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The Generalized Theory of Thermo-Magnetoelectroelasticity

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The governing equations for thermo-magnetoelectroelasticity are given for the heat-flux-dependent theory of Lebon. First, we establish a reciprocal relation using a new method of proof, which involves two thermoelastic processes at different instants. We show that this relation can be used to obtain reciprocity, uniqueness and continuous dependence theorems. The reciprocal theorem avoids both the use of the Laplace transform and the incorporation of initial conditions into the equations of motion. The uniqueness theorem is derived avoiding both the use of the definiteness assumption on the thermoelastic coefficients and the restriction that the conductivity tensor is positive definite. There are also no restrictions on the piezoelectric moduli, piezomagnetic moduli, and the thermal coupling coefficients other than symmetry conditions. We prove also that the reciprocal relation leads to a continuous dependence theorem studied on external body loads and heat supply, which ensures that the mathematical model for the generalized problem is well posed.

1 Introduction

Recently, with increasing wide use of piezoelectric and piezomagnetic materials in the aerospace and automotive industries, etc., the study of the mechanics and physics of magnetoelectroelastic solids has attracted much attention. They are also extensively used as electric packaging, sensors and actuators, e.g., magnetic field probes, acoustic/ultrasonic devices, hydrophones, and transducers with the responsibility of electromagnetomechanical energy conversion (Wu and Huang, 2000; Zhou et al., 2005). Consequently, the magnetoelectroelastic materials offer great opportunities to develop new sensing and actuating devices working in desired ways.

Actually, as is well known, the term generalized usually refers to thermodynamic theories based on hyperbolic (wave-type) heat equations, so that a finite speed for propagation of thermal signals is admitted (the so-called second sound effect). In this connection, several kinds of generalization have been performed, typically modifying the entropy production inequality and/or the set of dependent and independent constitutive variables. As regards in particular the heat conduction issue in elastic bodies, we can mention the papers of Lord and Shulman (1967), Müller (1971), Green and Laws (1972) and Green and Lindsay (1972) for an exhaustive survey of methods and results. In Hetnarski and Ignaczak (1999) historical outlines on the subject along with further remarks and references can be found. More recently, Lebon (1980, 1982) has proposed a new approach towards a generalized formulation. It is based upon the idea to consider the heat flux as a constitutive independent variable, and to add a rate-type evolution equation for the heat flux to the system of constitutive equations. Application of the general principles of continuum thermomechanics, among which the entropy inequality in the form proposed by Müller (1971), then yields the thermodynamic restrictions on the constitutive relations. In the linear approximation, the main equations become analogous to those previously considered by Lord and Shulman (1972); in particular, the counterpart of the classical Fourier law for heat conduction, arising from the above evolution equation, is given by an equation of Cattaneo-Maxwell-Vernotte type.

Apart from the admission of second sound effect, we emphasize that generalized theories resulting in hyperbolic heat equations should be physically more suitable than the conventional ones in connection with problems involving very short intervals of time and/or very high heat fluxes.

The generalized approach by Lebon has been carried onto micropolar and piezoelectricity in an elastic context by Chandrasekharaiah (1986, 1987, 1988), onto a porous thermoelastic context by Ciarletta and Scarpetta (1996), and onto a porous micropolar context by Passarella (1996). The resulting theories appear to be similar to that empirically developed by Kaliski (1965 a, b), where the physical relevance for incorporation of second sound effects into piezoelectricity is also stressed. Nowacki (1964, 1965) derived an extensive investigation on piezoelectricity for thermoelastic bodies.

To the author's knowledge, no work has been done regarding the study of the qualitative properties of the solution of the theory of thermo-magnetoelectroelasticity in the frame of a generalized theory, though similar research in classical thermo-magnetoelectroelasticity (Li, 2003), in microstrech piezoelectricity (Iesan, 2006 a) and in classical microstrech thermopiezoelectricity (Iesan, 2006 b), has been popular in recent years. So, this paper aims to fill up this vacancy. We are interested in reciprocity, uniqueness and continuous theorems. This should permit to model the behaviour of several kinds of thermoelastic and thermopiezoelectric materials.

In the last years, various methods for the study of the qualitative properties of the solution in elasticity or thermoelasticity have been elaborated. A uniqueness theorem in thermoelastodynamics of homogeneous and isotropic bodies has been derived by Weiner (1957). This result has been extended by Cazimir (1964) on the anisotropic bodies. However, these uniqueness theorems rest on the positive definiteness assumption on the elasticity tensor. Brun (1969) was the first to establish uniqueness without the definiteness assumption on the elastic coefficients. Knops and Payne (1970) and Rionero and Chirita (1987) reached a similar conclusion, but their results are based on the assumption that the conductivity tensor is positive definite. The latter assumption is not a consequence of the second law of thermodynamics because that law implies that the conductivity tensor is only positive semi-definite. Iesan (1989) was the first to establish a uniqueness theorem in classical thermoelasticity avoiding both the use of the definiteness assumption on the elasticity tensor and the restriction that the conductivity tensor is positive definite.

By using the method presented by Iesan (1989, 2006a, b) for different classical thermoelastic problems, we establish a reciprocal relation which involves two thermoelastic processes at different instants for a generalized problem under Lebon's model. This relation forms the basis of reciprocity, uniqueness and continuous dependence theorems.

The new contributions of this paper compared to those cited previously are:

(i) Li (2003) and Iesan (1989, 2006a, b) established reciprocity and uniqueness theorems for classical thermoelastic problems. However, in this paper both theorems are derived in the frame of a generalized theory, which is considered physically more realistic that the classical one.

(ii) In this paper, the reciprocity and uniqueness theorems are established on the basis of results obtained from the reciprocal relation. The uniqueness theorem is derived avoiding both the use of the definiteness assumption on the thermoelastic coefficients and the restriction that the conductivity tensor is positive definite. The reciprocal theorem avoids both the use of the Laplace transform and the incorporation of the initial conditions into the equations of motion. However the same corresponding theorems are derived by Li (2003) as two independent problems by incorporation of the Laplace transform and initial conditions into the field equations and with restrictions on the conductivity tensor.

(iii) We have derived a continuous dependence theorem within a generalized thermoelastic theory on the basis of results obtained from reciprocity and uniqueness. However, the same corresponding theorem has been derived by Iesan (2006b) under a classical thermoelastic theory as an independent problem without implementing reciprocity and uniqueness results in the study.

(iv) The method employed in this paper to derive the continuous dependence theorem is more general than that used by Iesan (2006b). It is applicable to isotropic media as well as to anisotropic media regardless wether the thermal effects are considered in the studied problem or not.

