\eta = 2\pi y/L$. The equilibrium equations (2) and the boundary conditions (6), (7) can be written as follows

$$\begin{aligned} &\left[1 + b_1 F \Phi(\eta)\right] u_{\xi\xi} + b_3 u_{\eta\eta} + \left(\nu_{12} + b_3\right) v_{\zeta\eta} = 0\,, \\ &b_2 v_{\eta\eta} + b_3 v_{\xi\xi} + \left(\nu_{21} + b_3\right) u_{\xi\eta} = 0\,, \end{aligned} \tag{10}$$

$$\left\{u_\eta + v_\xi = \pm e_1 v_{\eta\eta}\right\}\Big|_{x=0,h}\,, \tag{11}$$

$$\left[1 + b_1 \Phi(\eta)\right]\left\{u_\xi + \nu_{12} v_\eta - T_0 \cos\eta = \pm e_2 u_{\eta\eta\eta\eta}\right\}\Big|_{x=0,h}\,, \tag{12}$$

where $b_1 = 2\pi E_f F/\left(B_{11} L\right)$, $b_2 = B_{22}/B_{11}$, $b_3 = B_{33}/B_{11}$, $\nu_{12} = B_{12}/B_{11}$, $h = 2\pi H/L$, $e_1 = 2\pi E_1 F_1/\left(B_{33} L\right)$, $e_2 = \left(2\pi\right)^3 E_1 I/\left(B_{11} L\right)^3$, $T_0 = L T_{10}/\left(2\pi B_{11}\right)$, $\Phi(\eta) = \sum_{i=-\infty}^{\infty} \delta\left(\eta - i/n\right)$, $n = 2\pi L/b$.

In many practical cases it is possible to estimate

$$e_1 >> 1, \quad e_2 << 1. \tag{13}$$

In order to verify these estimations we consider the following numerical example. Let the matrix be isotropic; then $B_{11} = E h_2/\left(1 - \nu^2\right)$, where E is the Young's modulus and ν is the Poisson's ratio of the matrix. As it has been emphasized by Christensen (1979) (see Chapter 3.4), for most of the real fibrous composites, the rigidity of reinforcement elements fits the range: $E_1/E = 10^2 ... 10^3$. Let $E_1/E = 100$. For thin boundary beams it is natural to suppose $h_1 = h_2$. We also assume that the applied load changes slowly so that $L = 10\pi h_1$. Then $e_1 = 20$, $e_2 = 1/15$. Hence, in practical calculations e_1 can be considered as a large parameter and e_2 as a small one.

Using estimation (13), the boundary condition (11) can be reduced to a simplified form

$$v = 0 \quad \text{at } \xi = 0, h\,. \tag{14}$$

Condition (8) can be approximately represented as follows:
- for a part of the solution that is slowly varying in the η-direction

$$\left[1 + b_1 \Phi(\eta)\right] u_\xi = T_0 \cos\eta \quad \text{at } \xi = 0, h\,, \tag{15}$$

– for a part of the solution that is fast varying in the η -direction

$$u = 0 \quad \text{at } \xi = 0, h . \tag{16}$$

In order to clarify the terms “slowly” and “fast varying” let us analyse a part of the solution having the form $\varphi(a\xi)\cos m\eta$, where the parameter a is an m-dependent constant. Due the linearity of the governing problem a is of the order of m. Comparing the terms u_ξ and $e_2 u_{\eta\eta\eta\eta}$ in condition (10) we can estimate that for $m < m^*$ the solution is slowly varying and for $m > m^*$ the solution is fast varying; here $m^* = \sqrt[3]{1/e_2}$.

