

## Cylindrical Precessions of an Unbalanced Jeffcott Rotor with four Degrees of Freedom in Non-linear Elastic Supports

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*To study forward synchronous whirling motion (precession) of a Jeffcott rotor with 4 degrees of freedom, a new approach has been suggested. The rotor is considered to be statically and dynamically unbalanced. It is attached to a massless linear elastic shaft and supported in non-linear elastic bearings. Depending on the surface traced by undeformed axis of revolution in 3D space one can differentiate three types of precession – cylindrical, conic and hyperboloidal. In the last case this surface is one-sheet hyperboloid.*

*For a statically unbalanced rotor supported in isotropic bearings with Hertzian contact, a cylindrical precession has been studied in assumption that rotation occurs at constant spin speed. External and internal damping have been taken into account. Two non-linear resonances have been found and dynamic response has been built. The problem of stability loss of a forward synchronous cylindrical precession has been investigated for a full range of angular velocities. It has been shown that different types of stability loss take place. Within some range jump phenomena and bi-stability occur, but the steady-state motion remains to be the forward synchronous cylindrical precession. For some other values of the angular velocity stability loss is accompanied by inducing a hyperboloidal precession. The threshold angular velocity for autovibration has been found. By computational modeling the limit cycles and the strange attractor are determined. The results of numerical integration reveal transition "chaos to chaos" in the process of rotation.*

### 1 Introduction

Non-linear analysis of rotor system is mainly based on the simplest model of the Jeffcott rotor with two degrees of freedom, often including gyroscopic terms. Non-linear effects in rotor dynamics can be produced by the presence of clearances in the support system, by non-linear hydrodynamic forces in the lubricated bearings or non-linear restoring forces in the elastic bearings etc. When studying dynamics of the rotor supported in ball bearings many authors consider non-linear contacts in the supports to be of Hertz's type (see Kelzon et al. (1982), Merkin (1997), Tiwari et al. (2000)). There is a special theoretical interest to investigate rotor precession motion with Hertzian contact in supports as it is essentially non-linear and the problem cannot be considered in linear approximation.

A model of a Jeffcott rotor with four degrees of freedom attracts an active interest in recent years (see Genta (2006)). The first works dealing with a rigid rotor with four degrees of freedom, i.e. a rigid rotor supported in linear elastic bearings, were by Bläss (1926); Timoshenko (1928); Dizioglu (1951). But the rotor models in Bläss (1926) and Dizioglu (1951) were incorrect so self-centering was not found out. Moreover, the conclusion of nonoccurrence of self-centering was made in the last paper. An influence of radial clearance has been studied in Neilson and Barr (1988). A perfectly balanced rigid rotor supported in non-linear elastic bearings has been investigated in Kelzon and Meller (1992) and parameters of cylindrical and conical precessions were found.

By using the asymptotic methods some problems of the flexible rotor dynamics were considered in Grobov (1961) and Kushul (1963). Non-stationary vibrations of the 4 d.o.f. rotor supported in linear elastic bearings, isotropic and non-isotropic, were studied in Grobov (1961). There in case of non-linear elastic supports stationary and non-stationary vibrations were studied under assumption that one support was immovable. Autovibration of the overhanging rotor with 4 d.o.f. at supercritical spin speed was investigated in Kushul (1963) also by asymptotic method of Krylov – Bogolyubov.

A new approach to investigation of the forward synchronous precession (whirling motion) of a 4 d.o.f. unbalanced rigid rotor supported in non-linear elastic bearings has been suggested in Pasyukova, (1997, 1998, 2000). Depending on the surface traced by the revolution axis in 3D space a precession can be qualified as cylindrical, conic or

hyperboloidal. In the last case this surface is one-sheet hyperboloid. Dependence on parameters of a complex amplitude for different types of precessions (cylindrical, conic or hyperboloidal) has been defined. Notion of a set of non-linear resonances has been determined as a concept generalization of a resonance (or critical) angular velocity, that allows to investigate rotor dynamics problem with essentially non-linear characteristics of rotor supports. The dynamics response, stability and scenarios of stability loss for the cylindrical precession of the unbalanced rotor in non-linear supports with Hertzian contact have been studied in Pasyukova (2006).

Studying of a four-degrees-of-freedom Jeffcott rotor supported in non-linear elastic bearings based on the new method has been performed in Pasyukova (2005). The rotor was considered to be statically and dynamically unbalanced. The behavior of a massless shaft was assumed to be linear. It was shown that a symmetrical hyperboloidal precession could take place for some values of rotor parameters.

The present research of precessional motion of a statically unbalanced Jeffcott rotor mounted in non-linear elastic bearings is carried out by using the approach developed for rigid rotors and results obtained in Pasyukova (2005, 2006).

## 2 Model Description and Equations of Motion

Let us consider a Jeffcott rotor model with four degrees of freedom, i.e. a rigid body with axial symmetry attached to a massless elastic shaft (see Figure 1). Let the rotor be of mass  $M$  and length  $L_r$ . Its moments of inertia are  $J_p$  (polar) and  $J_t$  (transversal). Imbalance is characterized by a static eccentricity  $e$ , a dynamic eccentricity  $\delta$  and a phase angle  $\varepsilon$  of the dynamic eccentricity. The rotor is supported vertically in two immovable bearings in such a way that a point of attachment  $Q$  is placed at a distance equal to  $e_j L$ , from  $j$ -bearing ( $j = 1, 2$ ),  $L$  being a distance between bearings. If  $Q$  is between the bearings, then both  $e_j > 0$ . If  $Q$  is located outside the  $j$ -bearing, then  $e_j < 0$ , so that  $e_1 + e_2 = 1$  is always right. The angular velocity of the rotor is constant and equal to  $\omega$ . The rotor displacement along the axis of revolution is negligible.