(v) The results of this paper, and in particular the continuous dependence theorem, prove that in the motion following any sufficiently small change in the external system, the solution of the initial-boundary value problem is everywhere arbitrary small in magnitude. Consequently, the mathematical model proposed for the generalized thermo-magnetoelectroelastic problem under Lebon's model is well posed.

2 Basic Equations and Preliminaries

We consider a body that at some instant occupies the region V of the Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂V. The motion of the body is referred to the reference configuration V and a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In what follows we use a superposed dot to denote the partial differentiation with respect to the time t.

We consider the generalized theory of thermo-magnetoelectroelasticity under Lebon's model. The fundamental system of field equations consists of the equations of motion (Li, 2003; Coleman and Dill; 1971, Amendola, 2000)

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \tag{1}$$

the equations of the electric and magnetic fields

$$D_{i,i} = \varrho \quad B_{i,i} = \sigma \tag{2}$$

the energy equation

$$\rho T_0 \dot{\eta} = q_{i,i} + \rho h \tag{3}$$

the constitutive equations

$$\sigma_{ij} = c_{ijkl} e_{kl} + F_{ijk} \zeta_k + \lambda_{ijk} E_k - a_{ij} T \tag{4}$$

$$D_k = -\lambda_{kij} e_{ij} + \alpha_{ki} \zeta_i + \gamma_{ki} E_i + p_k T \tag{5}$$

$$B_k = -F_{kij} e_{ij} + A_{ki} \zeta_i + \alpha_{ki} E_i + m_k T \tag{6}$$

$$\rho \eta = a_{ij} e_{ij} + m_k \zeta_k + p_k E_k + cT \tag{7}$$

$$k_{ij} T_{,j} = q_i + \tau_0 \dot{q}_i \tag{8}$$

and the geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\psi_{,i}, \quad \zeta_i = -\varphi_{,i} \tag{9}$$

Here we have used the following notations: F_i, ϱ and σ are the body force, electric charge density, and electric current density, respectively; ρ is the mass density; h is the heat supply; u_i, ψ and φ are the displacement vector, the electric potential, and the magnetic potential, respectively. σ_{ij}, D_k, B_k and η are stress tensor, the dielectric displacement vector, the magnetic intensity, and the entropy density, respectively. e_{ij}, E_i, ζ_i and T are strain tensor, electric field, magnetic field, and temperature change to a reference temperature T_0, respectively. k_{ij} is the conductivity tensor. The coefficients c_{ijkl}, γ_{kj}, A_{kj} and c are constitutive moduli which directly connect similar fields (for example, stress to strain). On the other hand λ_{ijk}, F_{ijk}, α_{kj}, a_{ij}, p_i and m_i are coupling coefficients connecting dissimilar fields (for example, stress to electric field).

We notice that other sets of constitutive equations can be obtained by choosing different sets of independent variables. The constitutive parameters satisfy the following symmetry conditions :

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad \lambda_{ijk} = \lambda_{kij} = \lambda_{kji}, \quad F_{ijk} = F_{kij} = F_{kji}$$

$$a_{ij} = a_{ji}, \quad \gamma_{ij} = \gamma_{ji}, \quad \alpha_{ij} = \alpha_{ji}, \quad k_{ij} = k_{ji}, \quad \kappa_{ij} = \kappa_{ji} \tag{10}$$

Here κ_{ij} are constitutive coefficients defined by (Chandrasekharaiah, 1886; 1987; 1988)

$$T_0 \kappa_{ij} k_{jh} = \tau_0 \delta_{ih} \tag{11}$$

The coefficients κ_{ij} satisfy the inequality

$$\frac{1}{\tau_0} \kappa_{ij} q_i q_j \geq 0 \tag{12}$$

The main difference with the coupled model of the theory of thermo-magnetoelectroelasticity lies in the treatment of equation (8). Instead of the classical (coupled) equation $q_i = k_{ij} T_{,j}$, we now must deal with the generalizing ones (8), (11) and (12).

As pointed out by Lebon (1982) and Chandrasekharaiah (1988), equation (8) appears as an anisotropic version of the Maxwell- Cattaneo-Vernotte heat conduction law, postulated by Kaliski ($1965a, b$). The material coefficients τ_0 and k_{ij} should be interpreted as a thermal relaxation time and thermal conductivity tensor, respectively. Of course, letting $\tau \to 0$ gives the equations of the coupled model of the theory of thermo-magnetoelectroelasticity.

The components of surface traction, the normal component of the electrical displacement, the normal component of the magnetic intensity, and the heat flux at regular points of ∂V are given by

$$f_i = \sigma_{ji} n_j, \quad d = D_i n_i, \quad b = B_i n_i, \quad q = q_i n_i \tag{13}$$

respectively. We denote by n_j the outward unit normal of ∂V.

To the system of field equations we must add boundary conditions and initial conditions. Let S_m $(m = 1, 2, ..., 8)$ be subsets of ∂V so that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \bar{S}_7 \cup S_8 = \partial V, \;\; S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \oslash$.

We consider the following boundary conditions

$$u_i = \tilde{u_i} \text{ on } \bar{S}_1 \times (0, \infty), \;\; \psi = \tilde{\psi} \text{ on } \bar{S}_3 \times (0, \infty), \;\; \varphi = \tilde{\varphi} \text{ on } \bar{S}_5 \times (0, \infty), \;\; T = \tilde{T} \text{ on } \bar{S}_7 \times (0, \infty)$$

$$\sigma_{ji} n_j = \tilde{f}_i \text{ on } S_2 \times (0, \infty), \;\; D_j n_j = \tilde{d} \text{ on } S_4 \times (0, \infty), \quad B_j n_j = \tilde{b} \text{ on } S_6 \times (0, \infty)$$

$$q_j n_j = \tilde{q} \text{ on } S_8 \times (0, \infty) \tag{14}$$

where $\tilde{u_i}, \; \tilde{\psi}, \; \tilde{\varphi}, \; \tilde{T}, \; \tilde{f_i}, \; \tilde{d}, \; \tilde{b}$ and $\tilde{q}$ are prescribed functions. The initial conditions are

$$u_i(x, 0) = u_i^0(x), \;\; \dot{u}_i(x, 0) = v_i^0(x), \quad \eta(x, 0) = \eta^0(x), \;\; x \in \bar{V} \tag{15}$$

where $u_i^0, \; v_i^0,$ and η^0 are given. We assume that

(i) F_i, h, ϱ and σ are continuous on $\bar{V} \times (0, \infty)$

(ii) ρ, $u_i^0, \; v_i^0$ and η^0 are continuous on $\bar{V}$

(iii) the constitutive coefficients satisfy the symmetry relations (10)

