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Arbitrary Penetration of a Rigid Axially Symmetric Indenter into an Axially Heterogeneous Rigid-Perfectly-Plastic Half-Space

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This paper is concerned with the axially symmetric plastic flow of an axially heterogeneous rigid-perfectly-plastic nonhardening half-space. The directions of heterogeneity coincide with the axis of symmetry of indenter and the radial direction in cylindrical frame of references. The arbitrary depth of penetration of the rigid indenter is studied on the basis of the Haar and v. Karman hypothesis. The analytical distribution of contact stress is obtained. It allows for taking into account the local adhesion of an indenter surface and the surface of the half-space. The conical indenter is investigated as a particular case. The dependence between the applied force and the penetration depth of the conical indenter for several cases of heterogeneity is determined.

1 Introduction

Many numerical investigations of plastic flow of heterogeneous ideally plastic bodies have been made (Olszak W. et. al., 1962; Grigoriev O.D., 1969). The basic equations of axially symmetric plastic fields are well known. It has been shown that these equations are statically determined when the Haar and v. Karman hypothesis is satisfied (Grigoriev O.D., 1969; Ishlinsky A.J. et al., 2001). But the approach of analytical solution of axially symmetric contact problem has been applied only to initial plastic flow of homogeneous half-space (Kravchuk A.S. et. al., 2005).

This paper deals with analytical solution of the contact problem for an arbitrary penetration of a rigid indenter into an axially symmetric heterogeneous half-space. We determine the analytical distribution of contact stress for an arbitrary axially symmetric rigid indenter taking into account the variation of the yield stress in the meridian plane. The conical indenter is considered as a particular case. The value of the Meyer hardness and the applied force are determined by the depth of penetration of a conical indenter. The influence of heterogeneity on the hardness and the value of applied force has been investigated

2 Statement of the problem. Preliminary transformation of equations

The contact problem can be conveniently studied with the help of cylindrical polar co-ordinates (r,φ,z), where 0z is the axis of symmetry of bodies (Figure 1). The surface of the half-space in the plain z0r after indenter penetration is determined by the equation (Kravchuk A.S. et. al., 2005)

$$z = f(r)$$

where $f(r)$ is an equation of hafl-space surface. It is a differentiable function for $r \in [0,a) \cup (a,+\infty)$, $f'(r) < 0$ when $r \in [0,a)$ and $f'(r) = \frac{df(r)}{dr} = 0$ when $r \in [a,+\infty)$, a is the radius of contact area (Figure 1).

The stress distribution in the half-space involves the four stress components $\sigma_r(r,z)$, $\sigma_\theta(r,z)$, $\sigma_z(r,z)$, $\tau_{rz}(r,z)$. The circumferential stress $\sigma_\theta(r,z)$ is a principal stress. These components satisfy the equation of equilibrium (Shield R.T., 1955; Grigoriev O.D., 1969; Ishlinsky A.J. et al., 2001)

$$\frac{\partial \sigma_r(r,z)}{\partial r} + \frac{\partial \tau_{rz}(r,z)}{\partial z} + \frac{\sigma_r(r,z) - \sigma_\theta(r,z)}{r} = 0 ,$$
$$\frac{\partial \tau_{rz}(r,z)}{\partial r} + \frac{\partial \sigma_z(r,z)}{\partial z} + \frac{\tau_{rz}(r,z)}{r} = 0 . \tag{1}$$

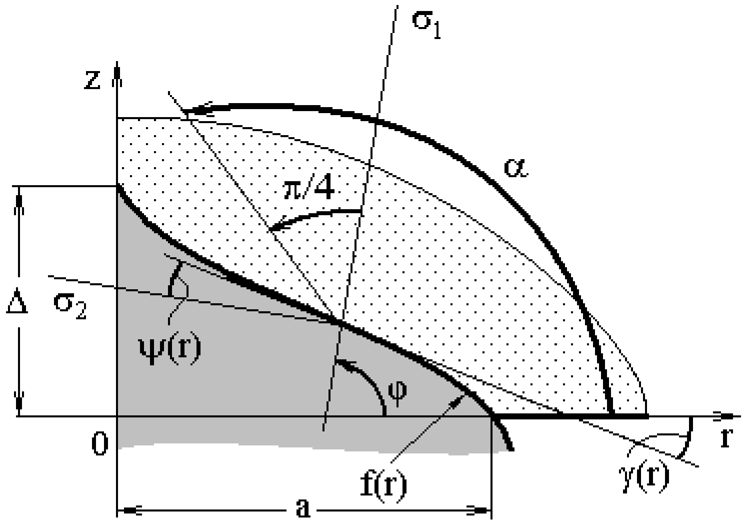

Figure 1. Plane section of contact of an axially symmetric indenter

Let us use the conditions of "full plasticity" for a heterogeneous body at the form (Grigoriev O.D., 1969)

