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The Stephenson - Kapitsa pendulum: Area of the Attraction of the Upper Positions of the Balance

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Here we consider the behavior of a pendulum with a vibrating axis of suspension which differs from the classical one by the presence of an additional degree of freedom: the pendulum contains some mass, fixed on a spring, which can be displaced with respect to the main mass in two mutually perpendicular directions. Experimental investigations have been performed to determine the stable equilibrium positions of the pendulum. The results of the experiment were in good agreement with those of the numerical solution of the complete equations of motions of the system. Both of them were in good agreement with the rough analytical solution, obtained before by I.I. Blekhman and L. Sperling. It has been confirmed that the presence of the inner degree of freedom of the additional mass in the "radial" direction results in an essential widening of the range of attraction of the upper vertical quasi-equilibrium position of the pendulum.

1 Introduction

The history of the investigations begins with the classical problem of a rigid pendulum with a vibrating axis of suspension (the *upside-down* pendulum). Stephenson (1908) proved that under the vibration of its axis the pendulum can preserve its stable upper vertical position. Later Kapitsa (1951)gave quite independently the solution of the same problem. He used a simple heuristic method which served as an impulse to the appearance of a new section of the theory of nonlinear oscillations to vibrational mechanics by Blekhman (2000). Acheson and Mullin (1993) found the conditions which are to be kept to provide the stability of the vertical position of the n-link upside-down pendulum. These conditions depend on the number of the links, on their weight and size. Thus a chain of three links (sticks) 0.19 m. long preserved its vertical position at the oscillation frequency of the support 40 Hz and the amplitude 0.01 m. The stability was not broken at the deviation of the position of the chain of sticks from the vertical at an angle equal to 45 degrees. The general statement of the problem and of the equation of motion of the pendulum with inner degrees of freedom under the vibration of its axis of suspension are given in the work of Blekhman and Sperling (2004a), Blekhman and Sperling (2004b). They have also determined analytically the domains of the stability of the equilibrium positions of such pendulum (ibid).
The presentation contains the results of the experimental investigations for determining the domains of stability of the equilibrium positions of the pendulum with a "radial" degree of freedom under a vertical harmonic vibration of its axis of suspension. There is a comparison with the results of the numerical calculations, obtained when solving the complete equations of motion of the pendulum (the *precise* solution) and the equations, based on the use of the method of direct separation of motions into the fast and slow components (the *approximate* solution). For the same purpose numerical solutions of complete equations were also made in the presence of the tangential degree of freedom of the additional mass.

2 Equations of Motion of the Pendulum with the Inner Degrees of Freedom

The scheme under consideration in accordance with Blekhman and Sperling (2004a) is shown in Figure 1.
Equations of motion of this system can be presented in the form

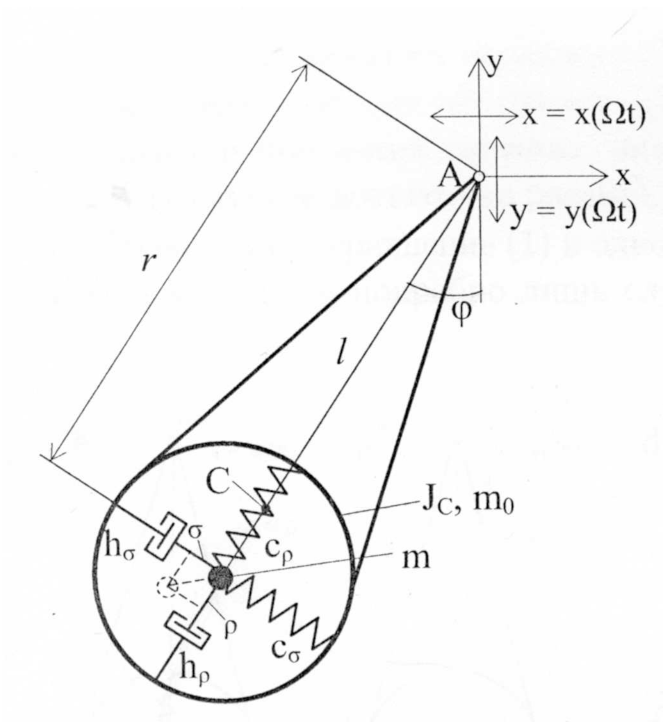

Figure 1. General scheme of the system.