(iv) the constitutive coefficients are continuous differentiable on $\bar{V}$

(v) $\tilde{u_i}, \; \tilde{\psi}, \; \tilde{\varphi}$ and $\tilde{T}$ are continuous on $\bar{S}_1 \times (0, \infty)$, $\bar{S}_3 \times (0, \infty)$, $\bar{S}_5 \times (0, \infty)$ and $\bar{S}_7 \times (0, \infty)$, respectively

(vi) $\tilde{f_i}, \; \tilde{d}, \; \tilde{b}$ and $\tilde{q}$ are continuous in time and piecewise regular on $S_2 \times (0, \infty)$, $S_4 \times (0, \infty)$, $S_6 \times (0, \infty)$ and $S_8 \times (0, \infty)$, respectively

Let M and N be non-negative integers. We say that f is of class $C^{M,N}$ on $V \times (0, \infty)$ if f is continuous on $V \times (0, \infty)$ and the functions

$$\frac{\partial^m}{\partial x_i \partial x_j \cdots \partial x_p} \left(\frac{\partial^n f}{\partial t^n}\right), \;\; m \in \{1, 2, \cdots, M\}, \;, \;\; n \in \{1, 2, \cdots, N\}, \; m + n \leq \max\{M, N\}$$

exist and are continuous on $V \times (0, \infty)$. We write C^M for $C^{M,M}$.

By an admissible process $p = \{u_i, \psi, \varphi, T, e_{ij}, \sigma_{ij}, \zeta_i, E_i, B_i, D_i, \eta, q_i\}$ we mean an ordered array of functions $u_i, \psi, \varphi, T, e_{ij}, \sigma_{ij}, \zeta_i, E_i, B_i, D_i, \eta,$ and q_i defined on $\bar{V} \times [0, \infty)$ with the following properties: (i) $u_i \in C^2$, $\psi, \varphi \in C^{2,0}$, $T \in C^{2,1}$, $\eta \in C^{0,1}$, $q_i \in C^{0,1}$, $e_{ij}, \; \zeta_i \in C^{1,0}$, $\sigma_{ij} \in C^{1,0}$, $E_i, \zeta_i \in C^{1,0}$ on $V \times (0, \infty)$; (ii) $u_i, \; \dot{u}_i, \; \ddot{u}_i, \; u_{i,j}, \; \varphi, \; \varphi_{,i}, \; \psi, \; \psi_{,i}, \; e_{ij}, \; \sigma_{ij}, \; \sigma_{ij,i}, \; D_i, \; D_{i,i}, \; B_i, \; B_{i,i} \; T, \; \dot{T}, \; T_{,i}, \; \eta, \; \dot{\eta}, \; q_i$ and $q_{i,i}$, are continuous on $\bar{V} \times [0, \infty)$.

By a solution of the mixed problem we mean an admissible process which satisfies equations (1)-(9) on $V \times (0, \infty)$, the boundary conditions (14), and the initial conditions (15).

3 Reciprocal Theorem

In this section we establish a reciprocity relation which involves two processes at different instants. This relation forms the basis of reciprocity, uniqueness, and continuous dependence theorems. The proof of the reciprocal theorem avoids both the use of the Laplace transform and the incorporation of the initial conditions in the equations of motion.

Let u and v be functions on $V \times (0, \infty)$ that are continuous in time. In the following we use $*$ to indicate the time convolution:

$$[u * v](x,t) = \int_0^t u(x, t-\tau) v(x,\tau) d\tau$$

Let ℓ be the function on $[0, \infty)$ defined by

$$\ell(t) = 1, \quad t \in [0, \infty) \tag{16}$$

We introduce the notation $\bar{f}$ for $\ell * f$, that is

$$\bar{f}(x,t) = \int_0^t f(x,\tau) d\tau = [\ell * f](x,t), \quad x \in V, \;\; t \in (0,\infty) \tag{17}$$

Following Iesan (1989, 2006a, b), equation (3) is equivalent to

$$\rho T_0 \eta = \bar{q}_{i,i} + W, \quad \text{on } V \times [0,\infty) \tag{18}$$

where

$$W = \rho \bar{h} + \rho T_0 \eta^0 \tag{19}$$

We consider two external data systems

$$\pounds^{(\alpha)} = \{F_i^{(\alpha)},\; \varrho^{(\alpha)},\; \sigma^{(\alpha)},\, h^{(\alpha)},\, \tilde{u_i}^{(\alpha)},\, \tilde{\phi}^{(\alpha)},\, \tilde{\psi}^{(\alpha)},\, \tilde{T}^{(\alpha)},\, \tilde{f_i}^{(\alpha)},\, \tilde{d}^{(\alpha)},\, \tilde{b}^{(\alpha)},\, \tilde{q}^{(\alpha)},\, {u_i}^{0(\alpha)},\, {v_i}^{0(\alpha)},\, \eta^{0(\alpha)}\}$$

($\alpha = 1, 2$). Let

$$p^{(\alpha)} = \{u_i^{(\alpha)},\, \psi^{(\alpha)},\, \varphi^{(\alpha)},\, T^{(\alpha)},\, e_{ij}^{(\alpha)},\, \sigma_{ij}^{(\alpha)},\, \zeta_i^{(\alpha)},\, E_i^{(\alpha)},\, B_i^{(\alpha)},\, D_i^{(\alpha)},\, \eta^{(\alpha)},\, {q_i}^{(\alpha)}\}$$

be a solution corresponding to $\pounds^{(\alpha)}$ $(\alpha = 1, 2)$. We denote

$$f_i^{(\alpha)} = \sigma_{ji}^{(\alpha)} n_j, \quad d^{(\alpha)} = D_i^{(\alpha)} n_i, \quad b^{(\alpha)} = B_i^{(\alpha)} n_i$$

$$q^{(\alpha)} = q_i^{(\alpha)} n_i, \quad W^{(\alpha)} = \rho \ell * h^{(\alpha)} + \rho T_0 \eta^{0(\alpha)}. \tag{20}$$