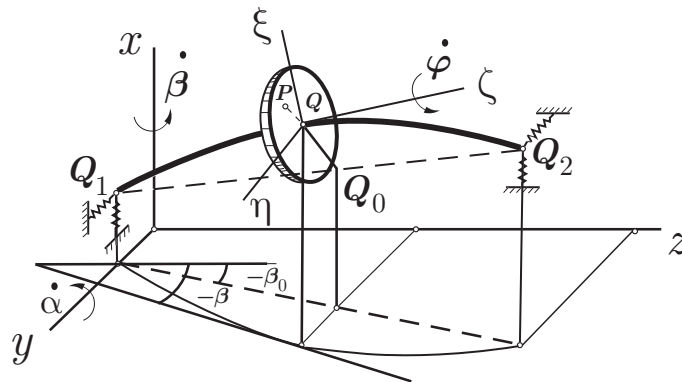


Figure 1: Model of a four-degree-of-freedom Jeffcott rotor supported in non-linear elastic bearings

Let us introduce the following frames of reference: the inertial frame  $Oxyz$  with  $Oz$ -axis coinciding with the rotation axis in its equilibrium state; the frame  $Q\xi\eta\zeta$  fixed with the rotor and  $Q\zeta$ -axis directed along the tangent to the deflected shaft.

The rotor's state can be parameterized by eight parameters:

- $(x, y)$  - Cartesian coordinates of  $Q$ -point;
- $(\alpha, \beta)$  - angles characterizing the direction of  $Q\zeta$ -axis;
- $(x_j, y_j)$  - Cartesian coordinates of  $Q_j$ -point, ( $j = 1, 2$ ).

The kinetic energy of the rotor accurate to linear terms of imbalance  $e$ ,  $\delta$  and quadratic terms of small angles  $\alpha, \beta$  and its derivatives  $\dot{\alpha}, \dot{\beta}$  can be computed as in Genta (1999):

$$T = \frac{1}{2}M (\dot{x}^2 + \dot{y}^2 + e\omega(\dot{y} \cos(\omega t) - \dot{x} \sin(\omega t))) + \frac{1}{2}J_p(\omega^2 - 2\omega\dot{\beta}\alpha) + \frac{1}{2}J_t(\dot{\alpha}^2 + \dot{\beta}^2) + (J_p - J_t)\delta\omega (\dot{\alpha} \sin(\omega t - \varepsilon) - \dot{\beta} \cos(\omega t - \varepsilon)). \quad (1)$$

The shaft is assumed to be linear elastic, so the potential energy of a deflected shaft is given by

$$\Pi_{\hat{a}} = \frac{1}{2}c_{11} ((x - x_0)^2 + (y - y_0)^2) + \frac{1}{2}c_{22} ((\alpha - \alpha_0)^2 + (\beta - \beta_0)^2) + c_{12} ((x - x_0)(\alpha - \alpha_0) + (y - y_0)(\beta - \beta_0)). \quad (2)$$

Here we denote  $C = \{c_{lm}\}$ ,  $(l, m = 1, 2)$  – a stiffness matrix of the elastic shaft supported in the rigid bearings;  $(x_0, y_0)$  – Cartesian coordinates of  $Q_0$ -point; angles  $(\alpha_0, \beta_0)$  characterize the direction of the straight line  $Q_1Q_2$ . Parameters  $(x_0, y_0, \alpha_0, \beta_0)$  define the rotor displacement as a rigid body (see Figure 1) and can be determined as functions of Cartesian coordinates  $x_j, y_j$ :

$$\begin{aligned} x_0 &= e_2 x_1 + e_1 x_2, & y_0 &= e_2 y_1 + e_1 y_2, \\ \alpha_0 &= (x_2 - x_1)/L, & \beta_0 &= (y_2 - y_1)/L. \end{aligned} \quad (3)$$

For the bearings they are considered to be isotropic and non-linear elastic. Bearings isotropy allows to introduce complex variables

$$\begin{aligned} S &= x + iy, & S_j &= x_j + iy_j, \quad (j = 0, 1, 2) \\ \gamma &= \alpha + i\beta, & \gamma_0 &= \alpha_0 + i\beta_0. \end{aligned} \quad (4)$$

The restoring forces only have radial components. The  $j$ -bearing reaction can be written as

$$\mathbf{R}_j^{\text{el}} = -F_j(|S_j|) \mathbf{n}_j. \quad (5)$$

Here  $S_j$  is a displacement of  $Q_j$ -point from its equilibrium position,  $\mathbf{n}_j$  is a unit vector of  $S_j$ -direction. Functions  $F_j(|S_j|)$  are continuously differentiable and  $F_j(0) = 0$ .

Let forces of external damping be given by a dissipative function  $\Phi_1$  and forces of internal damping from material of the shaft be given by a dissipative function  $\Phi_2$  (see, e.g., Grobov (1961)):

$$\Phi_1 = \frac{1}{2}\tilde{\mu}_e (\dot{S}^2 + L^2\dot{\gamma}^2), \quad \Phi_2 = \frac{1}{2}\tilde{\mu}_i (|\dot{S} - i\omega S|^2 + L^2|\dot{\gamma} - i\omega\gamma|^2). \quad (6)$$

Lagrange equations with respect to scalar variables include four differential equations and four algebraic equations, which can be written with respect to complex variables as

$$\begin{aligned} M\ddot{S} + (\tilde{\mu}_e + \tilde{\mu}_i)\dot{S} - i\omega\tilde{\mu}_i S + c_{11}(S - S_0) + c_{12}(\gamma - \gamma_0) &= M e \omega^2 \exp(i\omega t), \\ J_t\ddot{\gamma} - iJ_p\omega\dot{\gamma} + (\tilde{\mu}_e + \tilde{\mu}_i)L^2\dot{\gamma} - i\omega\tilde{\mu}_i L^2\gamma + c_{12}(S - S_0) + c_{22}(\gamma - \gamma_0) &= \\ &= (J_t - J_p)\delta\omega^2 \exp(i(\omega t - \varepsilon)), \end{aligned} \quad (7)$$

$$\begin{aligned} \left(c_{11}e_2 - \frac{c_{12}}{L}\right)(S - S_0) + \left(c_{12}e_2 - \frac{c_{22}}{L}\right)(\gamma - \gamma_0) &= F_1(|S_1|) \frac{S_1}{|S_1|}, \\ \left(c_{11}e_1 + \frac{c_{12}}{L}\right)(S - S_0) + \left(c_{12}e_1 + \frac{c_{22}}{L}\right)(\gamma - \gamma_0) &= F_2(|S_2|) \frac{S_2}{|S_2|}. \end{aligned} \quad (8)$$