$$\sigma_1(r,z) = \sigma_3(r,z) - 2\cdot K(r,z)\,,\ \sigma_2(r,z) = \sigma_3(r,z)$$

where $\sigma_{i,i\in\overline{1,3}}(r,z)$ is a component of a principal stress, $K(r,z) = \sigma_s(r,z)/2$, $\sigma_s(r,z)$ is the yield stress. The following equations are valid (Olszak W. et. al., 1962; Grigoriev O.D., 1969) (Figure 1)

$$\begin{aligned} &\sigma_r(r,z) = \sigma(r,z) - K(r,z)\cdot\sin(2\cdot\alpha(r,z)), \quad &&\sigma_z(r,z) = \sigma(r,z) + K(r,z)\cdot\sin(2\cdot\alpha(r,z)),\\ &\sigma_\theta(r,z) = \sigma(r,z) + K(r,z)\,, &&\tau_{rz}(r,z) = K(r,z)\cdot\cos(2\cdot\alpha(r,z)), \end{aligned} \tag{2}$$

where $\alpha(r,z) = \varphi(r,z) + \pi/4$ (Shield R.T., 1955), $\sigma(r,z) = \frac{1}{2}(\sigma_1(r,z) + \sigma_3(r,z))$ (Grigoriev O.D., 1969; Ishlinsky A.J. et al., 2001), $\varphi(r,z)$ is the angle between a direction 0r and the first principle stress (Figure 1) (Shield R.T., 1955).

Let us consider the function $\alpha(r,z)$. It is defined on the contact surface by the boundary condition

$$\alpha(r,f(r)) = \gamma(r) + \omega(r)\,, \tag{3}$$

where (Figure 1, 2)

$$\omega(r) = \begin{cases} \omega^- = \dfrac{3\pi}{4} + \psi(r),\ r\in[0,a), \\ \omega^+ = \dfrac{\pi}{4},\ r\in[a,\infty), \end{cases} \tag{4}$$

$\gamma(r) = arctg(f'(r))$ is the angle of the tangent slope to the surface of indenter (Figure 1), $f'(r) = \frac{d}{dr}f(r)$, $\psi(r)\in[0,\pi/4]$ is a variable angle which is defined by the direction of plastic shear in the contact area. The variable angle $\psi(r)$ allows for taking into account the local adhesion of an indenter surface which contacts with a half-space surface. It is supposed that $f'(r) = 0$, when $[a,\infty)$.

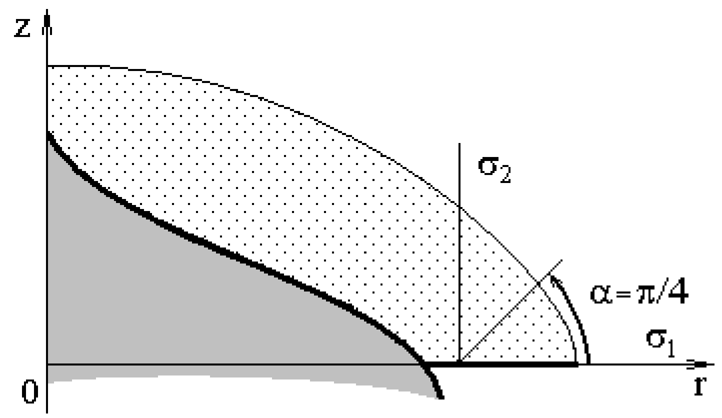

Figure 2. Orientation of principal stress on the free boundary (Shield R.T., 1955)

Substituting (2) into (1), we obtain the following system

$$\begin{gathered}
\frac{\partial \sigma(r,z)}{\partial r} - 2\cdot K(r,z)\cdot\cos(2\cdot\alpha(r,z))\frac{\partial \alpha(r,z)}{\partial r} - 2\cdot K(r,z)\cdot\sin(2\cdot\alpha(r,z))\frac{\partial \alpha(r,z)}{\partial z} - \\
-\frac{K(r,z)}{r}(1+\sin(2\cdot\alpha(r,z))) + \\
+\left\{\left(\frac{\partial}{\partial z}K(r,z)\right)\cdot\cos(2\cdot\alpha(r,z)) - \left(\frac{\partial}{\partial r}K(r,z)\right)\cdot\sin(2\cdot\alpha(r,z))\right\} = 0 \\
\frac{\partial \sigma(r,z)}{\partial z} - 2K(r,z)\cdot\sin(2\cdot\alpha(r,z))\frac{\partial \alpha(r,z)}{\partial r} + 2\cdot K(r,z)\cdot\cos(2\cdot\alpha(r,z))\frac{\partial \alpha(r,z)}{\partial z} + \\
+\frac{K(r,z)}{r}\cos(2\cdot\alpha(r,z)) + \\
\left\{\left(\frac{\partial}{\partial r}K(r,z)\right)\cdot\cos(2\cdot\alpha(r,z)) + \left(\frac{\partial}{\partial z}K(r,z)\right)\cdot\sin(2\cdot\alpha(r,z))\right\} = 0.
\end{gathered} \tag{5}$$

