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Application of a Hybrid WKB-Galerkin Method in Control of the Dynamic Instability of a Piezolaminated Imperfect Column

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This paper deals with the problem of dynamic instability of a piezolaminated column subjected to an arbitrary axial load. The aim of the analysis is to obtain the closed form solution for the equilibrium equation of the loaded column considering a damping coefficient variable in time, presented as function of time. A solution of the problem is obtained using a hybrid WKB-Galerkin method.

1 Introduction

The branch of mechanics which considers piezoelectric materials is highly developing at present since piezoelectric actuators and sensors are compact and do not need supporting mechanisms to bear reacting forces. Moreover piezo structures are reliable since the actuators transfer forces to the structural member according to the magnitude of the excitation voltage. They are also attractive due to the greatest advantage, i.e. the activation response time is fast and takes only few milliseconds. Meressi and Paden (1992) showned that the magnitude of the first critical load of a flexible beam can be increased using piezoelectric actuators and strain sensors. Faria and Almedia (1999) used the finite element method to predict the pre-buckling behavior of a composite beam with geometric imperfections. In their research piezoelectric actuators were also used. By the application of the same method the nonlinear behavior of piezolaminated beams was analyzed by Mukherjee and Chaudhuri (2001). The set of experiments was conducted by Thomson and Loughan (1995) who examined the overall flexural buckling control of smart composite column stripes using piezoelectric actuators.

As we can see investigations devoted to the stability of piezo structures are evolving. Thus the main objective of this paper is to investigate the problem of control of dynamic instability of the piezolaminated imperfect column, paying attention to the fact that the damping coefficient can be expressed as a in time function. Therefore the equilibrium equation of the column is presented taking into consideration the mentioned fact. According to the investigations presented in the cited literature it is obvious that the finite element method and numerical methods are often used in this field of mechanics. Thus it is important to investigate the problem of the dynamic instability of a piezolaminated column by using an analytical method. In this paper the hybrid (Wentzel-Kramer-Brillouin) WKB-Galerkin method is used because its advantages have already been shown in different branches of mechanics and it will give us an opportunity to obtain the approximate solution as an asymptotic one.

2 Basic Concepts of the Hybrid WKB-Galerkin Method

Differential equations with variable coefficients and boundary problems in mechanics can be solved by integration only in individual cases. Therefore it is partienlarely important to solve the problem using approximate methods.

Along with numerical methods also approximate methods have been developed. Among these methods such analytical methods as variational and non-variational (Bubnov-Galerkin method) should be mentioned.

If the differential equation of the problem contains a dimensionless parameter ε (small or large), it is reasonable to find the approximate solution as an asymptotic one with a predetermined value of the parameter. The solution obtained in this case runs up to a fair accuracy in a small interval of parameter variation. For this reason the purpose of asymptotic mathematics is to find the methods based on classical ones which can improve an approximate solution. There are quite a number of hybrid approaches which are based on the idea of building a better solution. The last one should be obtained using a set of coordinate asymptotic functions and undetermined parameters. If it is possible to define the parameters then a more exact hybrid solution of the problem can be very close to the exact one in a large interval of parameter variation. It should be noted that perturbation-Galerkin and WKB-Galerkin methods are hybrid approaches. However, according to results obtained in different branches of mechanics, the hybrid WKB-Galerkin method shows a higher accuracy of solution compared to the perturbation-Galerkin method.

Hybrid methods have proved to be useful in a wide variety of applications such as structural mechanics problems, applications to slender-bodies, thermal and structure problems. Geer and Andersen (1989, 1990, 1991) used the two steps hybrid perturbation-Galerkin method to obtain a solution for some types of differential equations and to solve some mechanical problems. The WKB-Galerkin method for a numerical solution of mechanical problems was also used by Steel (1971, 1989). Gristchak (1979) showed that the WKB-method can be also used in some linear mechanical problems, in solution of nonlinear bending problems, and in problems of oscillation of homogeneously structured system. The hybrid WKB-Galerkin method was successfully used in the solution of mechanical boundary problems which contain a linear differential equation with variable coefficients and a parameter near the highest order derivative. The obtained solution has a pinpoint accuracy and can be useful in a wide variety of applications. However, the algebra of the solution process following the WKB method becomes typically more and more tedious as higher and higher order terms are computed, and frequently the computational effort rises so fast from term to term that even with computational assistance very few terms can be computed. Thus for cases where higher order terms may have a significant effect, it is important to get as much use of the information contained in the lower order terms as possible. The hybrid WKB-Galerkin method seems to greatly extend the power and usefulness of the WKB method without significant computational effort.