4 Exact Solution

Firstly, let us consider a half-plane. In this case the boundary conditions can be written as follows

$$\left[1 + b_1\Phi(\eta)\right]u_\xi = T_0 \cos\eta \quad \text{at } \xi = 0 , \tag{17}$$

$$v = 0 \quad \text{at } \xi = 0 , \tag{18}$$

$$u, v \to 0 \quad \text{at } x \to \infty . \tag{19}$$

Applying to the first equation of system (10) the cos-Fourier transforms and to the second one the sin-Fourier transforms (Tranter, 1971)

$$\overline{u}(p,\eta) = \int_0^\infty u(\xi,\eta)\cos(p\xi)d\xi , \quad \overline{v}(p,\eta) = \int_0^\infty v(\xi,\eta)\sin(p\xi)d\xi , \tag{20}$$

and taking into account the boundary condition (17), one obtains the following system of two ODEs:

$$\begin{aligned} &\left[1 + b_1 F\Phi(\eta)\right] p^2\overline{u} - b_3\overline{u}_{\eta\eta} + \left(\nu_{12} + b_3\right) p\overline{v}_\eta = -T_0 \cos\eta , \\ &b_2\overline{v}_{\eta\eta} - \left(\nu_{21} + b_3\right) p\overline{u}_\eta - b_3 p^2\overline{v} = 0 . \end{aligned} \tag{21}$$

The periodicity of the governing problem in the η -direction allows for representing the solution of system (21) in the following form

$$\overline{u}(p,\eta) = \sum_{i=1}^\infty A_i(p)\cos(i\eta) , \quad \overline{v}(p,\xi) = \sum_{i=1}^\infty D_i(p)\sin(i\eta) . \tag{22}$$

After substitution of expressions (22) into the system (21) and splitting it with respect to trigonometric functions, one obtains for the coefficients A_i , D_i an infinite system of coupled linear algebraic equations

$$D_i(p) = \frac{\left(b_3 + \nu_{12}\right) pi}{b_3 p^2 + b_2 i^2} A_i(p) , \tag{23}$$

$$A_1 + Q_1 \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) \sum_{j=1}^\infty A_j \cos\left(j\frac{2\pi k}{n}\right) = -C_1 , \tag{24}$$

$$A_i + Q_i \sum_{k=0}^{n-1} \cos\left(i\frac{2\pi k}{n}\right) \sum_{j=1}^\infty A_j \cos\left(j\frac{2\pi k}{n}\right) = 0 , \quad i = 2,3,4,\ldots , \tag{25}$$

where

$$Q_i = \frac{p^2 b_1 \left(b_3 p^2 + b_2 i^2\right)}{\pi \left\{ b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2 \right] + b_3 p^4 \right\}}, \tag{26}$$

$$C_1 = \frac{p^2 b_1 \left(b_3 p^2 + b_2 i^2\right) T_0}{\left\{ b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2 \right] + b_3 p^4 \right\}}.$$

Due the symmetry the infinite system (24), (25) can be solved exactly

$$A_1 = C_1 - Q_1 \frac{n C_1}{2 + n \left(Q_1 + \sum_{d=1}^{\infty} Q_{dn\pm 1} \right)}, \tag{27}$$

$$A_{dn\pm 1} = -Q_{dn\pm 1} \frac{n C_1}{2 + n \left(Q_1 + \sum_{d=1}^{\infty} Q_{dn\mp 1} \right)}, \tag{28}$$

$$A_i = 0, \quad i \neq dn \pm 1, \quad d = 1, 2, 3, \ldots \; . \tag{29}$$

Now let us pass to the case of the matrix-strip. For the strip it is possible to use the finite integral transforms (Tranter, 1971)

$$\bar{u}(p,\eta) = \int_0^h u(\xi,\eta) \cos(\pi p \xi / h) d\xi, \quad \bar{v}(p,\eta) = \int_0^h v(\xi,\eta) \sin(\pi p \xi / h) d\xi . \tag{30}$$