Having made some transformation in system (5) we get the equation

$$\begin{gathered}
\left\{\frac{\partial \sigma(r,z)}{\partial r} + \frac{\partial \sigma(r,z)}{\partial z}f'(r)\right\} - \\
-2\cdot K(r,z)\left[\{\cos(2\,\alpha(r,z)) + \sin(2\,\alpha(r,z))\cdot f'(r)\}\frac{\partial \alpha(r,z)}{\partial r} + \right. \\
\left. + \{\sin(2\,\alpha(r,z)) - \cos(2\,\alpha(r,z))\cdot f'(r)\}\frac{\partial \alpha(r,z)}{\partial z}\right] + F(r,z) = 0
\end{gathered} \tag{6}$$

where

$$\begin{gathered}
F(r,z) = -\frac{K(r,z)}{r}(1+\sin(2\cdot\alpha(r,z))) + \left(\left(\frac{\partial}{\partial z}K(r,z)\right)\cdot\cos(2\cdot\alpha(r,z)) - \left(\frac{\partial}{\partial r}K(r,z)\right)\cdot\sin(2\cdot\alpha(r,z))\right) + \\
+\left\{\frac{K(r,z)}{r}\cos(2\cdot\alpha(r,z)) + \left(\frac{\partial}{\partial r}K(r,z)\right)\cdot\cos(2\cdot\alpha(r,z)) + \left(\frac{\partial}{\partial z}K(r,z)\right)\cdot\sin(2\cdot\alpha(r,z))\right\}\cdot f'(r).
\end{gathered}$$

Taking into account (3) after some permutations we obtain the equation

$$
\begin{gathered}
\left\{\left.\frac{\partial\sigma(r,z)}{\partial r}\right|_{z=f(r)}+\left.\frac{\partial\sigma(r,z)}{\partial z}\right|_{z=f(r)}\cdot f'(r)\right\}-\\
-2\cdot K(r,f(r))\cdot\left[\{cos(2\cdot\gamma(r))+sin(2\cdot\gamma(r))\cdot f'(r)\}\cdot cos(2\cdot\omega(r))-\right.\\
\left.-\left\{sin(2\cdot\gamma(r))\cdot(f'(r))^{-1}-cos(2\cdot\gamma(r))\right\}\cdot f'(r)\cdot sin(2\cdot\omega(r))\right]\cdot\left.\frac{\partial\alpha(r,z)}{\partial r}\right|_{z=f(r)}+\\
-2\cdot K(r,f(r))\cdot\left[\left\{sin(2\cdot\gamma(r))\cdot(f'(r))^{-1}-cos(2\cdot\gamma(r))\right\}\cdot f'(r)\cdot cos(2\cdot\omega(r))+\right.\\
\left.+\{cos(2\cdot\gamma(r))+sin(2\cdot\gamma(r))\cdot f'(r)\}\cdot sin(2\cdot\omega(r))\right]\cdot\left.\frac{\partial\alpha(r,z)}{\partial z}\right|_{z=f(r)}+F(r,f(r))=0,\\
r\in[0,a)\cup(a,\infty)
\end{gathered}
$$

The following equalities are valid

$$
\begin{gathered}
cos(2\cdot\gamma(r))+sin(2\cdot\gamma(r))\cdot f'(r)=cos(2\cdot\gamma(r))+2\cdot sin(\gamma(r))\cdot cos(\gamma(r))\cdot tg(\gamma(r))=1\\
sin(2\cdot\gamma(r))\cdot(f'(r))^{-1}-cos(2\cdot\gamma(r))=sin(2\cdot\gamma(r))\cdot arctg(\gamma(r))-cos(2\cdot\gamma(r))=1
\end{gathered}
$$