In the analysis of complex mechanical models with nonhomogenous structure it is also essential to obtain a solution of a system of differential equations which contain undetermined parameters and some large or small parameters. In these problems it is not possible to obtain some general exact solution. Thus it is necessary to use approximate methods such as hybrid methods.

The hybrid WKB-Galerkin method enables us to obtain especially good results in an approximate solution of the differential equation which contains a parameter near the highest order derivative. For this dimensionless parameter we can choose the natural frequency, the ratio of the largest and the smallest measurements of the structure or others. To solve the linear differential equations the method is applied in two stages: obtaining the WKB-solution of the problem and applying the Bubnov-Galerkin method taking into consideration asymptotic coefficients. In step one the functions of x are determined using the WKB method by forming an expansion in ε. In step two the functions of ε are determined by the classical Bubnov-Galerkin method. The resulting hybrid method has the potential of overcoming some of the drawbacks of the WKB and the Bubnov-Galerkin methods applied separately, while combining some of the good features of each.

It is necessary to obtain the solution $u(x,\varepsilon)$ of the boundary problem

$$L[u(x,\varepsilon),x,\varepsilon]=0 \tag{1}$$

where L is some linear differentiation operator of nth order, ε is a parameter near the highest order derivative, x is located in some interval $[a,b]$, and $u(x,\varepsilon)$ is satisfied by the given boundary conditions.

In the first step the solution $u(x,\varepsilon)$, according to the WKB-procedure, can be expressed as

$$u(x,\varepsilon)=\exp\left(\int_a^x \sum_{i=0}^{\infty} u_i(x)\gamma_i(\varepsilon)dx\right) \tag{2}$$

where $\gamma_i(\varepsilon)$ are asymptotic sequences ($\gamma_i(\varepsilon)=\varepsilon^{i-1}$) and every $u_i(x)$ is determined by the standard WKB-method. Approximate functions $u_i(x)$ should be chosen as coordinate functions to go through the solution procedure by the Bubnov-Galerkin method. The more exact solution $\widetilde{u}_i(x,\varepsilon)$ for $u(x,\varepsilon)$ is defined as

$$\widetilde{u}_i(x,\varepsilon)=\exp\left(\int_a^x \sum_{i=0}^{N} u_i(x)\delta_i(\varepsilon)dx\right) \tag{3}$$

where the undetermined parameters $\delta_i(\varepsilon)$ are complex functions of ε, and all $u_i(x)$ are approximate coordinate functions which were found at the first step. To find the undetermined coefficients $\delta_i(i=0..N)$, we should substitute (3) into (1).

$$L[\widetilde{u}_i(x,\varepsilon),x,\varepsilon] = \exp\left(\int_a^x \sum_{i=0}^{N} \delta_i(\varepsilon)u_i(x)dx\right) R\left(\delta_0,...,\delta_N,u_0,...,u_N,u'_0,...,u_N^{(n-1)},x,\varepsilon\right) \quad (4)$$

Taking into consideration equation (1), it should be noticed the that right part of the last equation must satisfy the condition

$$R\left(\delta_0,\ldots,\delta_N,u_0,\ldots,u_N,u'_0,\ldots,u_N^{(n-1)},x,\varepsilon\right) \to 0. \quad (5)$$

Therefore it should be marked that *R* must be orthogonal to the *N+1* coordinate functions in the interval [*a,b*]

$$\int_a^b R\left(\delta_0,...,\delta_N,u_0,...,u_N,u'_0,...,u_N^{(n-1)},x,\varepsilon\right)u_i(x)dx = 0 \quad (6)$$

where *i=0,...,N.*

Equation (6) is the system of *N+1* equations with *N+1* undetermined coefficients $\delta_i(\varepsilon)$. If δ_k are complex functions $\delta_k = \delta_{1k} + i\delta_{k2}$ then the obtained system has *2(N+1)* equations with *2(N+1)* undetermined coefficients. Equation (6) can be solved by numerical methods.