Theorem 1 *Suppose that the symmetry relations (10) hold. Let*

$$\begin{aligned}
\Omega_{\alpha\beta}(a,b) &= \int_V \Big[F_i^{(\alpha)}(a) u_i^{(\beta)}(b) - \varrho^{(\alpha)}(a)\psi^{(\beta)}(b) - \sigma^{(\alpha)}(a)\varphi^{(\beta)}(b) - \frac{1}{T_0} W^{(\alpha)}(a) T^{(\beta)}(b)\Big] dV \\
&- \int_V \Big[\rho \ddot{u}_i^{(\alpha)}(a) u_i^{(\beta)}(b) - \frac{1}{T_0}\bar{q}_i^{(\alpha)}(a) T_{,i}^{(\beta)}(b)\Big] dV \\
&+ \int_{\partial V} \Big[f_i^{(\alpha)}(a) u_i^{(\beta)}(b) + d^{(\alpha)}(a)\psi^{(\beta)}(b) + b^{(\alpha)}(a)\varphi^{(\beta)}(b) - \frac{1}{T_0}\bar{q}^{(\alpha)}(a) T^{(\beta)}(b)\Big] dS
\end{aligned} \tag{21}$$

for all $a, b \in (0, \infty)$. Then

$$\Omega_{\alpha\beta}(a,b) = \Omega_{\beta\alpha}(b,a), \quad \alpha, \beta = 1, 2, \quad \textit{for all } a, b \in (0,\infty) \tag{22}$$

Proof. Let

$$\Gamma_{\alpha\beta}(a,b)=\sigma_{ij}^{(\alpha)}(a)e_{ij}^{(\beta)}(b)-D_k^{(\alpha)}(a)E_k^{(\beta)}(b)-B_k^{(\alpha)}(a)\zeta_k^{(\beta)}(b)-\rho\eta^{(\alpha)}(a)T^{(\beta)}(b) \tag{23}$$

where, for convenience, we have suppressed the argument x. In view of equations (5)-(8), from (23) we get

$$\begin{aligned}\Gamma_{\alpha\beta}(a,b)&=c_{ijkl}e_{kl}^{(\alpha)}(a)e_{ij}^{(\beta)}(b)-\gamma_{ki}E_i^{(\alpha)}(a)E_k^{(\beta)}(b)-A_{ki}\zeta_i^{(\alpha)}(a)\zeta_k^{(\beta)}(b)-cT^{(\alpha)}(a)T^{(\beta)}(b)\\&+F_{kij}\Big(\zeta_k^{(\alpha)}(a)e_{ij}^{(\beta)}(b)+e_{ij}^{(\alpha)}(a)\zeta_k^{(\beta)}(b)\Big)+\lambda_{kij}\Big(E_k^{(\alpha)}(a)e_{ij}^{(\beta)}(b)+e_{ij}^{(\alpha)}(a)E_k^{(\beta)}(b)\Big)\\&-\alpha_{ki}\Big(\zeta_i^{(\alpha)}(a)E_k^{(\beta)}(b)+E_i^{(\alpha)}(a)\zeta_k^{(\beta)}(b)\Big)-a_{ij}\Big(T^{(\alpha)}(a)e_{ij}^{(\beta)}(b)+e_{ij}^{(\alpha)}(a)T^{(\beta)}(b)\Big)\\&-p_k\Big(E_k^{(\alpha)}(a)T^{(\beta)}(b)+T^{(\alpha)}(a)E_k^{(\beta)}(b)\Big)-m_k\Big(\zeta_k^{(\alpha)}(a)T^{(\beta)}(b)+T^{(\alpha)}(a)\zeta_k^{(\beta)}(b)\Big)\end{aligned} \tag{24}$$

If we use the symmetry relation (10), then we find that

$$\Gamma_{\alpha\beta}(a,b)=\Gamma_{\beta\alpha}(b,a),\quad \alpha,\beta=1,2,\quad \text{for all } a,b\in(0,\infty) \tag{25}$$

On the other hand, in view of equations (1)-(3), (9) and (23), we obtain

$$\begin{aligned}\Gamma_{\alpha\beta}(a,b)&=\sigma_{ij}^{(\alpha)}(a)u_{j,i}^{(\beta)}(b)+D_j^{(\alpha)}(a)\psi_{,j}^{(\beta)}(b)+B_j^{(\alpha)}(a)\varphi_{,j}^{(\beta)}(b)-\frac{1}{T_0}\Big(\bar{q}_{i,i}^{(\alpha)}(a)+W^{(\alpha)}(a)\Big)T^{(\beta)}(b)\\&=[\sigma_{ji}^{(\alpha)}(a)u_i^{(\beta)}(b)+D_j^{(\alpha)}(a)\psi^{(\beta)}(b)+B_j^{(\alpha)}(a)\varphi^{(\beta)}(b)-\frac{1}{T_0}\bar{q}_j^{(\alpha)}(a)T^{(\beta)}(b)]_{,j}\\&+F_i^{(\alpha)}(a)u_i^{(\beta)}(b)-\varrho^{(\alpha)}(a)\psi^{(\beta)}(b)-\sigma^{(\alpha)}(a)\varphi^{(\beta)}(b)-\frac{1}{T_0}W^{(\alpha)}(a)T^{(\beta)}(b)\\&-\rho\ddot{u}_i^{(\alpha)}(a)u_i^{(\beta)}(b)+\frac{1}{T_0}\bar{q}_i^{(\alpha)}(a)T_{,i}^{(\beta)}(b)\end{aligned} \tag{26}$$

By using the divergence theorem and equations (21), (24) and (26), we obtain

$$\int_V \Gamma_{\alpha\beta}(a,b)dV=\Omega_{\alpha\beta}(a,b) \tag{27}$$

In view of equation (25), we obtain the desired result.

Theorem 1 forms the basis of the following reciprocity theorem.

Theorem 2 *Assume that the symmetry relations (10) hold. Let $p^{(\alpha)}$ be a solution corresponding to the external data system $\mathcal{L}^{(\alpha)}$, $(\alpha=1,2)$. Then we have*

$$\begin{aligned}&\int_V\Big[\Psi_i^{(1)}*u_i^{(2)}-g*\varrho^{(2)}*\psi^{(1)}-g*\sigma^{(2)}*\varphi^{(1)}-\frac{1}{T_0}g*W^{(1)}*T^{(2)}+g*\ell*\lambda_{ij}q_i^{(1)}*\dot{q}_j^{(2)}\Big]dV\\&+\int_{\partial V}g*\Big[f_i^{(1)}*u_i^{(2)}+d^{(1)}*\psi^{(2)}+b^{(1)}*\varphi^{(2)}-\frac{1}{T_0}\ell*q^{(1)}*T^{(2)}\Big]dS\\&=\int_V\Big[\Psi_i^{(2)}*u_i^{(1)}-g*\varrho^{(1)}*\psi^{(2)}-g*\sigma^{(1)}*\varphi^{(2)}-\frac{1}{T_0}g*W^{(2)}*T^{(1)}+g*\ell*\lambda_{ij}q_i^{(2)}*\dot{q}_j^{(1)}\Big]dV\\&+\int_{\partial V}g*\Big[f_i^{(2)}*u_i^{(1)}+d^{(2)}*\psi^{(1)}+b^{(2)}*\varphi^{(1)}-\frac{1}{T_0}\ell*q^{(2)}*T^{(1)}\Big]dS\end{aligned} \tag{28}$$

where

$$\Psi_i^{(\alpha)}=g*F_i^{(\alpha)}+\rho(tv_i^{0(\alpha)}+u_i^{0(\alpha)}),\qquad g(t)=t,\quad t\in[0,\infty)$$