Therefore

$$
\begin{gathered}
\left\{\left.\frac{\partial\sigma(r,z)}{\partial r}\right|_{z=f(r)}+\left.\frac{\partial\sigma(r,z)}{\partial z}\right|_{z=f(r)}\cdot f'(r)\right\}-\\
-2\cdot K(r,f(r))\cdot\{cos(2\cdot\omega(r))-f'(r)\cdot sin(2\cdot\omega(r))\}\cdot\left.\frac{\partial\alpha(r,z)}{\partial r}\right|_{z=f(r)}+\\
-2\cdot K(r,f(r))\cdot\{f'(r)\cdot cos(2\cdot\omega(r))+sin(2\cdot\omega(r))\}\cdot\left.\frac{\partial\alpha(r,z)}{\partial z}\right|_{z=f(r)}+F(r,f(r))=0,
\end{gathered}
$$

Taking into account the results of Kravchuk et al., (2005), we can consider additional equation

$$
\left\{\left.\frac{\partial\alpha(r,z)}{\partial r}\right|_{z=f(r)}\cdot(-f'(r))+\left.\frac{\partial\alpha(r,z)}{\partial z}\right|_{z=f(r)}\right\}=-ctg(\omega(r))\left\{\left.\frac{\partial\alpha(r,z)}{\partial r}\right|_{z=f(r)}+\left.\frac{\partial\alpha(r,z)}{\partial z}\right|_{z=f(r)}f'(r)\right\}. \tag{7}
$$

Using (6), (7), we obtain that the function σ satisfies the differential equation on the bound of half-space

$$
\frac{d\,\sigma}{dr}+2K(r,f(r))\cdot\frac{d\alpha}{dr}=-F(r,f(r)),\ r\in[0,a)\cup(a,\infty)\,. \tag{8}
$$

Taking into account (3) and (4) after transformations we obtain the solution of (8)

$$
\begin{gathered}
\sigma=-2\cdot K(r,f(r))\cdot\alpha(r,f(r))+((\sigma(a,f(a))+2\cdot K(a,f(a))\cdot\alpha(a,f(a))))-\\
-\int_a^r F(r,f(r))dr+2\int_a^r\left(\frac{d}{dr}K(r,f(r))\right)\cdot\alpha(r,f(r))\,dr,r\in[0,a)
\end{gathered}
$$

where $\sigma(a,f(a))$ is a constant.

It is known that $\sigma_z=0$ when $r\in[a,\infty)$. Therefore we obtain the boundary condition from (2) and (4) (Shield R.T., 1955; Ishlinsky A.J. et al., 2001)

$$
\sigma(r,0)=-K(r,0),\ \ r\in[a,\infty) \tag{9}
$$

Hence $\sigma(a,f(a))=-K(a,f(a))=-K(a,0)$.

Thus (3), (4), (9), in the area of contact we obtain the following equation

$$\begin{aligned}
\sigma_z(r, f(r)) = -2 \cdot \left(arctg\big(f'(r)\big) + \frac{3\pi}{4} + \psi(r) \right) \cdot K\big(r, f(r)\big) + 2 \cdot \left(\frac{\pi}{4} - \frac{1}{2} \right) \cdot K\big(a, f(a)\big) - \\
- \int_a^r F\big(r, f(r)\big) dr + 2 \cdot \int_a^r \left(arctg\big(f'(r)\big) + \frac{3\pi}{4} + \psi(r) \right) \cdot \left(\frac{d}{dr} K\big(r, f(r)\big) \right) dr + \\
+ sin\left(2 \cdot \left(arctg\big(f'(r)\big) + \frac{3\pi}{4} + \psi(r) \right) \right) \cdot K(r, f(r)), \\
r \in [0, a)
\end{aligned} \tag{10}$$

The applied force F is determined by the equation (Ishlinsky A.J. et. al., 2001)

$$F = 2\pi \int_0^a \big(-\sigma_z(r) \big) \cdot r dr \tag{11}$$

Taking into account (10), (11), Meyers hardness (*HM*) is defined by the following equation

$$HM = \frac{F}{\pi a^2} = \frac{1}{\pi a^2} \left(2\pi \int_0^a \big(-\sigma_z(r) \big) \cdot r dr \right). \tag{12}$$

3 Conical Indenter

The boundary of conical indenter is determined by equation:

$$f(r) = tg(\beta) \cdot r + \Delta .$$

In this case $f'(r) = tg(\beta)$, where β is the negative angle between the element of the cone and the direction $0r$ (Figure 3). The penetration depth of indenter is determined by the condition $f(a) = 0$ (Figure 1, 3). It means that the depth of penetration Δ is defined by equation (Figure 1, 3)

$$\Delta = -tg(\beta) \cdot a .$$

It is supposed that $\psi(r) = \psi$, where $\psi \in [0, \pi/4]$ is a constant. In this case equation (10) can be simplified