The algebraic equations (8) reflect an equality between elastic forces from the deflected shaft and restoring forces  $F_1(|S_1|) \frac{S_1}{|S_1|}, F_2(|S_2|) \frac{S_2}{|S_2|}$  from deformed bearings. These equations can be considered as constraints equations of coordinates  $S, \gamma$  and  $S_1, S_2$ . Either sets of variables  $(S, \gamma)$  or  $(S_1, S_2)$  can be chosen as generalized coordinates. If we choose the  $(S, \gamma)$ -set, then we obtain differential equations with modified stiffness matrix. If we choose the

$(S_1, S_2)$ -set as generalized coordinates, then we obtain differential equations with modified matrix of inertia. By choosing the  $(S_1, S_2)$ -set we can apply methods and results of the investigation received for rigid and flexible rotors with 4 d.o.f. supported in non-linear elastic bearings in (Pasyukova, 1997, 1998, 2005, 2006). So let  $(S_1, S_2)$  be generalized coordinates.

Equations (8) are linear with respect to  $(S, \gamma)$ . Thus we can find an exact solution:

$$\begin{aligned} S &= \tilde{a}_1(|S_1|) S_1 + \tilde{a}_2(|S_2|) S_2, & \gamma &= -\tilde{b}_1(|S_1|) \frac{S_1}{L} + \tilde{b}_2(|S_2|) \frac{S_2}{L}; \\ \tilde{a}_j &= e_{3-j} + \kappa_{1j} \frac{F_j(|S_j|)}{|S_j|}, & \tilde{b}_j &= 1 + \kappa_{2j} L \frac{F_j(|S_j|)}{|S_j|}; \\ \kappa_{1j} &= c_{11}^* + (-1)^j c_{12}^* e_j L, & \kappa_{2j} &= c_{22}^* e_j L + (-1)^j c_{12}^*, \end{aligned} \quad (9)$$

$C^* = \{c_{lm}^*\}$  – a compliance matrix. Values  $c_{lm}^*$  depend on a way of shaft supporting. They are well-known and reported in many handbooks, for example in Dimentberg (1961).

By substituting solution (9) into equations (7) one can obtain differential equations with respect to  $S_1, S_2$ . Denote the differential operators as

$$\begin{aligned} \mathcal{M}_t(*) &= M \frac{d^2}{dt^2} (*) + (\tilde{\mu}_e + \tilde{\mu}_i) \frac{d}{dt} (*) - i\omega \tilde{\mu}_i (*), \\ \mathcal{J}_t(*) &= J_t \frac{d^2}{dt^2} (*) + (-i\omega J_p + (\tilde{\mu}_e + \tilde{\mu}_i) L^2) \frac{d}{dt} (*) - i\omega \tilde{\mu}_i L^2 (*), \end{aligned} \quad (10)$$

then differential equations (7) can be rewritten as

$$\begin{aligned} \sum_{j=1,2} \left( \mathcal{M}_t(\tilde{a}_j(|S_j|) S_j) + F_j(|S_j|) \frac{S_j}{|S_j|} \right) &= M e \omega^2 \exp(i\omega t), \\ \sum_{j=1,2} (-1)^j \left( \mathcal{J}_t(\tilde{b}_j(|S_j|) S_j) + e_j L^2 F_j(|S_j|) \frac{S_j}{|S_j|} \right) &= (J_t - J_p) L \delta \omega^2 \exp(i\omega t - \varepsilon). \end{aligned} \quad (11)$$

It is convenient to rewrite the system (11) in non-dimensional form by introducing non-dimensional variables and non-dimensional time

$$s_j = S_j/h, \quad \tau = \omega_0 t. \quad (12)$$

Here  $h$  is a certain small length, e.g., the static eccentricity  $e$  or  $L \delta$ . The choice of  $\omega_0$  depends on the nonlinearity  $F_j(|S_j|)$ .

Operators (10) with respect to non-dimensional time are reduced to

$$\begin{aligned} \mathcal{M}_\tau(*) &= \frac{d^2}{d\tau^2} (*) + (\mu_e + \mu_i) \frac{d}{d\tau} (*) - i\Omega \mu_i (*), \\ \mathcal{J}_\tau(*) &= \frac{d^2}{d\tau^2} (*) + (-i\Omega \lambda + k l (\mu_e + \mu_i)) \frac{d}{d\tau} (*) - i\Omega k l \mu_i (*). \end{aligned} \quad (13)$$

Here the following denotations have been introduced

$$\Omega = \frac{\omega}{\omega_0}, \quad \lambda = \frac{J_p}{J_t}, \quad l = 1 - \lambda, \quad k = \frac{M L^2}{J_t - J_p}, \quad \mu_e = \frac{\tilde{\mu}_e}{M \omega_0}, \quad \mu_i = \frac{\tilde{\mu}_i}{M \omega_0}. \quad (14)$$

Then equations (11) can be obtained in form

$$\begin{aligned} \sum_{j=1,2} \left( \mathcal{M}_\tau(a_j(|s_j|) s_j) + f_j(|s_j|) \frac{s_j}{|s_j|} \right) &= d_1 \Omega^2 \exp(i\Omega \tau), \\ \sum_{j=1,2} (-1)^j \left( \mathcal{J}_\tau(b_j(|s_j|) s_j) + k l e_j f_j(|s_j|) \frac{s_j}{|s_j|} \right) &= l d_2 \Omega^2 \exp(i(\Omega \tau - \varepsilon)). \end{aligned} \quad (15)$$