3 The Behaviour of a Piezolaminated Imperfect Column under Arbitrary Axial Load

Mukherjee and Chaudhuri (2002) used the imperfection approach to find the problem solutions for the control of instability of piezolaminated columns under axial loads. They investigated the control of steel columns with surface bonded lead zirconated titanate layers under axial loading. The authors took into consideration the initial imperfection of the column which was subjected to static, periodic, and arbitrary axial loads to obtain a closed form solution.

According to Mukherjee and Chaudhuri (2002) we consider the initially imperfect simply supported column.

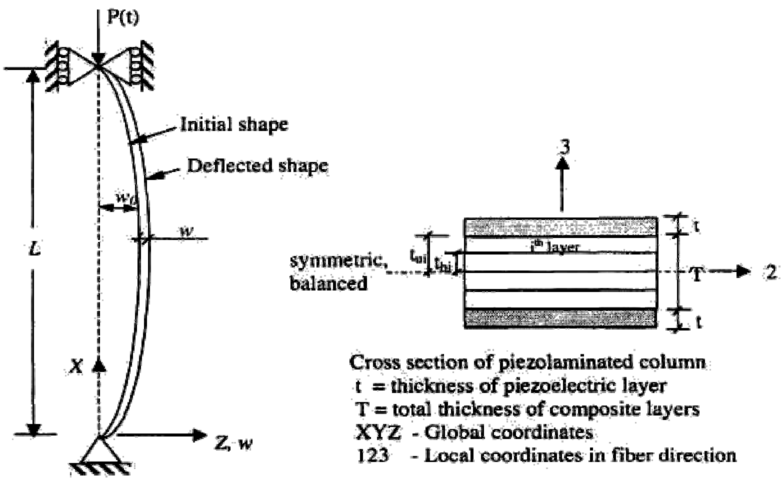

Figure 1. Piezolaminated Column

This column is subjected to an axial loading as we can see in Figure 1. The initial imperfection of the column is w_0 and deflects to w under the load. This initial imperfection for the analyzed column can be written as

$$w_0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \quad (7)$$

Following Mukherjee and Chaudhuri (2002), the equilibrium equation for the system can be expressed as

$$\overline{D}\frac{\partial^4 w}{\partial x^4}+P(t)\frac{\partial^2}{\partial x^2}\left(w+w_0\right)+m\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}=0 \tag{8}$$

where *m* is mass per unite length, *c* is the damping coefficient, *t* is the time, *P(t)* is an arbitrary dynamic axial load, w_0 is defined by (7), and $\overline{D}$ is the modified stiffness due to subcritical actuation and can be expressed as

$$\overline{D}=D\left(1+\frac{Ge_{31}^2(T+t)^2 tb}{2D\xi_{33}}\right) \tag{9}$$

where e_{31} is the piezoelectric stress constant and ξ_{33} is the dielectric constant of the piezoelectric material. The equilibrium equation of a piezolaminated simply supported column under arbitrary axial load determined by Mukherjee and Chaudhuri (2002) can be written as

$$\overline{D}\frac{\partial^4 w}{\partial x^4}+P(t)\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}=-P(t)\frac{\partial^2 w_0}{\partial x^2} \tag{10}$$

In this paper it is essential to take into consideration the damping coefficient which will be presented as $c(t)=c_0\varphi(t)$. In this case the equilibrium equation (10) can be rewritten as

$$\overline{D}\frac{\partial^4 w}{\partial x^4}+P(t)\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial t^2}+c(t)\frac{\partial w}{\partial t}=-P(t)\frac{\partial^2 w_0}{\partial x^2} \tag{11}$$