Using for the functions $\bar{u}(p,\eta)$ and $\bar{v}(p,\eta)$ expressions (22), we can derive

$$D_i = \frac{\left(b_3 + \nu_{12}\right)\left(\pi p / h\right) i}{b_3 \left(\pi p / h\right)^2 + b_2 i^2} A_i \tag{31}$$

and come to an infinite algebraic system, which is similar to the system (24), (25) and, thus, possesses the exact solution in the form (27)–(29), where

$$Q_i = \frac{\left(\pi p / h\right)^2 b_1 \left[b_3 \left(\pi p / h\right)^2 + b_2 i^2 \right]}{\pi \left\{ b_2 b_3 i^4 + i^2 \left(\pi p / h\right)^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2 \right] + b_3 \left(\pi p / h\right)^4 \right\}}, \tag{32}$$

$$C_1 = C_{11} T_0 \left[1 + (-1)^{p+1} \right], \quad C_{11} = \frac{\left(\pi p / h\right)^2 b_1 \left[b_3 \left(\pi p / h\right)^2 + b_2 i^2 \right]}{\left\{ b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2 \right] + b_3 \left(\pi p / h\right)^4 \right\}}.$$

Applying now the inverse transforms (Tranter, 1971)

$$u(\xi,\eta) = \frac{2}{h} \sum_{p=1}^{\infty} \bar{u}(p,\eta) \cos(\pi p \xi / h), \quad v(\xi,\eta) = \frac{2}{h} \sum_{p=1}^{\infty} \bar{v}(p,\eta) \sin(\pi p \xi / h) \tag{33}$$

to the expressions (27)–(29), (31), and (32), we obtain

$$A_1(\xi) = \frac{4T_0}{h} \sum_{p=1,3,5,...} \left[C_{11} - Q_1 \frac{nC_{11}}{2+n\left(Q_1 + \sum\limits_{d=1}^{\infty} Q_{dn\pm1}\right)} \right] \cos(\pi p\xi/h)\,, \tag{34}$$

$$A_{dn\pm1}(\xi) = -\frac{4T_0}{h} \sum_{p=1,3,5,...} Q_{dn\pm1} \frac{nC_{11}\cos(\pi p\xi/h)}{2+n\left(Q_1 + \sum\limits_{d=1}^{\infty} Q_{dn\mp1}\right)}\,, \tag{35}$$

$$A_i = 0\,, \quad i \neq dn \pm 1\,, \quad d = 1,2,3,...\ . \tag{36}$$

$$D_i(\xi) = \sum_{p=1,3,5...} \frac{pi\left(b_3 + \nu_{12}\right)A_i(p)}{b_3 p^2 + b_2 i^2} \sin(\pi p\xi/h)\,. \tag{37}$$

5 Homogenized Solution for the Half-Plane

The homogenization approach is widely used in the theory of composite materials (Bakhvalov and Panasenko, 1989; Bensoussan et al., 1978). Here we have a rare possibility to compare a homogenized solution with correctors based on the exact solution of the 2D problem. In this section we derive the homogenized solution for the half-plane from the solution (23), (27), (28). Let us define a natural small parameter ε characterizing the rate of heterogeneity of the composite structure, $\varepsilon = 1/n$, and assume $\varepsilon \ll 1$. The sums $\sum_{d=1}^{\infty} Q_{dn\pm1}$ are of the order ε^2. Expanding expressions (23), (27), and (28) in powers of ε, applying the inverse transforms

$$u(\xi,\eta) = \frac{1}{2\pi}\int_0^\infty \overline{u}(p,\eta)\cos(p\xi)d\xi\,, \quad v(\xi,\eta) = \frac{1}{2\pi}\int_0^\infty \overline{v}(p,\eta)\sin(p\xi)d\xi \tag{38}$$

and keeping terms of the order ε^0 we obtain the homogenized solution

$$u^{(0)} = T_1\left(\frac{\gamma^2}{\beta}\mathrm{e}^{-\beta\xi} - \frac{\beta^2}{\gamma}\mathrm{e}^{-\gamma\xi}\right)\cos\eta\,, \tag{39}$$

$$v^{(0)} = T_1\left(\mathrm{e}^{-\beta\xi} - \mathrm{e}^{-\gamma\xi}\right)\sin\eta\,, \tag{40}$$