$$\begin{aligned}
\sigma_z(r, f(r)) = -2 \cdot \left(\beta + \frac{3\pi}{4} + \psi \right) \cdot K\big(r, f(r)\big) + 2 \cdot \left(\frac{\pi}{4} - \frac{1}{2} \right) \cdot K\big(\Delta/(-tg(\beta)), 0\big) - \int_{\Delta/(-tg(\beta))}^{r} F\big(r, f(r)\big) dr + \\
+ 2 \cdot \left(\beta + \frac{3\pi}{4} + \psi \right) \cdot \Big(K\big(r, f(r)\big) - K\big(\Delta/(-tg(\beta)), 0\big) \Big) + sin\left(2 \cdot \left(\beta + \frac{3\pi}{4} + \psi \right) \right) \cdot K(r, f(r)), \\
r \in [0, \Delta/(-tg(\beta)))
\end{aligned} \tag{13}$$

where

$$F(r,f(r)) = -\left(1+\sin\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)-\cos\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)\cdot tg(\beta)\right)\frac{K(r,f(r))}{r}+$$
$$+\cos\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)\cdot\left(\frac{\partial}{\partial z}K(r,z)\right)\Bigg|_{z=f(z)}-\sin\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)\cdot\left(\frac{\partial}{\partial r}K(r,z)\right)\Bigg|_{z=f(r)}+$$
$$+tg(\beta)\cdot\cos\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)\cdot\left(\frac{\partial}{\partial r}K(r,z)\right)\Bigg|_{z=f(r)}+$$
$$+tg(\beta)\cdot\sin\left(2\cdot\left(\beta+\frac{3\pi}{4}+\psi\right)\right)\cdot\left(\frac{\partial}{\partial z}K(r,z)\right)\Bigg|_{z=f(z)},$$
$$r\in[0,\Delta/(-tg(\beta)))$$

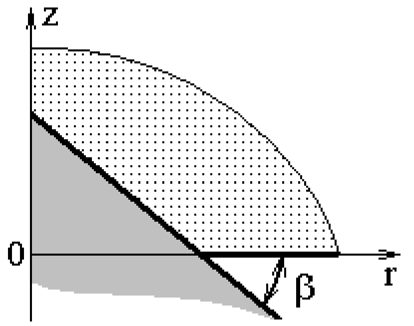

Figure 3. Conical indenter

The hardness and diagram F/Δ (force/(penetration depth)) have been determined in some cases of heterogeneity. The periodical variations of yield stress in direction 0z have been considered (Figure 4, 5, 6)

$$\sigma_y(r,z) = 2\cdot K(r,z) = 2\cdot K(z) = 2\cdot K_z\cdot(1+\mu_z\cdot\sin(\lambda_z\cdot z)). \tag{14}$$

The decreasing of yield stress in direction 0z has been investigated (Figure 7, 8, 9)

$$\sigma_y(r,z) = 2\cdot K(r,z) = 2\cdot K(z) = 2\cdot K_z\cdot\left(1+\nu_z\cdot e^{-\xi_z\cdot z^2}\right). \tag{15}$$

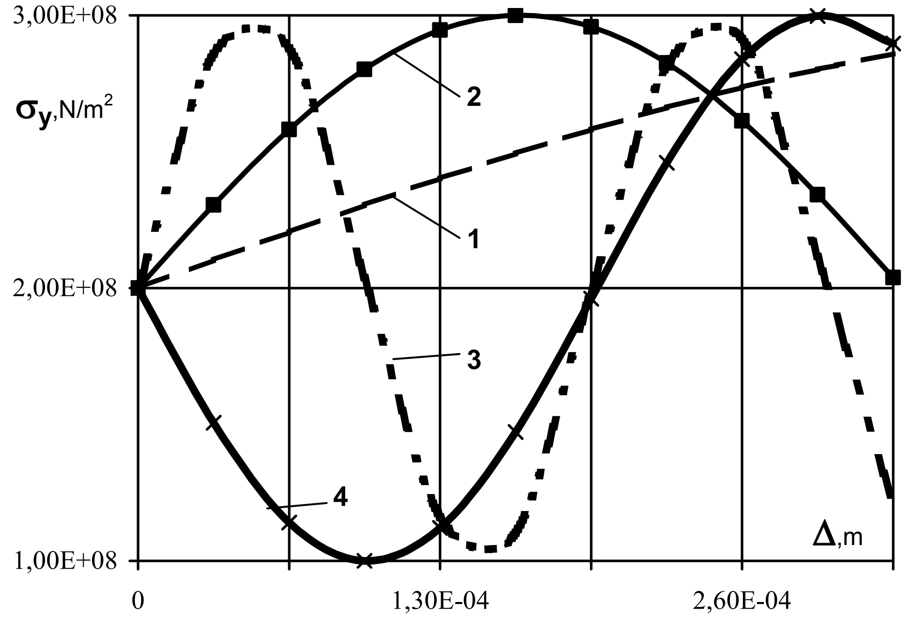

Figure 4. Periodic variation of the yield stress σ_y (14) ($K_z = 10^8$):