The general solution for the problem can be expressed as

$$w(x,t)=\sum_{n=1}^{\infty} f(t)\sin\frac{n\pi x}{L} \tag{12}$$

By us substituting (12) and (7) into equation (11) and considering only the *n*th term, it can be written as

$$\left[mf''+c(t)f'+\overline{D}\frac{\pi^4 n^4 f}{L^4}-P(t)\frac{n^2\pi^2 f}{L^2}\right]\sin\frac{n\pi x}{L}=P(t)\frac{a_n n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}\quad . \tag{13}$$

Using the definition for *c(t)* it is possible to rewrite the above equation as

$$\varepsilon^2 f''+\varphi(t)f'+\left(\widetilde{D}-\overline{c}P(t)\right)f=a_n\overline{c}P(t) \tag{14}$$

where $c(t)=c_0\varphi(t)$; $\varepsilon^2=\frac{m}{c_0}$; $\widetilde{D}=\frac{\overline{D}}{c_0}\left(\frac{n\pi}{L}\right)^4$; $\overline{c}=\frac{1}{c_0}\left(\frac{n\pi}{L}\right)^2$; $c_0>m$.

It is essential that the derivative is taken in the time domain.

Supposing $\overline{P}(t)=\widetilde{D}-\overline{c}P(t)$ and assuming that $\overline{P}(t)>0$, we obtain

$$\varepsilon^2 f''+\varphi(t)f'+\overline{P}(t)f=a_n\overline{c}P(t) \tag{15}$$

The general solution of equation (15) consists of a complementary function and a particular integral.

The homogeneous equation of the general equation (15) can be expressed as

$$\varepsilon^2 f'' + \varphi(t) f' + \overline{P}(t) f = 0 \tag{16}$$

As it was mentioned in the hybrid WKB-Galerkin method description, the solution procedure consists of two steps.

The WKB solution of equation (16) taking into consideration only two first terms of the WKB-expansion can be written as

$$f(t,\varepsilon) = \exp\left(\int_a^t \left(\frac{1}{\varepsilon} u_0 + u_1 \right) dt \right) \tag{17}$$

Substituting (17) into (16) and forming a system according to the ε th power we can get

$$\begin{cases} {u_0}^2 + \varphi(t) u_1 + \overline{P}(t) = 0 \\ 2u_0 u_1 + u_0' = 0 \end{cases} \tag{18}$$

or

$$\begin{cases} u_1 = -\dfrac{u_0'}{2u_0} \\ 2u_0^3 - \varphi(t) u_0' + 2u_0 \overline{P}(t) = 0 \end{cases} \tag{19}$$

As a solution of the system (19) we obtain

$$\begin{cases} u_0 = \pm \dfrac{1}{2} i e^{2\int \frac{\overline{P}(t)}{\varphi(t)} dt} \left[\displaystyle\int \frac{e^{4\int \frac{\overline{P}(t)}{\varphi(t)} dt}}{\varphi(t)} dt \right]^{-1/2} \\ u_1 = -\dfrac{1}{2} \dfrac{d}{dt} \ln u_0 \end{cases} \tag{20}$$

In this case the general WKB solution of equation (16) with respect to (17) can be expressed as

$$f(t,\varepsilon) = \exp\left(\int_a^t \frac{1}{\varepsilon} u_0 dt - \frac{1}{2} \ln u_0 \right) \tag{21}$$

where u_0 is defined in (20). Thus it is possible to rearrange the WKB solution of equation (16) into

$$f(t,\varepsilon) = \exp\left(\int_a^t \pm \frac{i}{2\varepsilon} e^{2\int \frac{\overline{P}(t)}{\varphi(t)} dt} \left[\int \frac{e^{4\int \frac{\overline{P}(t)}{\varphi(t)} dt}}{\varphi(t)} dt \right]^{-1/2} - \frac{1}{2} \ln \left[\frac{1}{2} e^{2\int \frac{\overline{P}(t)}{\varphi(t)} dt} \left(\int \frac{e^{4\int \frac{\overline{P}(t)}{\varphi(t)} dt}}{\varphi(t)} dt \right)^{-1/2} \right] \right) \tag{22}$$

However, it was mentioned above the hybrid WKB-Galerkin method enables us to achieve a better result.