Non-dimensional statical and dynamic eccentricities  $d_1, d_2$  and non-dimensional nonlinear forces  $f_j(|s_j|)$  can be computed as

$$d_1 = \frac{e}{h}, \quad d_2 = \frac{L\delta}{h}, \quad f_j(|s_j|) = \frac{1}{h M \omega_0^2} F_j(h|s_j|).$$

We denote

$$\begin{aligned} a_j(|s_j|) &= e_{3-j} + \sigma_{1j} \frac{f_j(|s_j|)}{|s_j|}, & \sigma_{1j} &= \kappa_{1j} M \omega_0^2, \\ b_j(|s_j|) &= 1 + \sigma_{2j} \frac{f_j(|s_j|)}{|s_j|}, & \sigma_{2j} &= \kappa_{2j} L M \omega_0^2, \quad j = 1, 2. \end{aligned} \quad (16)$$

One can see that equations (15) admit an exact solution in a form

$$s_j = R_j \exp(i\varphi_j) \exp(i\Omega\tau), \quad j = 1, 2, \quad (17)$$

which is a forward synchronous circle precession. The forward synchronous precession can be cylindrical, conic or hyperboloidal according to the surface traced by the straight line  $Q_1 Q_2$  in 3D space. If  $\varphi_1 = \varphi_2$  and  $R_1 = R_2$ , then (17) represents a cylindrical precession. If  $\varphi_1 = \varphi_2$  or  $\varphi_1 = \varphi_2 + \pi$  for  $\forall R_1, R_2$ , it is a conic precession, and if  $\varphi_1 \neq \varphi_2$  for  $\forall R_1, R_2$ , it is a hyperboloidal one. In the last case the surface traced by  $Q_1 Q_2$  is one-sheet hyperboloid (see Pasynkova (2005)).

### 3 Cylindrical precessions

Let the rotor be only statically unbalanced, i.e.  $e \neq 0$  and  $\delta = 0$ . Let us consider the elastic bearings with non-linear characteristics of Hertz's type, so restoring forces are given by formula  $F_j(|S_j|) = c_j |S_j|^{3/2}$ . In this case one can choose

$$h = e, \quad \omega_0^2 = \frac{c_1 \sqrt{e}}{M}.$$

Then the system (15) can be written as

$$\begin{aligned} \sum_{j=1,2} \left( \mathcal{M}_\tau(a_j(|s_j|)s_j) + \nu_j \sqrt{|s_j|} s_j \right) &= \Omega^2 \exp(i\Omega\tau), \\ \sum_{j=1,2} (-1)^j \left( \mathcal{J}_\tau(b_j(|s_j|)s_j) + k l \nu_j e_j \sqrt{|s_j|} s_j \right) &= 0, \end{aligned} \quad (18)$$

where

$$\begin{aligned} a_j(|s_j|) &= e_{3-j} + \sigma_{1j} \sqrt{|s_j|}, & b_j(|s_j|) &= (1 + \sigma_{2j} \sqrt{|s_j|}), \\ \nu_j &= c_j/c_1, & \sigma_{1j} &= \kappa_{1j} c_j \sqrt{e}, & \sigma_{2j} &= \kappa_{2j} L c_j \sqrt{e}, \quad j = 1, 2. \end{aligned} \quad (19)$$

By substituting (17) into (18) one can obtain an inhomogeneous algebraic system with respect to the complex amplitude  $R_j \exp(i\varphi_j)$ :

$$\begin{aligned} A_1 R_1 \exp(i\varphi_1) + A_2 R_2 \exp(i\varphi_2) &= \Omega^2, \\ -B_1 R_1 \exp(i\varphi_1) + B_2 R_2 \exp(i\varphi_2) &= 0, \end{aligned} \quad (20)$$

with coefficients

$$\begin{aligned} A_j &= \nu_j \sqrt{R_j} - a_j(R_j) \Omega^2 + i \mu_e a_j(R_j) \Omega, \\ B_j &= k \nu_j e_j \sqrt{R_j} - b_j(R_j) \Omega^2 + i k \mu_e b_j(R_j) \Omega. \end{aligned} \quad (21)$$

Note that the imaginary parts of  $A_j, B_j$  only depend on the damping coefficient  $\mu_e$  in a linear manner.

It is convenient to denote

$$X = \Omega^2, \quad Y_j = \sqrt{R_j}. \quad (22)$$

Then the system (20) can be considered with respect to  $\exp(i\varphi_j)$  and the solution can be written as

$$\exp(i\varphi_j) = \frac{X B_{3-j}}{Y_j^2 \Delta_\mu}, \quad \Delta_\mu \neq 0, \quad (23)$$

where  $\Delta_\mu$  is a determinant of the system (20). Under  $\mu_e = 0$  the denominator in (23) turns into

$$\Delta = \Re(A_1)\Re(B_2) + \Re(A_2)\Re(B_1), \quad (24)$$

where  $\Re(*)$  – a real part of  $(*)$ .

The surface  $\Delta = 0$  determines a set of non-linear resonances in  $\{X, Y_1, Y_2\}$ -space. The definition of a set of non-linear resonances was introduced in Pasyukova (1997, 2005).

If the rotor is attached at the midspan, then  $e_1 = e_2 = 1/2$ . In this case elements of the compliance matrix  $c_{p,q}^*$ , ( $p, q = 1, 2$ ) are given by well-known formulae (see for example Dimentberg (1961))

$$c_{12}^* = c_{21}^* = 0, \quad c_{11}^* = \frac{L^3}{48EJ}, \quad c_{22}^* = \frac{L}{12EJ}, \quad (25)$$

with  $E$  – Young's modulus,  $J$  – moment of inertia of the shaft's cross-section.

If the bearings are identical, then  $c_1 = c_2$ . One can obtain

$$\sigma_{11} = \sigma_{12} = c_{11}^* c_1 \sqrt{e} = \sigma, \quad \sigma_{21} = \sigma_{22} = c_{22}^* L^2 e_1 c_1 \sqrt{e} = 2\sigma. \quad (26)$$

Under these assumptions one can find out the solution  $s_1 = s_2$ , which presents a cylindrical precession. Then the second equation in (18) is satisfied identically.