Proof. We take in equation (22), $a = \tau$ and $b = t - \tau$ and integrate from 0 to t, we arrive with the aid of equation (21) at

$$\int_V \left[F_i^{(1)} * u_i^{(2)} - \varrho^{(1)} * \psi^{(2)} - \sigma^{(1)} * \varphi^{(2)} - \frac{1}{T_0} W^{(1)} * T^{(2)}\right] dV$$

$$- \int_V \rho \ddot{u}_i^{(1)} * u_i^{(2)} dV + \frac{1}{T_0} \int_V \ell * q_i^{(1)} * T_{,i}^{(2)} dV$$

$$+ \int_{\partial V} \left[f_i^{(1)} * u_i^{(2)} + d^{(1)} * \psi^{(2)} + b^{(1)} * \varphi^{(2)} - \frac{1}{T_0} \ell * q^{(1)} * T^{(2)}\right] dS$$

$$= \int_V \left[F_i^{(2)} * u_i^{(1)} - \varrho^{(2)} * \psi^{(1)} - \sigma^{(2)} * \varphi^{(1)} - \frac{1}{T_0} W^{(2)} * T^{(1)}\right] dV$$

$$- \int_V \rho \ddot{u}_i^{(2)} * u_i^{(1)} dV + \frac{1}{T_0} \int_V \ell * q_i^{(2)} * T_{,i}^{(1)} dV$$

$$+ \int_{\partial V} \left[f_i^{(2)} * u_i^{(1)} + d^{(2)} * \psi^{(1)} + b^{(2)} * \varphi^{(1)} - \frac{1}{T_0} \ell * q^{(2)} * T^{(1)}\right] dS \tag{29}$$

If we take the convolution of equation (8) with $\kappa_{hi} q_h$, and using equation (11), we get

$$\frac{1}{T_0} q_i * T_{,i} = \kappa_{ij} q_i * \dot{q}_j + \frac{1}{\tau_0} \kappa_{ij} q_i * q_j. \tag{30}$$

Taking the convolution of the relation (30) with ℓ and substituting the resulting equation into (29), we obtain

$$\int_V \left[F_i^{(1)} * u_i^{(2)} - \varrho^{(1)} * \psi^{(2)} - \sigma^{(1)} * \varphi^{(2)} - \frac{1}{T_0} W^{(1)} * T^{(2)}\right] dV$$

$$- \int_V \rho \ddot{u}_i^{(1)} * u_i^{(2)} dV + \int_V \ell * \kappa_{ij} q_i^{(1)} * \dot{q}_j^{(2)} dV + \frac{1}{\tau_0} \int_V \ell * \kappa_{ij} q_i^{(1)} * q_j^{(2)} dV$$

$$+ \int_{\partial V} \left[f_i^{(1)} * u_i^{(2)} + d^{(1)} * \psi^{(2)} + b^{(1)} * \varphi^{(2)} - \frac{1}{T_0} \ell * q^{(1)} * T^{(2)}\right] dS$$

$$= \int_V \left[F_i^{(2)} * u_i^{(1)} - \varrho^{(2)} * \psi^{(1)} - \sigma^{(2)} * \varphi^{(1)} - \frac{1}{T_0} W^{(2)} * T^{(1)}\right] dV$$

$$- \int_V \rho \ddot{u}_i^{(2)} * u_i^{(1)} dV + \int_V \ell * \kappa_{ij} q_i^{(2)} * \dot{q}_j^{(1)} dV + \frac{1}{\tau_0} \int_V \ell * \kappa_{ij} q_i^{(2)} * q_j^{(1)} dV$$

$$+ \int_{\partial V} \left[f_i^{(2)} * u_i^{(1)} + d^{(2)} * \psi^{(1)} + b^{(2)} * \varphi^{(1)} - \frac{1}{T_0} \ell * q^{(2)} * T^{(1)}\right] dS \tag{31}$$

where

$$g * \ddot{u}_i^{(\alpha)} = u_i^{(\alpha)} - t v_i^{(0\alpha)} - u_i^{0(\alpha)}. \tag{32}$$

Taking the convolution of the relation (31) with g, we conclude with the aid of relation (32), that (28) holds.

Corollary 1 *Assume that the symmetry relations (10) hold. With homogeneous boundary conditions, the two sets of causes and responses satisfy*

$$\int_V \left[\Psi_i^{(1)} * u_i^{(2)} - g * \varrho^{(2)} * \psi^{(1)} - g * \sigma^{(2)} * \varphi^{(1)} - \frac{1}{T_0} g * W^{(1)} * T^{(2)} + g * \ell * \lambda_{ij} q_i^{(1)} * \dot{q}_j^{(2)}\right] dV$$

$$= \int_V \left[\Psi_i^{(2)} * u_i^{(1)} - g * \varrho^{(1)} * \psi^{(2)} - g * \sigma^{(1)} * \varphi^{(2)} - \frac{1}{T_0} g * W^{(2)} * T^{(1)} + g * \ell * \lambda_{ij} q_i^{(2)} * \dot{q}_j^{(1)}\right] dV$$

This result follows easily from Theorem 2 since the surface integrals vanish for homogeneous boundary conditions.

4 Uniqueness Theorem

In this section we prove that the reciprocity relation established in the last section leads to a uniqueness result theorem. The uniqueness theorem is established avoiding both the use of the definiteness assumption on the elasticity tensor and the restriction that the conductivity tensor is positive definite.

The following theorem is a consequence of Theorem 1.