According to the solution procedure for the hybrid WKB-Galerkin method, the obtained WKB-solution will be used. Therefore, according to (3), we consider the solution in the form

$$\tilde{f}(t,\varepsilon) = \exp\left(\int_a^t \left(\delta_{01}(\varepsilon) + i\delta_{02}(\varepsilon) \right) u_0(t) dt \right) \tag{23}$$

Therefore, considering the procedure described by (4), (5) and (6), we obtain

$$\varepsilon^2\left[\left(-\delta_{01}^2 i+2\delta_{01}\delta_{02}+\delta_{02}^2 i\right)\overline{u}_0^3+\left(-\delta_{01}-i\delta_{02}\right)\overline{u}_0\overline{u}_0'\right]+\varphi(t)\left(-\delta_{01}-i\delta_{02}\right)\overline{u}_0^2+i\overline{P}(t)\overline{u}_0=0 \qquad (24)$$

where $u_0 = i\overline{u}_0$.

According to the steps of solution, separating *real* and *imaginary* terms, we obtain the following system of equations

$$\begin{cases} 2A\delta_{01}\delta_{02}-B\delta_{01}=0 \\ A\delta_{02}^2-A\delta_{01}^2-B\delta_{02}+\widetilde{P}(t)=0 \end{cases} \qquad (25)$$

where $A=\int_a^b \varepsilon^2\overline{u}_0^3 dt$; $B=\int_a^b\left(\overline{u}_0\overline{u}_0'\varepsilon^2+\varphi(t)\overline{u}_0^2\right)dt$; $\widetilde{P}(t)=\int_a^b \overline{u}_0\overline{P}(t)dt$.

Solving system (25) we get

$$\begin{cases} \delta_{01}=\dfrac{\sqrt{4A\widetilde{P}(t)+B^2}}{2A} \\ \delta_{02}=\pm\dfrac{B}{2A} \end{cases} \qquad (26)$$

Concluding all calculations and making substitutions according to (23), we finally obtain the hybrid solution of equation (16) as

$$\widetilde{f}(t,\varepsilon)=\exp\left(\int_a^t\left(\delta_{01}(\varepsilon)+i\delta_{02}(\varepsilon)\right)\left(\pm\frac{1}{2}ie^{2\int\frac{\overline{P}(t)}{\varphi(t)}dt}\left[\int\frac{e^{4\int\frac{\overline{P}(t)}{\varphi(t)}dt}}{\varphi(t)}dt\right]^{-1/2}\right)dt\right) \qquad (27)$$

where δ_{01},δ_{02} are defined by (26).

4 Numerical Example

To validate the obtained solution, it is important to present the graphical result of the problem for predetermined parameters. Supposing that

$$a_n=2,\ \widetilde{D}=0.5,\ \varphi(t)=e^{2t},\ \varepsilon=0.5,\ \ L=2,\ \ \overline{P}(t)=2e^{2t},\ \ P(t)=\frac{1}{\overline{c}}(\widetilde{D}-2e^{2t}),\ \ \overline{c}=\frac{1}{2}\left(\frac{n\pi}{L}\right)^2,\ \ \widetilde{D}=\frac{\overline{D}}{2}\left(\frac{n\pi}{L}\right)^4,$$

the equilibrium equation can be expressed as follows

$$\varepsilon^2 f''+\varphi(t)f'+\overline{P}(t)f=a_n(\widetilde{D}-2e^{2t}) \qquad (28)$$

or

$$\varepsilon^2 f''+e^{2t}f'+2e^{2t}f=1-4e^{2t} \qquad (29)$$

The homogeneous equation of equation (29) is

$$\varepsilon^2 f'' + e^{2t} f' + 2e^{2t} f = 0 \tag{30}$$

The solution of equation (29) consists of a complementary function and a particular integral. In order, to find the complementary function (22), we need to define u_0 using expressions determined below.