There are equalities  $R_1 = R_2 = R$  and  $\varphi_1 = \varphi_2 = \varphi$  for a cylindrical precession. Denote  $Y_1 = Y_2 = Y$  and coefficients  $A_1 = A_2 = A(X, Y)$ ,  $B_1 = B_2 = B(X, Y)$ , where

$$\begin{aligned} A(X, Y) &= Y - a(Y)X + i\mu_e a(Y)\sqrt{X}, \quad a(Y) = \frac{1}{2} + \sigma Y, \\ B(X, Y) &= \frac{k}{2}Y - b(Y)X + i\mu_e b(Y)\sqrt{X}, \quad b(Y) = 1 + 2\sigma Y. \end{aligned} \quad (27)$$

By using a property  $|\exp(i\varphi)| = 1$ , one can obtain amplitude-frequency and phase-frequency responses from the formula (23) as

$$Y^2 |A(X, Y)| = X/2, \quad \tan(\varphi) = -\frac{\Im(A(X, Y))}{\Re(A(X, Y))},$$

or in form

$$Y^2 \sqrt{(Y - a(Y)X)^2 + (\mu_e a(Y))^2 X} = X/2, \quad \tan(\varphi) = -\frac{\mu_e a(Y)\sqrt{X}}{Y - a(Y)X}, \quad (28)$$

From formulae (28) it results, that the self-centering takes place at the large spin speed. The limit value of  $Y_\infty$  under  $X \rightarrow \infty$  is equal to a positive root of the following equation

$$2\sigma Y_\infty^3 + Y_\infty^2 - 1 = 0. \quad (29)$$

It is easy to check that under  $\sigma > 0$  this equation always has one positive and two negative roots. If one returns to dimensional parameters then the value  $Y_\infty > 0$  corresponds to the location of mass center on the  $Oz$ -axis, i.e. on the supports line.

The set of non-linear resonances  $\Delta = 0$  degenerates into a set  $\Re(A(X, Y))\Re(B(X, Y)) = 0$ . In the plane  $(X, Y)$  every resonance curve

$$A = Y - \frac{1}{2}(1 + 2\sigma Y)X = 0, \quad B = \frac{kY}{2} - (1 + 2\sigma Y)X = 0 \quad (30)$$

presents a part of the half-hyperbola with vertical asymptotes  $X = 1/\sigma$  and  $X = k/(4\sigma)$ .

Let us consider a dynamically prolate rotor (as a long rotor), then  $\lambda < 1$  and  $k > 0$ . On Figure 2 the dynamic responses of cylindrical precessions are shown with thick lines, non-linear resonances  $A(X, Y) = 0$ ,  $B(X, Y) = 0$  are shown with thin lines, and the limit value  $Y_\infty = 0.919$  is shown with dashed line.

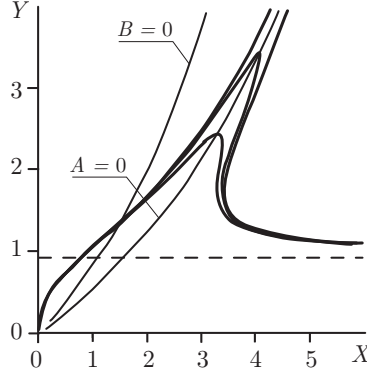


Figure 2: Dynamic responses of cylindrical precessions.  
Parameters are  $k = 2.8$ ,  $\sigma = 0.1$ ,  $\mu_e = 0$  (an unclosed curve),  $\mu_e = 0.1$ ,  $\mu_e = 0.2$ .

As it was shown in Pasyukova (1997, 2006), non-linear resonances for a rigid rotor are parameterized by straight lines in the  $(X, Y)$ -plane and under  $\mu_e = 0$  infinite large radii of precession only exist at infinite large angular velocity. For the rotor attached to a flexible shaft unlimited radii of precession can be at a finite angular velocity, as the non-linear resonances have vertical asymptotes. It follows from the assumption on linear behavior of the shaft.

#### 4 Stability loss in the proximity to non-linear resonances

The linear analysis of stability can be applied to study cylindrical precessions. To obtain equations of the linear approximation in the immediate vicinity of the cylindrical precession with parameters  $R$ ,  $\psi$  under the rotor angular velocity  $\Omega$ , one can let

$$s_j = (R + r_j) \exp(i(\psi + \alpha_j)) \exp(i\Omega\tau), \quad (31)$$

where  $r_j$ ,  $\alpha_j$  are the new variables. Note that  $|s_j| = R + r_j$ .

Substituting these values for  $s_j$  into equations (18) we get the equations for perturbed motion. Non-linear terms are expanded into power series in  $r_j$  and  $\alpha_j$ . Then, by confining ourselves to terms of the first order with respect to  $r_j$ ,  $\alpha_j$ , we get the equations of the linear approximation. At first let us write out linear components of the non-linear terms in equations (18):

$$\begin{aligned} \mathcal{L}(a_j(|s_j|) s_j) &= (a_0 r_j + i a_1 \alpha_j) \exp(i\psi) \exp(i\Omega\tau), \\ a_0 &= \frac{1}{2} (1 + 3\sigma\sqrt{R}), \quad a_1 = \frac{1}{2} R(1 + 2\sigma\sqrt{R}), \\ \mathcal{L}(b_j(|s_j|) s_j) &= (b_0 r_j + i b_1 \alpha_j) \exp(i\psi) \exp(i\Omega\tau), \\ b_0 &= 1 + 3\sigma\sqrt{R}, \quad b_1 = R(1 + 2\sigma\sqrt{R}). \end{aligned} \quad (32)$$

Linear components of the non-linear forces are

$$\mathcal{L}(\sqrt{|s_j|} s_j) = \left( \frac{3}{2} \sqrt{R} r_j + i R \sqrt{R} \alpha_j \right) \exp(i\psi) \exp(i\Omega\tau). \quad (33)$$