$$[J_A + m(r+\rho)^2 + m\sigma^2]\ddot{\varphi} + k\dot{\varphi} + m[(r+\rho)^2 + \sigma^2]^{\cdot}\dot{\varphi} - [S_A + m(r+\rho)](\ddot{x}\cos\varphi - \ddot{y}\sin\varphi) + \\ + m\sigma(\ddot{x}\sin\varphi + \ddot{y}\cos\varphi) + m[(r+\rho)\ddot{\sigma} - \sigma\ddot{\rho}] + [S_A + m(r+\rho)]g\sin\varphi + mg\sigma\cos\varphi = 0, \quad (1)$$

$$\ddot{\rho} + 2\delta_\rho\dot{\rho} + \omega_\rho^2\rho = \sigma\ddot{\varphi} + 2\dot{\sigma}\dot{\varphi} + (r+\rho)\dot{\varphi}^2 + \ddot{x}\sin\varphi + (g+\ddot{y})\cos\varphi, \quad (2)$$

$$\ddot{\sigma} + 2\delta_\sigma\dot{\sigma} + \omega_\sigma^2\sigma = -(r+\rho)\ddot{\varphi} - 2\dot{\rho}\dot{\varphi} + \sigma\dot{\varphi}^2 + (\ddot{x}\cos\varphi - (\ddot{y}+g)\sin\varphi). \quad (3)$$

Here we have

$$J_A = J_C + m_0 l^2, \quad S_A = m_0 l, \quad \omega_\rho^2 = c_\rho/m, \quad \omega_\sigma^2 = c_\sigma/m, 2\delta_\rho = h_\rho/m, \quad 2\delta_\sigma = h_\sigma/m, \quad (4)$$

g – is the free fall acceleration. For the case, corresponding to the vertical harmonic vibration of the axis of suspension and the "radial" degree of freedom of the additional mass of the pendulum, it is sufficient to consider just two first differential equations of the system (1-3) and assume that $\sigma \equiv 0, x \equiv 0, y = A\cos\Omega t$, where A is the amplitude, and Ω is the vibration frequency.
Then the system of equations is

$$[J_A + m(r+\rho)^2]\ddot{\varphi} + k\dot{\varphi} + 2m(r+\rho)\dot{\rho}\dot{\varphi} + [S_A + m(r+\rho)](g - A\Omega^2\cos\Omega t)\sin\varphi = 0 \quad (5)$$

$$\ddot{\rho} + 2\delta_\rho\dot{\rho} + \omega_\rho^2\rho = (r+\rho)\dot{\varphi}^2 + (g - A\Omega^2\cos\Omega t)\cos\varphi. \quad (6)$$

For the approximate solution of this problem the method of direct separation of motions by Blekhman (2000) is used. Then the solution of the problem will be

$$\varphi = \alpha(t) + \psi(t, \Omega t), \quad \rho = \tilde{\varepsilon}(t) + \kappa(t, \Omega t),$$

where α and $\tilde{\varepsilon}$ – are the slow component, and ψ and κ – are the fast ones, 2π – periodical in "fast" time $\tau = \Omega t$, while their average values by τ are equal to zero

$$\langle\psi(t,\tau)\rangle = 0, \quad \langle\kappa(t,\tau)\rangle = 0$$

(the angular brackets denote here the averaging by τ).