$$\int \frac{\overline{P}(t)}{\varphi(t)} dt = \int \frac{2e^{2t}}{e^{2t}} dt = 2t,\ e^{2\int \frac{\overline{P}(t)}{\varphi(t)} dt} = e^{4t},\ e^{4\int \frac{\overline{P}(t)}{\varphi(t)} dt} = e^{8t},\ \int \frac{e^{4\int \frac{\overline{P}(t)}{\varphi(t)} dt}}{\varphi(t)} dt = \int \frac{e^{8t}}{e^{2t}} dt = \frac{e^{6t}}{6} \tag{31}$$

It follows that u_0 can be written as

$$u_0 = \pm \frac{i}{2} e^{4t} \frac{\sqrt{6}}{e^{3t}} = \pm \frac{\sqrt{6}}{2} i e^{t} \tag{32}$$

Summarizing all obtained expressions and according to (22), the WKB solution of the homogeneous equation (30) becomes

$$f(t) = \sqrt{\frac{2}{\sqrt{6}}} e^{-\frac{t}{2}} \left(C_1 \cos(\sqrt{6} e^t) + C_2 \sin(\sqrt{6} e^t) \right) \tag{33}$$

where C_1, C_2 are arbitrary constants.

To find a particular solution, we use variation of parameters, considering the obtained complementary function (33). It should be noticed that in use only two terms of the infinite series. In this case we get

$$\begin{cases} C_1 = -3.088t^5 - 12.684t^4 - 14.708t^3 - 4.416t^2 - 0.002t + \overline{c}_1 \\ C_2 = 4.727t^4 + 14.406t^3 + 11.706t^2 + 3.602t + \overline{c}_2 \end{cases} \tag{34}$$

The boundary conditions may be written as

$$\begin{cases} f'(0) = 0 \\ f(0) = 0 \end{cases} \tag{35}$$

Therefore, it is possible to write the general solution as

$$f(t) = \sqrt{\frac{2}{\sqrt{6}}} e^{-\frac{t}{2}} \Big(\left(-3.088t^5 - 12.684t^4 - 14.708t^3 - 4.416t^2 - 0.002t + 0.5984 \right) \cos(\sqrt{6} e^t) + \\ + \left(4.727t^4 + 14.406t^3 + 11.706t^2 + 3.602t + 0.7227 \right) \sin(\sqrt{6} e^t) \Big) \tag{36}$$

The graph of the obtained WKB solution is presented below.

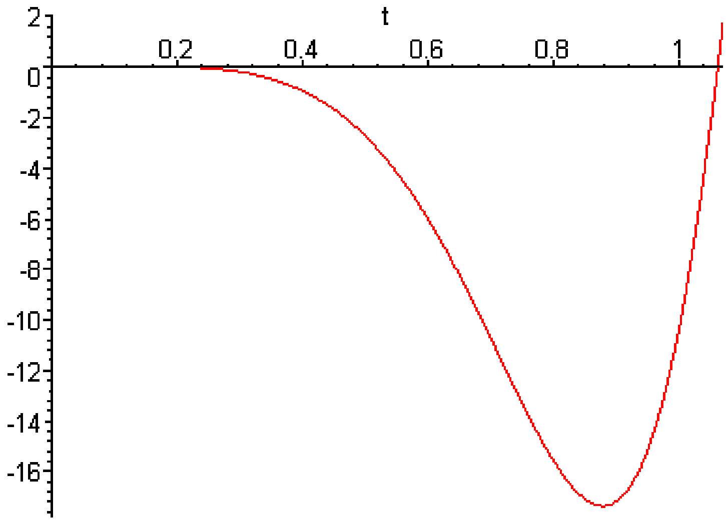

Figure 2. The WKB Solution of the Problem

To obtain the hybrid solution we find the necessary expressions according to (26)

$$\begin{cases} \delta_{01} = -2.472479336 \\ \delta_{02} = 7.497478321 \end{cases} \tag{37}$$

According to the closed form solution (27), simplifying the obtained expression we get the complementary function of the general solution in the following form

$$\tilde{f}(t) = \exp\left(-\frac{\delta_{02}\sqrt{6}}{2}e^t\right)\left(c_1\cos\left(\frac{\sqrt{6}\delta_{01}e^t}{2}\right) + c_2\sin\left(\frac{\sqrt{6}\delta_{01}e^t}{2}\right)\right) \tag{38}$$

which is similar to the WKB solution (33) but should give more accurate result.