The system of the first approximation can be written in complex form as

$$\sum_{j=1,2} (\mathcal{L}_{1j} + i \mathcal{L}_{2j}) = 0, \quad \sum_{j=1,2} (-1)^j (\mathcal{L}_{3j} + i \mathcal{L}_{4j}) = 0, \quad (34)$$

or in real form as

$$\begin{aligned} \mathcal{L}_{11} + \mathcal{L}_{12} &= 0, & -\mathcal{L}_{31} + \mathcal{L}_{32} &= 0, \\ \mathcal{L}_{21} + \mathcal{L}_{22} &= 0, & -\mathcal{L}_{41} + \mathcal{L}_{42} &= 0. \end{aligned} \quad (35)$$

By using notation (22) one can compute  $\mathcal{L}_{mj}$ ,  $m = \overline{1,4}$ ,  $j = \overline{1,2}$  as

$$\begin{aligned} \mathcal{L}_{1j} &= a_0 \ddot{r}_j + (\mu_e + \mu_i) a_0 \dot{r}_j + \left( \frac{3}{2} Y - a_0 X \right) r_j - a_1 \sqrt{X} (2 \dot{\alpha}_j + \mu_e \alpha_j), \\ \mathcal{L}_{2j} &= a_0 \sqrt{X} (2 \dot{r}_j + \mu_e r_j) + a_1 \ddot{\alpha}_j + (\mu_e + \mu_i) a_0 \dot{\alpha}_j + (Y^3 - a_1 X) \alpha_j, \end{aligned} \quad (36)$$

$$\begin{aligned}
\mathcal{L}_{3j} &= a_0 \ddot{r}_j + k l (\mu_e + \mu_i) a_0 \dot{r}_j + l \left( \frac{3}{4} k Y - a_0 X \right) r_j - a_1 \sqrt{X} ((1+l) \dot{\alpha}_j + k l \mu_e \alpha_j), \\
\mathcal{L}_{4j} &= a_0 \sqrt{X} ((1+l) \dot{r}_j + k l \mu_e r_j) + a_1 \ddot{\alpha}_j + k l (\mu_e + \mu_i) a_0 \dot{\alpha}_j + l \left( \frac{k}{2} Y^3 - a_1 X \right) \alpha_j.
\end{aligned} \tag{37}$$

An eighth-order characteristic polynomial of the system (35) splits up into two polynomials of the fourth order as in the case of a rigid rotor (see Pasynkova (2006)). We denote these polynomials as  $M(p), N(p)$ . Applying an usual procedure, the coefficients of  $M(p)$

$$\begin{aligned}
M(p) &= m_0 p^4 + m_1 p^3 + \dots + m_4 = 0, \\
m_0 &= a_0 a_1, \quad m_1 = 2(\mu_e + \mu_i) a_0 a_1, \\
m_2 &= (a_0 Y^3 + \frac{3}{2} a_1 Y) + 2 a_0 a_1 X + (\mu_e + \mu_i)^2 a_0 a_1, \\
m_3 &= (\mu_e + \mu_i) (a_0 Y^3 + \frac{3}{2} a_1 Y) + 2 a_0 a_1 (\mu_e - \mu_i) X, \\
m_4 &= (\frac{3}{2} Y - a_0 X)(Y^3 - a_1 X) + a_0 a_1 \mu_e^2 X
\end{aligned} \tag{38}$$

and the coefficients of  $N(p)$  can be computed as

$$\begin{aligned}
N(p) &= n_0 p^4 + n_1 p^3 + \dots + n_4 = 0, \\
n_0 &= b_0 b_1, \quad n_1 = 2 k l (\mu_e + \mu_i) b_0 b_1, \\
n_2 &= \frac{1}{2} k l (b_0 Y^3 + \frac{3}{2} b_1 Y) + (1 + l^2) b_0 b_1 X + k^2 l^2 (\mu_e + \mu_i)^2 b_0 b_1, \\
n_3 &= \frac{1}{2} k^2 l^2 (\mu_e + \mu_i) (b_0 Y^3 + \frac{3}{2} b_1 Y) + 2 k l b_0 b_1 (\mu_e - l \mu_i) X, \\
n_4 &= l^2 (\frac{3}{4} k Y - b_0 X)(\frac{k}{2} Y^3 - b_1 X) + k^2 l^2 b_0 b_1 \mu_e^2 X.
\end{aligned} \tag{39}$$

One can see that the first three coefficients of  $M(p), N(p)$  are positive. Coefficients  $m_3, n_3$  can be negative for sufficiently large values of  $X$  and limited values of  $Y$ , if  $\mu_i > \max(\mu_e, \mu_e/l)$ .

Absolute terms  $m_4, n_4$  do not depend on the internal damping.

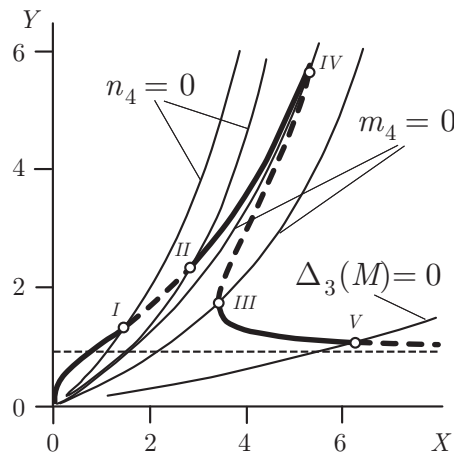


Figure 3: Dynamic response and a set of bifurcation. Parameters are  $k = 2.8, l = 0.7, \sigma = 0.1, \mu_e = 0.03$ .