Theorem 3 *Suppose that the symmetry relations (10) hold. Let*

$$p = \{u_i,\ \psi,\ \varphi,\ T\ e_{ij},\ \sigma_{ij},\ \zeta_i,\ E_i,\ D_i,\ B_i,\ \eta,\ q_i\}$$

be a solution corresponding to the external data system

$$\{F_i,\ \varrho,\ \sigma,\ h,\ \tilde{u_i},\ \tilde{\psi},\ \tilde{\varphi},\ \tilde{T},\ \tilde{f_i},\ \tilde{d},\ \tilde{b},\ \tilde{q},\ {u_i}^0,\ {v_i}^0,\ \eta^0\}$$

and let

$$\Lambda(a,b) = \int_{\partial V}\Big[f_i(a)u_i(b) + d(a)\psi(b) + b(a)\varphi(b) - \frac{1}{T_0}\bar{q}(a)T(b)\Big]dS$$

$$+\int_V\Big[F_i(a)u_i(b) - \varrho(a)\psi(b) - \sigma(a)\varphi(b) - \frac{1}{T_0}W(a)T(b)\Big]dV \tag{33}$$

for all $a, b \in (0,\infty)$. *Then*

$$\frac{d}{dt}\Big(\int_V[\rho u_i u_i + \kappa_{ij}\overline{q}_j\overline{q}_i]dV + \frac{1}{\tau_0}\int_0^t\int_{\partial V}\kappa_{ij}\overline{q}_j\overline{q}_i dtdV\Big) = \int_0^t[\Lambda(t-\tau,t+\tau) - \Lambda(t+\tau,t-\tau)]dV$$

$$+\int_V\Big(\rho(\dot{u}_i(2t)u_i(0) + \dot{u}_i(0)u_i(2t)) + \frac{1}{\tau_0}\kappa_{ij}\overline{q}_j(0)\overline{q}_i(2t) + \kappa_{ij}[q_j(2t)\overline{q}_i(0) + q_j(0)\overline{q}_i(2t)]\Big)dV \tag{34}$$

Proof. In view of Eq. (22)

$$\int_0^t\Omega_{11}(t+\tau,t-\tau)d\tau = \int_0^t\Omega_{11}(t-\tau,t+\tau)d\tau \tag{35}$$

Let us apply this relation to the process $p^{(1)} = p$. From equations (21), (30) and (33) we obtain

$$\int_0^t\Omega_{11}(t+\tau,t-\tau)d\tau = \int_0^t\Lambda(t+\tau,t-\tau)d\tau - \int_0^t\int_V\rho\ddot{u}_i(t+\tau)u_i(t-\tau)dVd\tau$$

$$+\frac{1}{\tau_0}\int_0^t\int_V\kappa_{ij}\overline{q}_i(t+\tau)\dot{\overline{q}}_j(t-\tau)dVd\tau + \int_0^t\int_V\kappa_{ij}\overline{q}_i(t+\tau)\ddot{\overline{q}}_j(t-\tau)dVd\tau \tag{36}$$

Similarly

$$\int_0^t\Omega_{11}(t-\tau,t+\tau)d\tau = \int_0^t\Lambda(t-\tau,t+\tau)d\tau - \int_0^t\int_V\rho\ddot{u}_i(t-\tau)u_i(t+\tau)dVd\tau$$

$$+\frac{1}{\tau_0}\int_0^t\int_V\kappa_{ij}\overline{q}_i(t-\tau)\dot{\overline{q}}_j(t+\tau)dVd\tau + \int_0^t\int_V\kappa_{ij}\overline{q}_i(t-\tau)\ddot{\overline{q}}_j(t+\tau)dVd\tau \tag{37}$$

If we use the relations

$$\int_0^t\ddot{f}(t+\tau)g(t-\tau)d\tau = \dot{f}(2t)g(0) - \dot{f}(t)g(t) + \int_0^t\dot{f}(t+\tau)\dot{g}(t-\tau)d\tau$$

$$\int_0^t \ddot{g}(t-\tau)f(t+\tau)d\tau = \dot{g}(t)f(t) - \dot{g}(0)f(2t) + \int_0^t \dot{g}(t-\tau)\dot{f}(t+\tau)d\tau$$

$$\int_0^t f(t+\tau)\dot{g}(t-\tau)d\tau = f(t)g(t) - g(0)f(2t) + \int_0^t \dot{f}(t+\tau)g(t-\tau)d\tau$$

then the relations (36) and (37) can be written

$$\int_0^t \Omega_{11}(t+\tau, t-\tau)d\tau = \int_0^t \Lambda(t+\tau, t-\tau)d\tau - \rho\int_V [\dot{u}_i(2t)u_i(0) - \dot{u}_i(t)u_i(t)]dV$$

$$-\rho\int_0^t\int_V \dot{u}_i(t+\tau)\dot{u}_i(t-\tau)dVd\tau$$

$$+\frac{1}{\tau_0}\int_V \kappa_{ij}\Big(\overline{q}_j(t)\overline{q}_i(t) - \overline{q}_j(0)\overline{q}_i(2t)\Big)dV + \frac{1}{\tau_0}\int_0^t\int_V \kappa_{ij}\dot{\overline{q}}_i(t+\tau)\overline{q}_j(t-\tau)dVd\tau$$

$$+\int_V \kappa_{ij}[\dot{\overline{q_j}}(t)\overline{q}_i(t) - q_j(0)\overline{q}_i(2t)]dV + \int_0^t\int_V \kappa_{ij}\dot{\overline{q}}_i(t+\tau)\dot{\overline{q}}_j(t-\tau)dVd\tau \tag{38}$$

$$\int_0^t \Omega_{11}(t-\tau, t+\tau)d\tau = \int_0^t \Lambda(t-\tau, t+\tau)d\tau - \rho\int_V [\dot{u}_i(t)u_i(t) - \dot{u}_i(0)u_i(2t)]dV$$

$$-\rho\int_0^t\int_V \dot{u}_i(t-\tau)\dot{u}_i(t+\tau)dVd\tau + \frac{1}{\tau_0}\int_0^t\int_V \kappa_{ij}\dot{\overline{q}}_j(t+\tau)\overline{q}_i(t-\tau)dVd\tau$$

$$+\int_V \kappa_{ij}[q_j(2t)\overline{q}_i(0) - \dot{\overline{q_j}}(t)\overline{q}_i(t)]dV + \int_0^t\int_V \kappa_{ij}\dot{\overline{q}}_i(t-\tau)\dot{\overline{q}}_j(t+\tau)dVd\tau \tag{39}$$

In view of equations (39) and (38), the relation (35) reduces to (34).

Theorem 3 implies the following uniqueness theorem.