We find the particular solution of equation (29) in the same way as it was done for the WKB solution, using the variation of parameters method. Thus we get

$$\begin{cases} c_1 = -13.856t^5 - 53.403t^4 - 59.313t^3 - 17.724t^2 - 1.839t + g_1 \\ c_2 = -17.512t^4 - 48.647t^3 - 32.907t^2 - 4.512t + g_2 \end{cases} \tag{39}$$

where g_1, g_2 are arbitrary constants.

According to the boundary conditions (35) and taking into consideration (38), (39) we can write the general solution of the problem (29) as

$$\tilde{f}(t) = \exp\left(-\frac{7.49747\sqrt{6}}{2}e^t\right)\left(\left(-13.856t^5 - 53.403t^4 - 59.313t^3 - 17.724t^2 - 1.839t + 0.1519336814\right)\times \right.$$
$$\left. \times\cos\left(\frac{\sqrt{6} - 2.47248e^t}{2}\right) + \left(-17.512t^4 - 48.647t^3 - 32.907t^2 - 4.512t - 0.8612330360\right)\sin\left(\frac{\sqrt{6} - 2.47248e^t}{2}\right)\right) \tag{40}$$

To express the solution of the problem we use the obtained function (40)

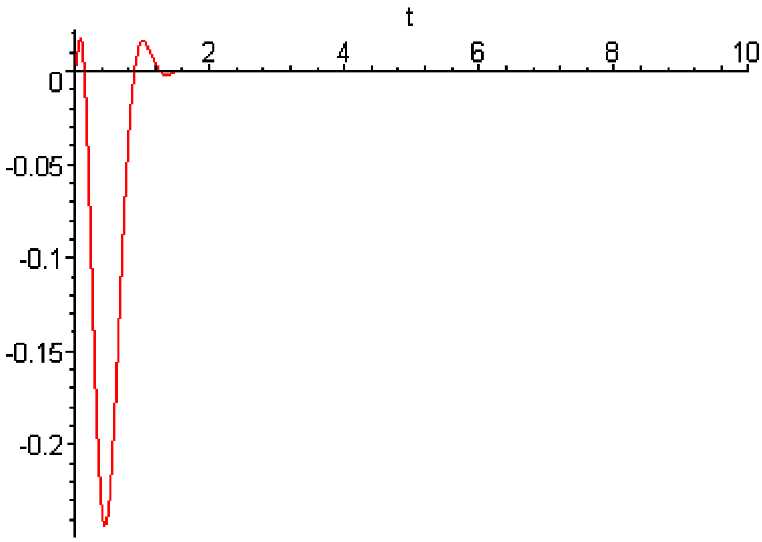

Figure 3. The Hybrid WKB-Galerkin Solution

To compare the obtained result with the numerical solution, we use the numerical method which was proposed as the optimal one in the Maple 7 commercial software for the second order nonhomogeneous differential equation (29). In this case, the solution of the mentioned problem is

$$f_{Num}(t) = \left\lfloor S_2 + \int \left(S_1 + 4t - 4e^{2t}\right) exp\left(2e^{2t}\right) dt \right\rfloor exp\left(-2e^{2t}\right) \tag{41}$$

where S_1, S_2 are arbitrary constants.

Simplifying expression (41) by using infinite series and considering two terms, with respect to the boundary conditions (35), we finally get

$$f_{Num}(t) = exp\left(-2-4t\right)\left(-11.7t - 13.8t^2 - 5.3t^3\right) \tag{42}$$

To compare the two functions of the presented solutions: the numerical one and the hybrid WKB-Galerkin solution, we plot them in one figure.