1. - $\mu_z = 0.5, \lambda_z = \dfrac{10000}{\pi}$; 2. - $\mu_z = 0.5, \lambda_z = \dfrac{30000}{\pi}$;

3. - $\mu_z = 0.5, \lambda_z = \dfrac{100000}{\pi}$; 4. - $\mu_z = -0.5, \lambda_z = \dfrac{50000}{\pi}$.

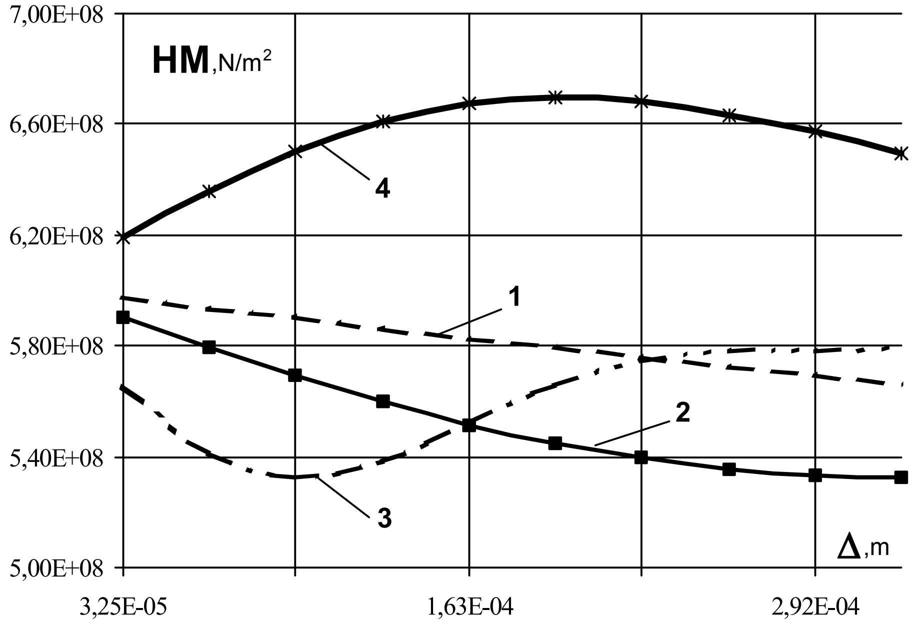

Figure 5. Variation of hardness of surface (12) in the case of heterogeneity (14) ($K_z = 10^8$, $\psi = \pi / 4, \beta = -0.1 \cdot \pi$ (13)): 1. - $\mu_z = 0.5, \lambda_z = \dfrac{10000}{\pi}$;

2. - $\mu_z = 0.5, \lambda_z = \dfrac{30000}{\pi}$; 3 - $\mu_z = 0.5, \lambda_z = \dfrac{100000}{\pi}$; 4. - $\mu_z = -0.5, \lambda_z = \dfrac{50000}{\pi}$