As a result of using the method, the following equations of slow motions are obtained

$$\left(J + \frac{1}{2}mA^2\eta^4V^2\cos^2\alpha\right)\ddot{\alpha} + (k + 2mr\dot{\varepsilon})\dot{\alpha} + S\left[g\left(1 - \frac{1}{4}\mu^2\sin^2\alpha\right) + \frac{\mu}{2}A\Omega^2\cos\alpha\right]\sin\alpha + \\ + \left[\frac{1}{2}(\mu g\cos\alpha + A\Omega^2)(1-\eta^2) - 2\mu\Omega D\eta\dot{\varepsilon}\right]mA\eta^2V^2\sin\alpha\cos\alpha = 0, \quad (7)$$

$$\ddot{\varepsilon} + 2\delta_\rho \dot{\varepsilon} + \omega_\rho^2 \varepsilon = r\dot{\alpha}^2 - 2\mu A\Omega D\eta^3 V^2 \sin\alpha\cos\alpha\,\dot{\alpha} + \frac{\mu^2}{2} r\Omega^2 \sin^2\alpha - \frac{\mu}{2}\left(\frac{\mu}{2} g\cos\alpha + A\Omega^2\right)\sin^2\alpha. \tag{8}$$

When deriving these equations it is used the same simplifying suppositions in the process of solving the equations of fast motions,that had been used in the work by Blekhman and Sperling (2004b). As result of those suppositions, the approximate equations of fast motions were accepted in the following form

$$J\ddot{\psi} = SA\Omega^2 \sin\alpha\cos\Omega t, \tag{9}$$

$$\ddot{\kappa} + 2\beta_\rho \dot{\kappa} + \omega_\rho^2 \kappa = -A\Omega^2 \cos\alpha\cos\Omega t, \tag{10}$$

here and above it is denoted

$$J = J_A + mr^2, \quad S = S_A + mr, \quad D = \frac{\delta_\rho}{\omega_\rho}, \quad \eta = \frac{\Omega}{\omega_\rho}, \tag{11}$$

$$V = \frac{1}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}, \quad \mu = \frac{SA}{J}, \quad \tilde{\varepsilon} = \varepsilon + \frac{mg\cos\alpha}{c_\rho}. \tag{12}$$

The ability to find, approximately, the fast motions, without any essential loss of accuracy in determining the slow motions, is one of the greatest advantages of the method of direct separation of motions, see Blekhman (2000).

3 Experimental Investigation of the Model of the Pendulum on a Vibrational Stand

Load 2 in Figure 2, placed on rod 4, was used in the experiments as a pendulum. The mass of the load was $m = 0.027$kg. The steel rod $\oslash 1.5$mm was passed through the hole $\oslash 1.6$mm passing through the center of gravity of the load. Thus the load could slide freely along the rod. The rod was fixed firmly on the axis of bearing 6, placed in rack 1. The load is linked to the rod by elastic elements 3 (springs) resting on the rings 5, fixed on the rod. The distance from the center of gravity of the pendulum to the axis of suspension is $l = r = 57.5$mm. Experiments were performed on the springs, operating, the general rigidity being equal to $c_\rho = 904$N/m.

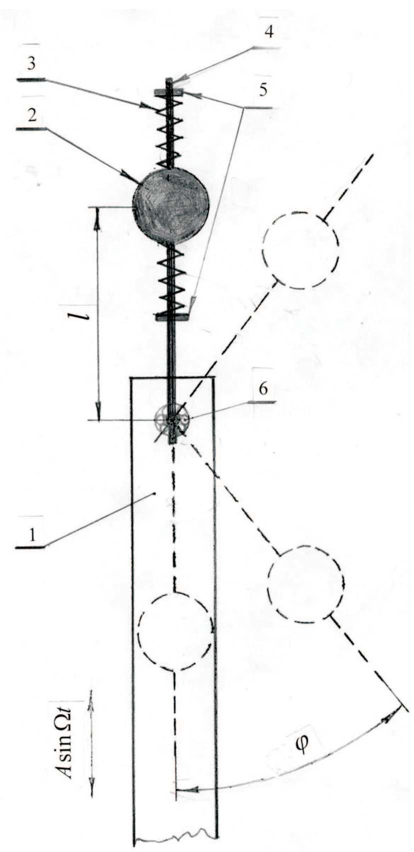

Figure 2. Scheme of the experimental installation.