In the  $(X, Y)$ -plane the curves  $m_4 = 0, n_4 = 0$  present a bifurcation set. The characteristic polynomials have zero-roots at cross-section points of the dynamic response and bifurcation curves. These points are critical and determine intervals of instability. On Figure 3 they are shown as  $I, II, III, IV$  and their coordinates are

$$\begin{cases} X_I = 1.54, \\ Y_I = 1.38, \end{cases} \quad \begin{cases} X_{II} = 2.88, \\ Y_{II} = 2.38, \end{cases} \quad \begin{cases} X_{III} = 3.42, \\ Y_{III} = 1.73, \end{cases} \quad \begin{cases} X_{IV} = 5.44, \\ Y_{IV} = 5.95. \end{cases}$$



On Figure 3 the stable modes of the cylindrical precession are shown with a thick solid line, unstable modes are shown with thick dashed lines. Curves  $m_4 = 0$ ,  $n_4 = 0$  are shown with thin solid lines.

Bifurcation of the critical points could be stiff or soft (see Arnold (1978); Neimark and Landa (1987)). The stiff bifurcation and phenomenon of amplitude jump is observed at the cross-section points of the dynamic response and the curve  $m_4 = 0$ , but the precession keeps on to be cylindrical. Note that under  $\mu_e = 0$  one half of the hyperbola  $m_4 = 0$  coincides with the non-linear resonance  $A = 0$ .

At the cross-section points of the dynamic response and the curve  $n_4 = 0$  the soft bifurcation takes place. Stability loss is accompanied by detachment of two new points of stable relative equilibrium and the type of stable precession changes to hyperboloidal one. The formulae (23) yield

$$\exp(i(\varphi_1 - \varphi_2)) = \frac{Y_2^2 B_2}{Y_1^2 B_1} \Rightarrow \tan(\varphi_1 - \varphi_2) = \frac{k^2 \mu_e \sqrt{X} Y_2^2 (Y_1 - Y_2)}{Y_1^2 \Re(B_2 \bar{B}_1)}. \quad (40)$$

If  $Y_1 \neq Y_2$ , then value  $\varphi_1 - \varphi_2$  is neither equal to 0, nor  $\pi$ . It confirms that new steady-state motion is a hyperboloidal precession.

To find parameters of the complex amplitude for a fixed  $X$  from the instability interval  $n_4 < 0$  one can solve the equations with respect to  $Y_1, Y_2$ :

$$|\exp(i(\varphi_1 - \varphi_2))| = 1, \quad |\exp(i\varphi_1)| = 1. \quad (41)$$

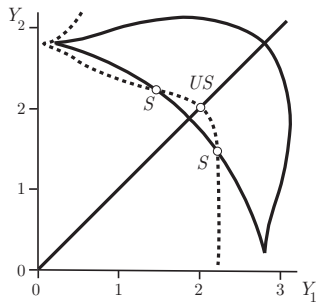


Figure 4: Location of stable and unstable points of relative equilibrium in the  $(Y_1, Y_2)$ -plane under  $X = 2.3$ .

On Figure 4 the curve  $|\exp(i(\varphi_1 - \varphi_2))| = 1$  is shown with a solid line. It consists of a bisectrix and a curve symmetrical with respect to the bisectrix. The curve  $|\exp(i\varphi_1)| = 1$  is shown with a dashed line. The parameters are the same ones as on Figure 3. An unstable cylindrical precession is parameterized by the point located on the bisectrix and its coordinates are  $Y_1 = Y_2 = 1.99$ . The coordinates of two stable points parameterizing a hyperboloidal precession are  $(Y_1 = 2.22, Y_2 = 1.22)$  and  $(Y_1 = 1.22, Y_2 = 2.22)$ .

## 5 Stability loss at the supercritical range of angular velocities. Results of numerical integration.

As it was shown in Pasyukova (2006) in case of a rigid rotor with 4 d.o.f. internal damping caused instability of self-centering and appearance of autovibration. In case of a flexible shaft one can observe similar situation. The first three coefficients (38), (39) of each polynomial  $M(p)$ ,  $N(p)$  are always positive. It can be easily checked that the two first Hurwitz's determinants are positive, too. If internal damping is significant, the coefficients  $m_3$ ,  $n_3$  could become negative for sufficiently large values of  $X$  and limited values of  $Y$ , but  $m_4 > 0$ ,  $n_4 > 0$  for these values of  $X$ ,  $Y$ . This fact indicates that self-centering cannot be stable. Moreover, if  $m_3 < 0$  and  $m_4 > 0$ , then the Hurwitz's determinant of third order  $\Delta_3(M) < 0$ ; likewise if  $n_3 < 0$  and  $n_4 > 0$ , then  $\Delta_3(N) < 0$  and modes of cylindrical precession for large values of  $X$  become unstable.

Let us consider a supercritical range of angular velocities. If  $\Delta_3(M) = 0$  and  $\Delta_3(N) = 0$ , then the polynomials  $M(p)$ ,  $N(p)$  have a pair of pure imaginary roots. In the  $(X, Y)$ -plane these curves determine a bifurcation set. While passing through the critical points of cross-section the dynamic response and the curves  $\Delta_3(M) = 0$  and  $\Delta_3(N) = 0$  a Hopf's bifurcation happens. It could be super- or subcritical bifurcation (see Gilmore (1981)). For the parameters corresponding to Figure 3 coefficient  $n_3 > 0$  and  $\Delta_3(N) > 0$  for all  $X > 0$ . The bifurcation set  $\Delta_3(M) = 0$  is shown with thin solid line on Figure 3. The coordinates of point  $V$  are  $\{x_V = 6.19, y_V = 1.07\}$ .

In case of a dynamically prolate rotor numerical computation reveals a supercritical bifurcation, i.e. stability loss is accompanied with detachment of a stable limit cycle.

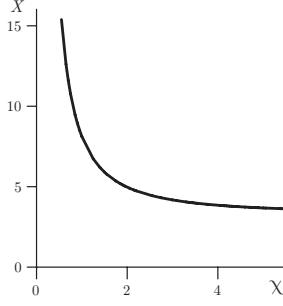


Figure 5: The threshold of autovibration excitation.