Theorem 4 *Assume that*

(i) *the symmetry relations (10) hold,*

(ii) ρ *is strictly positive,*

(iii) κ_{ij} *is positive semi-definite.*

(iv) γ_{ij}, α_{ij} *and* A_{ij} *are positive definite,*

Let $p^* = \{u_i{}^*,\ \psi^*,\ \varphi^*,\ e^*_{ij},\ \sigma^*_{ij},\ E^*_i,\ \zeta^*_i,\ D^*_k,\ B^*_k,\ T^*,\ \eta^*,\ q^*_i\}$ *be the difference of any two solutions of the mixed problem. Then*

$$u^*_i = 0, \quad T^* = 0, \quad \psi^* = const. \text{ and } \varphi^* = const., \quad \text{on } V\times(0,+\infty) \tag{40}$$

Moreover, if S_3 *and* S_5 *are nonempty, then the mixed problem has at most one solution.*

Proof. Clearly, the difference of any two solutions corresponds to null data. Thus, from equation (34) we conclude that

$$\frac{d}{dt}\Big(\int_V [\rho u^*_i u^*_i + \kappa_{ij}\overline{q^*}_j\overline{q^*}_i]dV + \frac{1}{\tau_0}\int_0^t\int_{\partial V} \kappa_{ij}\overline{q^*}_j\overline{q^*}_i dtdV\Big) = 0 \tag{41}$$

Since u_i^* vanish initially, the above relation implies that

$$\int_V [\rho u_i^* u_i^* + \kappa_{ij}\overline{q^*}_j\overline{q^*}_i]dV + \frac{1}{\tau_0}\int_0^t \int_{\partial V} \kappa_{ij}\overline{q^*}_j\overline{q^*}_i dtdV = 0 \tag{42}$$

In view of hypotheses $(i) - (iii)$, from equation (42) we get

$$u_i^* = 0, \quad \bar{q}_i^* = 0, \text{ on } V \times [0, +\infty) \tag{43}$$

Using equation $(43)_2$, equation (8) may be written as

$$T_{,i}^* = 0 \quad on\ V \times (0, +\infty) \tag{44}$$

So that $q_i^* = 0$ on $V \times (0, +\infty)$. The energy equation (3) yields $\rho T_0 \eta^* = 0$ and it follows that

$$\eta^* = 0 \tag{45}$$

On the other hand, from the constitutive equations (5)-(6), we have

$$D_j^* = \alpha_{ji}\zeta_i^* + \gamma_{ji}E_i^* + p_j T^* \tag{46}$$

$$B_j^* = A_{ji}\zeta_i^* + \alpha_{ji}E_i^* + m_j T^* \tag{47}$$

Moreover, by equation (2) and the null data

$$D_{j,j}^* = 0, \quad and \quad B_{j,j}^* = 0 \tag{48}$$

By using equations (44) and (48), and the divergence theorem, we get

$$\int_V D_j^* E_j^* dV = -\int_{\partial V} D_j^* n_j \psi^* dS + \int_V D_{j,j}^* \psi^* dV \tag{49}$$

$$\int_V B_j^* \zeta_j^* dV = -\int_{\partial V} B_j^* n_j \varphi^* dS + \int_V B_{j,j}^* \varphi^* dV \tag{50}$$

With the help of equations (46)-(48) and the boundary conditions, from equations (49)-(50) we obtain

$$\int_V \alpha_{ji}\zeta_i^* E_j^* dV + \int_V \gamma_{ji}E_i^* E_j^* dV = 0 \tag{51}$$

$$\int_V A_{ji}\zeta_i^*\zeta_j^* + \int_V \alpha_{ji}E_i^*\zeta_j^* dVdV = 0 \tag{52}$$

In view of hypothesis (iv), we find that

$$E_i^* = \zeta_i^* = 0, \quad \text{on } V \times (0, +\infty) \tag{53}$$

so that $\psi* = const.$ and $\varphi* = const.$ on $V \times (0, +\infty)$. Clearly, if $S_3 \neq \emptyset$ and $S_5 \neq \emptyset$ then we obtain $\psi* = 0$ and $\varphi* = 0$ on $V \times (0, +\infty)$. It follows from equations (45) and (53) that $T^* = 0$ on $V \times (0, +\infty)$. This complete the proof.

5 Continuous Dependence Theorem

In this section we prove that the results established in the last sections lead to a continuous dependence theorem of the solution of the mixed problem in the frame of a generalized theory on the external body loads and heat supply $\{F_i, \varrho, \sigma, h\}$. In this order, we employ a method suggested by the work of Rionero and Chirita (1987). Throughout this section, we assume that the hypotheses of Theorem 3 are satisfied and that the time interval $\wp = [0, t_1]$ is finite, i.e. $t_1 < \infty$.

We consider two external data systems which differ only by the body loads and heat supply

$$\pounds^{(\alpha)} = \{F_i^{(\alpha)},\ \varrho^{(\alpha)},\ \sigma^{(\alpha)},\ h^{(\alpha)},\ \tilde{u_i},\ \tilde{\psi},\ \tilde{\varphi},\ \tilde{T},\ \tilde{f_i},\ \tilde{d},\ \tilde{b},\ \tilde{q},\ {u_i}^0,\ {v_i}^0,\ \eta^0\}$$

$(\alpha = 1, 2)$, and let $p^{(\alpha)}$ be two solutions of the boundary-initial-value problem corresponding to $\pounds^{(\alpha)}$, respectively. We denote by

$$F_i = F_i^{(2)} - F_i^{(1)}, \quad \varrho = \varrho^{(2)} - \varrho^{(1)}, \quad \sigma = \sigma^{(2)} - \sigma^{(1)}, \quad h = h^{(2)} - h^{(1)}$$

Then, the difference of the two solutions $p = p^{(2)} - p^{(1)}$ represents a solution of the mixed problem for null boundary and initial data and for the body loads and heat supply $\{F_i, \varrho, \sigma, h\}$. By application of Theorem 3 for the solution p we deduce

$$\frac{d}{dt}\Big(\int_V [\rho u_i u_i + \lambda_{ij}\overline{q}_j\overline{q}_i]dV + \frac{1}{\tau_0}\int_0^t\int_V \lambda_{ij}\overline{q}_j\overline{q}_i dV d\tau\Big)$$

$$= \int_0^t\int_V \Big[F_i(t-\tau)u_i(t+\tau) - F_i(t+\tau)u_i(t-\tau) + \psi(t-\tau)\varrho(t+\tau) - \psi(t+\tau)\varrho(t-\tau)$$

$$+\varphi(t-\tau)\sigma(t+\tau) - \varphi(t+\tau)\sigma(t-\tau) + \frac{\rho}{T_0}[T(t-\tau)\bar{h}(t+\tau) - T(t+\tau)\bar{h}(t-\tau)]\Big]dV d\tau \tag{54}$$

for every $t \in (0, t_1/2)$. We shall use the identity (54) in order to prove the following continuous dependence theorem.