where $T_1 = \lambda_1 - \lambda_2 + 2\sqrt{1-\lambda_2/\lambda_1^2}$, $\gamma^2 = \lambda_1\left(1-\sqrt{1-\lambda_2/\lambda_1^2}\right)$, $\beta^2 = \lambda_2\left(1+\sqrt{1-\lambda_2/\lambda_1^2}\right)$, $\lambda_1 = \left[b_3(b_{11}+b_2) - \left(\nu_{12}+b_3\right)^2\right]\Big/\left(2b_{11}b_2\right)$, $\lambda_2 = b_3^2/b_{11}b_2$, $b_{11} = 1 + E_f F/\left(B_{11}b\right)$.

Keeping terms of the orders ε^0, ε^2 for u and of the orders ε^0, ε^3 for v, one derives the following correctors to the homogenized solution

$$u^{(1)} = b_{11}\varepsilon^2 T_1\beta\gamma\left(\gamma\mathrm{e}^{-\beta\xi} - \beta\mathrm{e}^{-\gamma\xi}\right)\cos\eta\sum_{i=1}^{m^*}\frac{\cos\left(j\eta/\varepsilon\right)}{j^2}\,, \tag{41}$$

$$v^{(1)} = b_{11}\varepsilon^3\frac{T_1\beta^2\gamma^2}{\lambda_2^2}\left(\mathrm{e}^{-\gamma\xi} - \mathrm{e}^{-\beta\xi}\right)\cos\eta\sum_{i=1}^{m^*}\frac{\sin\left(j\eta/\varepsilon\right)}{j^3}\,. \tag{42}$$

From expressions (3), (33)–(36) we obtain the following estimations for the displacements

$$u^{(0)} \sim \frac{E_1 F}{B_{11} b}\left(\frac{b}{L}\right)^2 u^{(1)}, \quad v^{(0)} \sim \left(\frac{E_1 F}{B_{11} b}\right)^2 \left(\frac{b}{L}\right)^3 v^{(1)} \tag{43}$$

and homogenized stresses

$$N_i^{(0)} \sim \frac{E_1 F}{B_{11} b}\left(\frac{b}{L}\right)^2 N_i^{(1)}, \quad N_{12}^{(0)} \sim \frac{E_1 F}{B_{11} b}\left(\frac{b}{L}\right) N_{12}^{(1)}, \quad i = 1,2 . \tag{44}$$

Expressions (43), (44) give a possibility to estimate the accuracy of the homogenized solution. We can observe that the main error of the homogenization approach arises in calculation of the shear stresses.

If from the very beginning we had the boundary conditions (14), (15) instead of the boundary conditions (11), (12), then in expressions (41), (42) we should take $m^* = \infty$. In this case evaluating the asymptotic series by means of formulae 5.4.2.12, 5.4.2.13 of Prudnikov et al. (1986) one obtains

$$u^{(1)} = \frac{1}{12} b_{11} T_1 \beta\gamma \left(\gamma \mathrm{e}^{-\beta\xi} - \beta \mathrm{e}^{-\gamma\xi}\right) \cos\left[\eta \psi_1(\eta)\right] \tag{45}$$

$$v^{(1)} = b_{11} \frac{T_1 \beta^2 \gamma^2}{12 \lambda_2^2} \left(\mathrm{e}^{-\gamma\xi} - \mathrm{e}^{-\beta\xi}\right) \cos\left[\eta \psi_2(\eta)\right] \tag{46}$$

where $\psi_1(\eta) = 3\eta^2 - 6\pi\varepsilon\eta + 2\varepsilon^2\pi^2$, $\psi_2(\eta) = \eta\left(3\eta^2 - 3\pi\varepsilon\eta + 2\varepsilon^2\pi^2\right)$ for $-\pi/\varepsilon \le \eta \le \pi/\varepsilon$. For other values of ε the functions $\psi_1(\eta)$, $\psi_2(\eta)$ continue periodically.