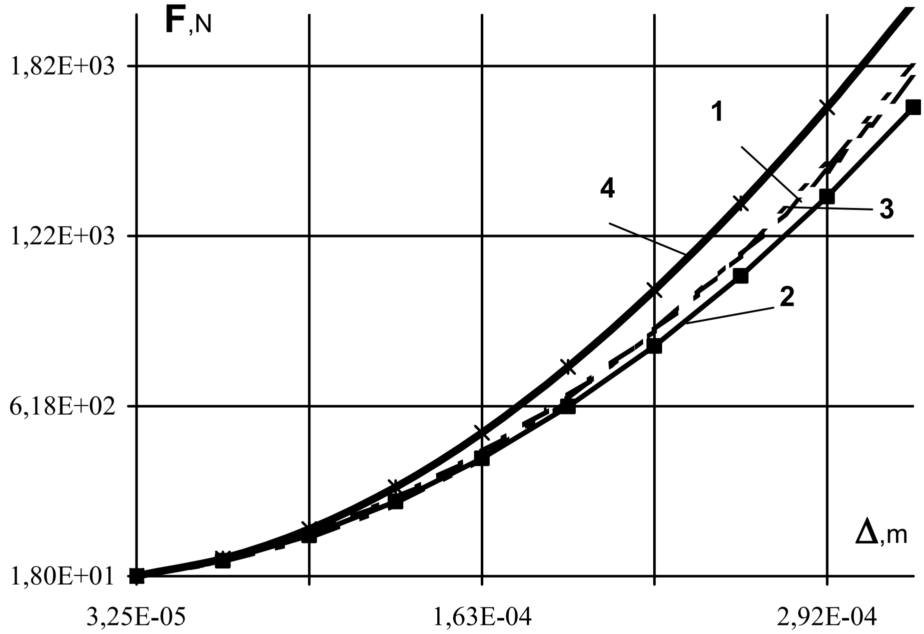

Figure 6. The diagram F / Δ (force/(penetration depth)) in the case of heterogeneity (14) ($K_z = 10^8$, $\psi = \pi / 4, \beta = -0.1 \cdot \pi$ (13)): 1. - $\mu_z = 0.5, \lambda_z = \dfrac{10000}{\pi}$; 2. - $\mu_z = 0.5, \lambda_z = \dfrac{30000}{\pi}$; 3 - $\mu_z = 0.5, \lambda_z = \dfrac{100000}{\pi}$; 4. - $\mu_z = -0.5, \lambda_z = \dfrac{50000}{\pi}$

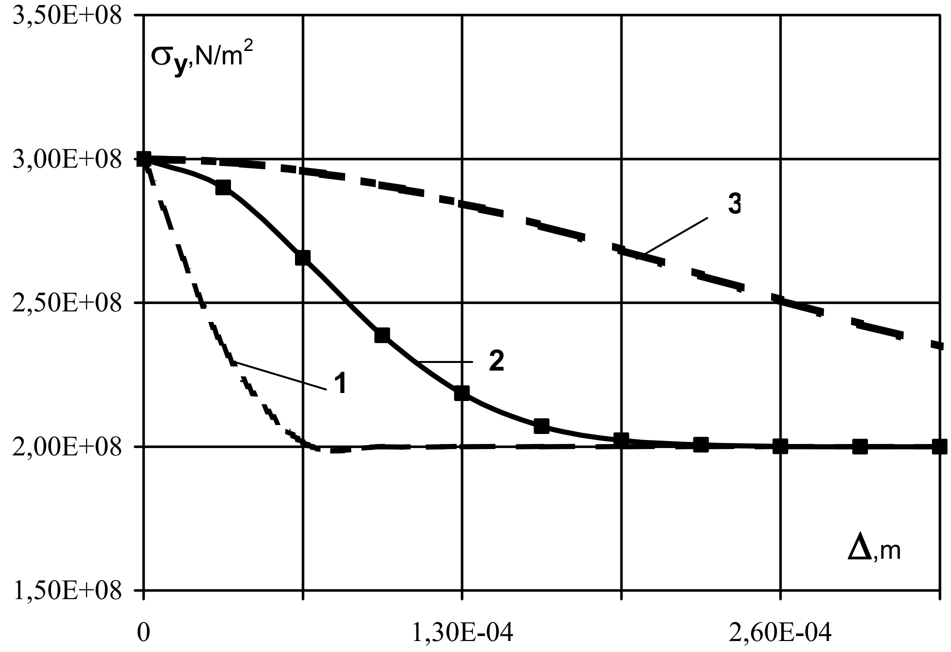

Figure 7. The decreasing of yield stress in direction 0z (15) ($K_z = 10^8$)
1 -$\nu_z = 0.5, \xi_z = 10^9$; 2 -$\nu_z = 0.5, \xi_z = 10^8$; 3 -$\nu_z = 0.5, \xi_z = 10^8$