Theorem 5 *We suppose that there exist* $t^* \in (0, t_1)$ *and some positive constants* $A,\ B$ *such that*

$$\int_0^{t^*}\int_V [\psi^2 + \varphi^2 + \frac{\rho}{T_0}T^2]dV d\tau \le A^2, \qquad \int_0^{t^*}\int_V u_i u_i dV d\tau \le B^2 \tag{55}$$

then, we have the inequality

$$\int_V [\rho u_i u_i + \lambda_{ij}\overline{q}_j\overline{q}_i]dV + \frac{1}{\tau_0}\int_0^t\int_V \lambda_{ij}\overline{q}_j\overline{q}_i dV d\tau$$

$$\le At^*\Big[\int_0^{t^*}\int_V (\varrho^2 + \sigma^2 + \frac{\rho}{T_0}\bar{h}^2)dV d\tau\Big]^{\frac{1}{2}} + Bt^*\Big[\int_0^{t^*}\int_V F_i F_i dV d\tau\Big]^{\frac{1}{2}} \tag{56}$$

for every $t \in [0, t^*/2]$.

Proof.

By the application the Cauchy-Schwartz inequality for the last terms of (54), we derive

$$\int_0^t\int_V \Big[\psi(t-\tau)\varrho(t+\tau) + \varphi(t-\tau)\sigma(t+\tau) + \frac{\rho}{T_0}T(t-\tau)\bar{h}(t+\tau)\Big]dV d\tau$$

$$\le \Big[\int_0^t\int_V [\psi^2(t-\tau) + \varphi^2(t-\tau) + \frac{\rho}{T_0}T^2(t-\tau)]dV d\tau\Big]^{\frac{1}{2}}\Big[\int_0^t\int_V [\varrho^2(t+\tau) + \sigma^2(t+\tau) +$$

$$+\frac{\rho}{T_0}\bar{h}^2(t+\tau)]dV d\tau\Big]^{\frac{1}{2}} \le A\Big[\int_0^{t^*}\int_V (\varrho^2 + \sigma^2 + \frac{\rho}{T_0}\bar{h}^2)dV d\tau\Big]^{\frac{1}{2}}, \quad \forall t \in [0, t^*/2]. \tag{57}$$

In the same way, we find the inequality

$$-\int_0^t\int_V \Big[\psi(t+\tau)\varrho(t-\tau) + \varphi(t+\tau)\sigma(t-\tau) + \frac{\rho}{T_0}T(t+\tau)\bar{h}(t-\tau)\Big]dV d\tau$$

$$\leq A\Big[\int_0^{t^*}\int_V(\varrho^2+\sigma^2+\frac{\rho}{T_0}\bar{h}^2)dVd\tau\Big]^{\frac{1}{2}},\quad \forall t\in[0,t^*/2] \tag{58}$$

From the above relations we get

$$\int_0^t\int_V\Big[\psi(t-\tau)\varrho(t+\tau)-\psi(t+\tau)\varrho(t-\tau)+\varphi(t-\tau)\sigma(t+\tau)-\varphi(t+\tau)\sigma(t-\tau)$$

$$+\frac{\rho}{T_0}[T(t-\tau)\bar{h}(t+\tau)-T(t+\tau)\bar{h}(t-\tau)]\Big]dVd\tau\leq 2A\Big[\int_0^{t^*}\int_V(\varrho^2+\sigma^2+\frac{\rho}{T_0}\bar{h}^2)dVd\tau\Big]^{\frac{1}{2}},\quad \forall t\in[0,t^*/2] \tag{59}$$

By the application the Cauchy-Schwartz inequality for the first term of (54), we derive

$$\int_0^t\int_V[F_i(t-\tau)u_i(t+\tau)-F_i(t+\tau)u_i(t-\tau)]dVd\tau$$

$$\leq\Big[\int_0^t\int_V F_i(t-\tau)F_i(t-\tau)dVd\tau\Big]^{\frac{1}{2}}\times\Big[\int_0^t\int_V u_i(t+\tau)u_i(t+\tau)dVd\tau\Big]^{\frac{1}{2}}$$

$$+\Big[\int_0^t\int_V F_i(t+\tau)F_i(t+\tau)dVd\tau\Big]^{\frac{1}{2}}\times\Big[\int_0^t\int_V u_i(t-\tau)u_i(t-\tau)dVd\tau\Big]^{\frac{1}{2}}$$

$$\leq 2B\Big[\int_0^{t^*}\int_V F_iF_idVd\tau\Big]^{\frac{1}{2}},\quad \forall t\in[0,t^*/2] \tag{60}$$

If we substitute the estimates given by (60) and (59) into the identity (54) and we integrate the relation thus obtained with respect to the time variable from 0 to t, then we obtain that (56) holds, for every $t\in[0,t^*/2]$. The proof is complete.

6 Conclusion

In this paper we have extended a new method of proof proposed by Iesan (1998, 2006a, b) for the classical thermoelastic theory to obtain reciprocity, uniqueness, and continuous dependence theorems in the frame of a generalized thermoelastic theory, which is considered physically more realistic that the classical one.

The advantages of this new method of proof compared to others dealing with thermoelastic problems under Lebon's approach (Chandrasekharaiah 1986, 1987, 1988; Ciarletta and Scarpetta, 1996; Passarella, 1996; Li, 2003) are :

(i) Reciprocity and uniqueness theorems are established in the above papers as two independent problems, while the same corresponding theorems are derived in this paper on the basis of results obtained from the reciprocal relation.

(ii) Uniqueness theorems in the above papers were derived under the definiteness assumptions on the thermoelastic coefficients and on the conductivity tensor, while the same corresponding theorem is established in this paper without recourse to these restrictions.

(iii) Reciprocity theorems in the above papers were derived by incorporation of the Laplace transform and initial conditions into the field equations, while the same corresponding theorem in this paper is established directly in the physical domain.

(iv) The continuous dependence theorem derived here and not considered in the above papers proves that in the motion following any sufficiently small change in the external system, the solution of the initial-boundary value problem is everywhere arbitrary small in magnitude. Consequently, the mathematical model proposed for a generalized thermo-magnetoelectroelastic problem under Lebon's model is well posed.

Finally, the obtained results are applicable for some special cases which can be deduced from our generalized model, such as thermoelasticity and thermopiezoelectricity problems in the frame of coupled and generalized models. Moreover, the new method presented in this paper should prove useful for researchers working on the qualitative properties of the solution in mechanics of solids and fluids.

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