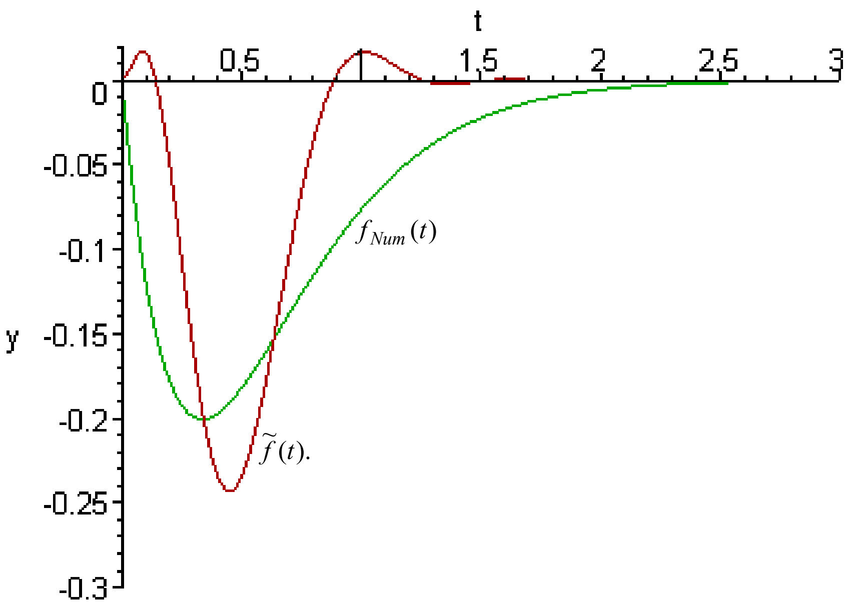

Figure 4. Comparison of the Hybrid WKB-Galerkin and Numerical Solutions

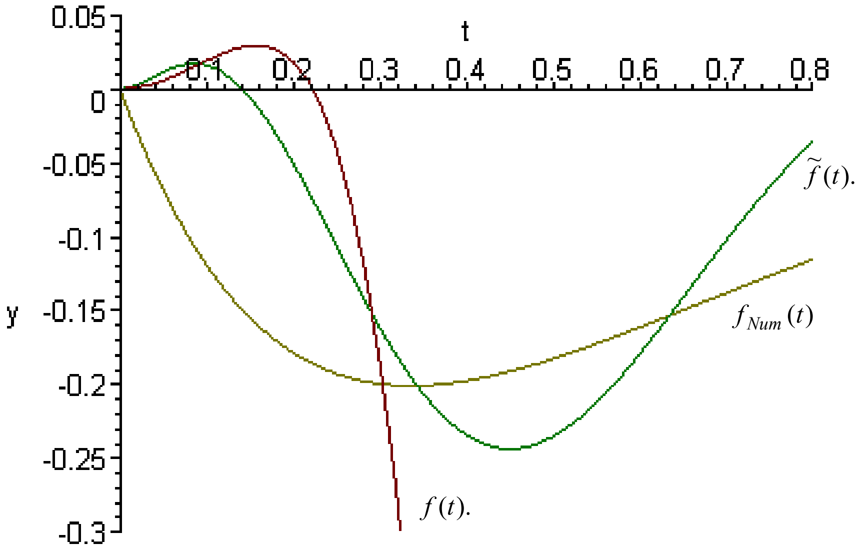

Figure 5. Comparison of the Hybrid ($\tilde{f}(t)$), WKB ($f(t)$) and Numerical ($f_{Num}(t)$) solutions

5 Concluding Remarks

The comparison of three obtained solutions illustrated in Figure 5 demonstrates that the best correlation of numerical solution and the obtained solution was achieved due to the hybrid WKB-Galerkin method. These results confirm that the application of the hybrid WKB-Galerkin method can be widely used in different branches of mechanics giving better results compared with the WKB method.

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