The axis of suspension of the pendulum was established horizontally, creating its vertical oscillations with a permanent amplitude $A = 6.5$mm and its frequency f $\left(f = \dfrac{\Omega}{2\pi}\right)$, changing within range 5-30Hz.

Results of the experiments are given in Table 1.

The frequency range in the experiments for the nonrigid pendulum was chosen from the condition of the proximity to the resonance frequency of the additional mass of the load $\omega_\rho = 183s^{-1}(f \approx 29Hz)$. The round numbers for the deviation angles in Table 1 are explained by the error in their measuring $(\pm 5^0)$ while the axis of suspension of the pendulum was under vibration.

Table 1. Domains of the attraction of the stable upper vertical position of the quasi-equilibrium of the pendulum

N	Vibration Frequency, Hz	The allowed angle of the deviation of the pendulum $\varphi* = \lvert 180^0 - \varphi \rvert$, degrees		
		Experiment	Calculation	
			By general equations	By approximate equations
1	23.3	–	26.5^0	32.2^0
2	24.2	30^0	33.9^0	42.5^0
3	25.0	40^0	37.9^0	46.5^0
4	26.0	50^0	40.2^0	49.9^0
5	26.6	60^0	39.6^0	50.5^0
6	27.7	70^0	35.0^0	45.9^0
7	28.3	80^0	27.6^0	38.5^0

For the sake of comparison, experiments were made on a rigid pendulum which had the same mass (0.027kg) and the same length (57.5mm) as the pendulum with an additional mass. With an oscillation amplitude of the axis of the pendulum being 6.5mm, the stable upper vertical position of the rigid pendulum was observed at the minimal frequency-30Hz. Then the greatest angle of the initial deviation of the pendulum, providing its return to the stable upper vertical position, was approximately 1.5^0.

It should be noted that the classical condition $A\Omega > \sqrt{2gl}$, which answers the stability of the upper vertical position of the quasi-equilibrium of the pendulum in the case of a rigid pendulum, is fulfilled under the above conditions

$$6.5 \cdot 10^{-3} \cdot 2\pi \cdot 30 > \sqrt{2 \cdot 9.8 \cdot 0.0575}, \quad i.e. \quad 1.22 > 1.06.$$

In the case of the pendulum with the inner degree of freedom this condition is "softened".

So for the minimal value of the oscillation frequency of the axis of the pendulum – 24.2Hz, providing the stability of the upper position, that condition is not met.

$$6.5 \cdot 10^{-3} \cdot 2\pi \cdot 24.2 < \sqrt{2 \cdot 9.8 \cdot 0.0575}, \quad i.e. \quad 0.987 < 1.06.$$

As is seen from the above inequalities, with the nonrigid pendulum the stability of the upper vertical position of the quasi-equilibrium of the pendulum is achieved with the lesser values of the oscillation speed of the axis $A\Omega = 0.987$ than in case of the rigid pendulum when $A\Omega = 1.22$.

As is seen from the table, the domain of attraction of the stable upper vertical position with respect to the initial deviation of the nonrigid pendulum is then considerably widened. The greatest angle of the initial deviation of the pendulum from the upper stable vertical position of the quasi-equilibrium at which it still can return to that stable position made 80^0.