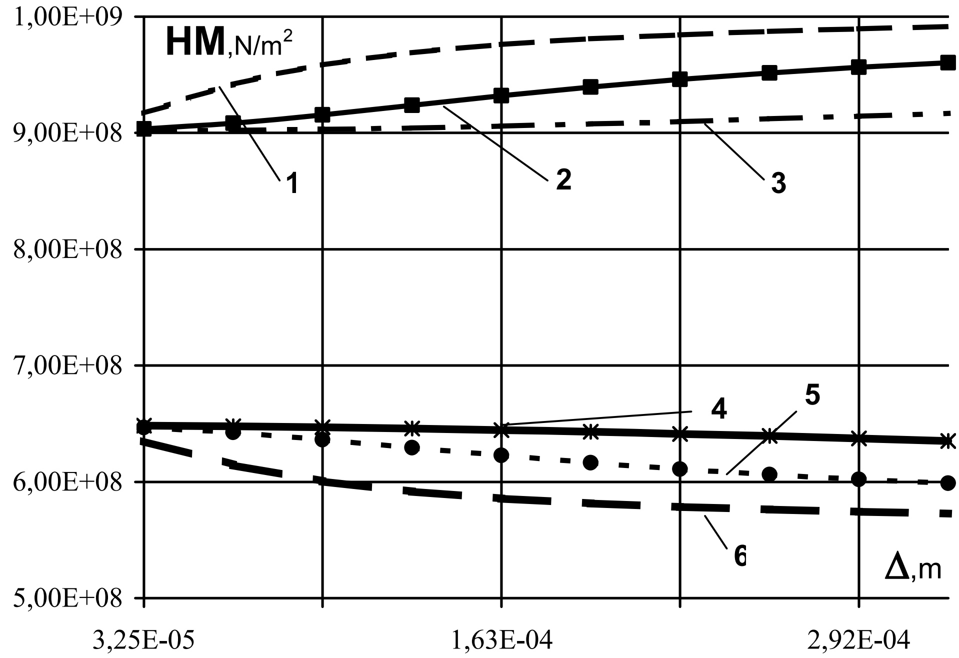

Figure 8. Variation of hardness of surface (12) in the case of heterogeneity (15) ($K_z = 10^8$, $\beta = -\dfrac{\pi}{10}, \nu_z = 0.5$): 1. - $\psi = \pi / 4, \xi_z = 10^9$; - $\psi = \pi / 4, \xi_z = 10^8$; 3. - $\psi = \pi / 4, \xi_z = 10^7$; 4. - $\psi = 0, \xi_z = 10^7$; 5. - $\psi = 0, \xi_z = 10^8$; 6. - $\psi = 0, \xi_z = 10^9$

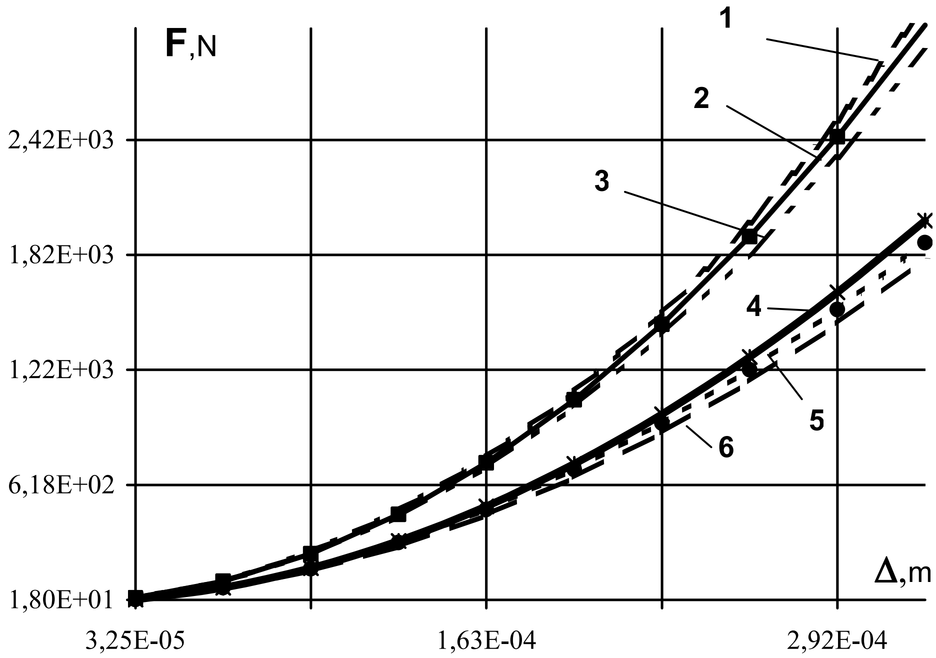

Figure 9. The diagram F / Δ (force/(penetration depth)) in the case of heterogeneity (15) ($K_z = 10^8$, $\beta = -\dfrac{\pi}{10}, \nu_z = 0.5$): 1. - $\psi = \pi / 4, \xi_z = 10^9$; 2. - $\psi = \pi / 4, \xi_z = 10^8$; 3. - $\psi = \pi / 4, \xi_z = 10^7$; 4. - $\psi = 0, \xi_z = 10^7$; 5. - $\psi = 0, \xi_z = 10^8$; 6. - $\psi = 0, \xi_z = 10^9$

4 Conclusions

An arbitrary penetration depth of a rigid indenter into a heterogeneous half space has been studied based on the Haar and v. Karman hypothesis. The analytical distribution of the contact stress has been obtained. It allows for taking into account the local adhesion between the indenter surface and the surface of half-space. The conical indenter has been investigated as a particular case. The dependence between the applied force and the penetration depth of the conical indenter for several cases of heterogeneity was determined.

5 Acknowledgments

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