6 Fibres with Weak Interface

When the fibre is glued to the matrix by an adhesive, a jump in displacements proportional to the transmission stress is observed although the continuity of the tensions is maintained for equilibrium. A natural way to model such a situation is to assume that the transmission stress τ depends linearly on the jump of the displacements (Lenci, 2000)

$$\tau = k\left(u_f - u\right) \tag{47}$$

The parameter k summarizes the mechanical characteristics of the interface, and it can be computed from the elastic moduli of the interface (Geymonat et al., 1999). Condition (47) replaces condition (1). Then the governing relations can be written as follows

$$B_{11} u_{xx} + B_{33} u_{yy} + \left(B_{12} + B_{33}\right) v_{xy} = 2 \sum_{j=-\infty}^{\infty} \tau(x, bj) \delta(y - jb), \tag{48}$$

$$B_{22} v_{yy} + B_{33} v_{xx} + \left(B_{21} + B_{33}\right) u_{xy} = 0 , \tag{49}$$

$$\tau(x, bj) = k\left[u_f(x) - u(x, bj)\right], \quad j = 0, \pm 1, \pm 2, \ldots , \tag{50}$$

$$2\tau(x, bj) = E_f F u_{f,xx}^{(j)} . \tag{51}$$

Let us consider the problem for the half-plane. Applying to the BVP (48)–(51) integral transforms (20) and using representations (22) with the above described analytical technique, one obtains the solution in the form (22)–(24), where

$$Q_i = \frac{p^2 b_1 \left(b_3 p^2 + b_2 i^2\right)}{\pi\left(1 + k_1 p^2\right)\left\{b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(v_{12} + b_3\right)^2\right] + b_3 p^4\right\}} \tag{52}$$

$$C_1 = \frac{p^2 b_1 \left(b_3 p^2 + b_2 i^2\right) T_0}{\left(1 + k_1 p^2\right)\left\{b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2\right] + b_3 p^4\right\}} \quad , \quad k_1 = \frac{4\pi E_f F}{kL^2} . \tag{53}$$

The solution for the strip can be derived in the form (23), (27)–(29), where

$$Q_i = \frac{\left(\pi p / h\right)^2 b_1 \left[b_3 \left(\pi p / h\right)^2 + b_2 i^2\right]}{\pi (1 + k_1 (\pi p / h)^2)\left\{b_2 b_3 i^4 + i^2 \left(\pi p / h\right)^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2\right] + b_3 \left(\pi p / h\right)^4\right\}} \quad , \tag{54}$$

$$C_{11} = \frac{\left(\pi p / h\right)^2 b_1 \left[b_3 \left(\pi p / h\right)^2 + b_2 i^2\right]}{(1 + k_1 (\pi p / h)^2)\left\{b_2 b_3 i^4 + i^2 p^2 \left[b_3^2 + b_2 + \left(\nu_{12} + b_3\right)^2\right] + b_3 \left(\pi p / h\right)^4\right\}} \quad . \tag{55}$$

Comparison of the obtained results for the perfect and weak interface reveals that in the last case the influence of the higher order harmonics on the final solution is essentially smaller (i.e., the weak interface cuts off the higher harmonics). Therefore, for the weak interface the proposed asymptotic simplification of governing boundary conditions is more accurate than for the perfect one.

7 Conclusions

In the present paper an asymptotic approach for a problem of load-transfer to an orthotropic fibre-reinforced composite strip via an elastic element is proposed. Asymptotic simplifications of the boundary conditions give a possibility to use integral transforms and to obtain the exact solution of the simplified problem. This solution is generalised for a weak interface problem. We also analyse the area of applicability of the homogenized solution.

Acknowledgements

Authors are grateful to Prof. D. Weichert, whose valuable suggestions and comments helped to improve the paper. We also would like to thank the anonymous reviewers for the very helpful criticism. This work was supported by the German Research Foundation (DFG), grant no. WE 736/25-1 (for I.V. Andrianov), and by the NATO reintegration grant no. NUKR.RIG.981491 (for V.V. Danishevs'kyy).

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