4 Numerical Experiment

The numerical solution of systems of equations (5), (6) and (7)-(10) was performed by the method of Runge-Kutta of the eighth order with an equi-stepwise lattice. The values of the parameters in the equations of motion correspond to the experimental ones and apart from those, mentioned above, were the following

$$J_A = 0.974 \cdot 10^{-4}\text{kg} \cdot \text{m}^2; \quad S_A = 0.0971 \cdot 10^{-2}\text{kg} \cdot \text{m}; \quad \delta_\rho = 30\text{s}^{-1}; \quad \omega_\rho = 183\text{s}^{-1}.$$

Calculations of the rigid pendulum have shown that the largest angle of the initial deviation of the pendulum, providing its return to the upper stable vertical position was approximately 10^0.
The difference from the experiment (approximately 1.5^0) is probably connected with a certain asymmetry in the experimental model of the pendulum. The asymmetry of the pendulum can be connected with the inaccuracy of its production and inaccuracy of the reproduction of the purely vertical vibration.
The results of the numerical calculations by exact and approximate equations for the non-rigid pendulum are given in Table 1.
The boundary of the angular deviation of the pendulum, below which there is a domain of attraction of the upper stable vertical position of the quasi-equilibrium, is shown in Figure 3 (the circles indicate the experimental results, the crosses show the calculation by the exact equations, the triangles indicate the calculation by the approximate equations). Experiments 2-4 (see Table 1 and Fig.3) are in good agreement with the results of the numerical calculations by both exact and approximate equations. The difference in the data of experiments 5-7 with the results of the numerical calculations is caused by the limitation of the strake of the additional mass of the pendulum in the experiment, and in experiment 1 - by the proximity of the boundary of stability.

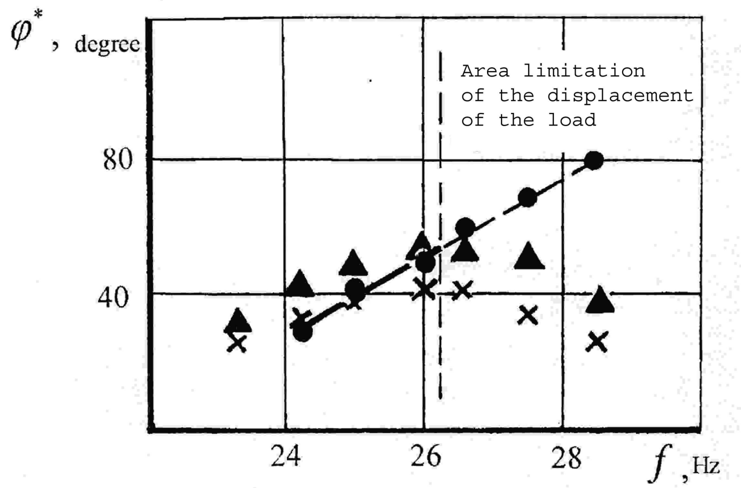

Figure 3. Dependence of the greatest possible deviation by the angle $\varphi*$, providing the return of the pendulum to the stable upper vertical position of the quasi-equilibrium on the oscillation frequency of the axis of its suspension.

Similar numerical calculations have been made for the exact equations (1), (3) with the tangential degree of freedom of the nonrigid pendulum ($\rho \equiv 0, x \equiv 0, y = A\cos\Omega t$) with the same values of the parameters (the mass of the load, the length of the pendulum, the rigidity of the springs). Calculation have shown that the presence of the tangential degree of freedom of the load of the pendulum does not provide a stable upper vertical position of the quasi-equilibrium at the preset amplitude value and frequency range.

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Conclusions

1. Numerical solutions of both exact and approximate equations of the motion of the pendulum under the vertical harmonic vibration of the axis of its suspension and the possibility of the displacement of the additional mass in the radial direction is in good agreement with the results of the experiments.
2. It has been shown that in the presence of the additional mass it is possible to stabilize the upper position of the pendulum at a considerably smaller value of the amplitude of the oscillation speed of the axis of the pendulum.

3. The presence of the additional degree of freedom of the load of the pendulum in the radial direction leads to the widening of the domain of attraction of the upper vertical position of the quasi-equilibrium of the pendulum (by the angular deviation from the vertical), and the presence of the tangential degree of freedom of the load in the frequency range under consideration does not provide the acquisition of the stable upper position of the pendulum. The results obtained can be used to facilitate the starting regimes of a number of vibrational machines.

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