For Jeffcott rotor with 2 d.o.f. supported in linear elastic bearings Smith (1933) showed that a threshold angular velocity only depends on a ratio of  $\mu_i/\mu_e$  and performed a simple formula. For Jeffcott rotor with 4 d.o.f. supported in non-linear elastic bearings the threshold of autovibration excitation can be found as a positive solution of equations:

$$Y^2 |A(X, Y)| = X/2, \quad \Delta_3(M) = 0. \quad (42)$$

Numerical solving of the system (42) practically confirms the Smith's result. For chosen parameters the plot  $X_{th} = X_{th}(\chi)$  with  $\chi = \mu_i/\mu_e$  is presented on Figure 5.

As a result of computational study of autovibration the stable limit cycles corresponding to cylindrical precession were found up to sufficiently large values  $X$ . At least, the upper bound for existence of autovibration was not specified.

On Figure 6 the limit cycles are shown on the  $(R_1, \dot{R}_1)$ -plane of the 6-dimensional phase space  $\{R_1, \dot{R}_1, R_2, \dot{R}_2, \varphi_1 - \varphi_2, \dot{\varphi}_1 - \dot{\varphi}_2\}$ . The limit cycle on the  $(R_2, \dot{R}_2)$ -plane is the same one. And the phase difference  $\varphi_1 - \varphi_2 = 0$ , that confirms a cylindrical type of precession.

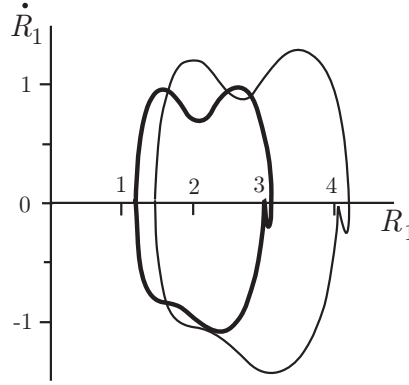


Figure 6: Limit cycles for the shaft's end  $Q_1$  (thick line) and for the point  $Q$  (thin line) under  $X = 6.6$ . Parameters are  $k = 2.8, l = 0.7, \sigma = 0.1, \mu_e = 0.03, \mu_i = 0.042$ .

It is important to note that a stable limit cycles is excited always if the initial position of the rotor is close enough to an element of cylinder, for example, initial conditions could be  $R_1 = R_2$  and other variables are equal to zero at  $\tau = 0$ . The initial conditions for the limit cycle on Figure 6 are  $R_1 = R_2 = 0.75, \dot{R}_1 = \dot{R}_2 = 0, \varphi_1 = \varphi_2 = 0, \dot{\varphi}_1 = \dot{\varphi}_2 = 0$ .

In computational modeling it has been revealed that the autovibration becomes sensitive to variation of initial conditions in a narrow interval of  $X$  close to the threshold value. The autovibration loses its stability and chaoticization of limit cycles happens. Close to the upper bound of this interval limit cycles with double-loop can be excited.

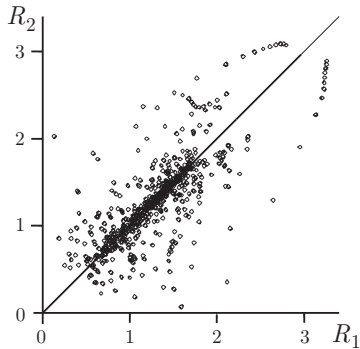


Figure 7: Poincaré's map in the plane  $\{R_1, R_2\}$  ( $\tau = 0[\text{mod}(2\pi/\Omega)]$ ).

If we change the initial values to  $R_1 = 0.75, R_2 = 3.75$ , then the chaotic revolution can be observed. Numerical integration of the system (18) has been done in the time interval corresponding to 700 revolutions of the rotor. A Fehlberg — Runge — Kutta method of fourth-fifth order was used.

The Poincaré's map confirms the chaos. A strange attractor is shown in a section of the 6-dimensional phase space  $\{R_1, \dot{R}_1, R_2, \dot{R}_2, \varphi_1 - \varphi_2, \dot{\varphi}_1 - \dot{\varphi}_2\}$  by the plane  $\{R_1, R_2\}$  ( $\tau = 0[\text{mod}(2\pi/\Omega)]$ ) on Figure 7.

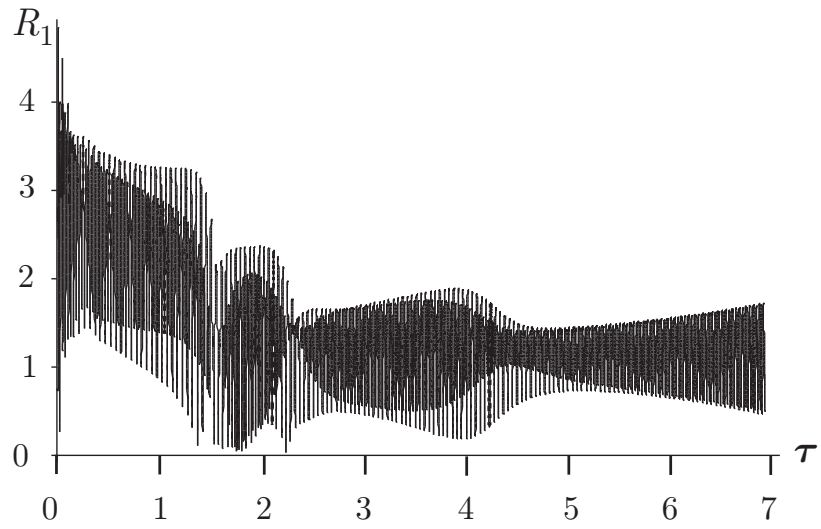


Figure 8: Time history of  $R_1$ .

On Figure 8 time history  $R_1 = R_1(\tau)$  is shown for the whole period of integration. One can clearly see four stages of different behavior of  $R_1$ . And in each stage the motion is chaotic as we cannot see any periodic pattern. Phase trajectories in the  $\{R_1, \dot{R}_1\}$ -plane are presented on Figure 9. One can see a transition "chaos to chaos" in time (see Neimark and Landa (1987)).

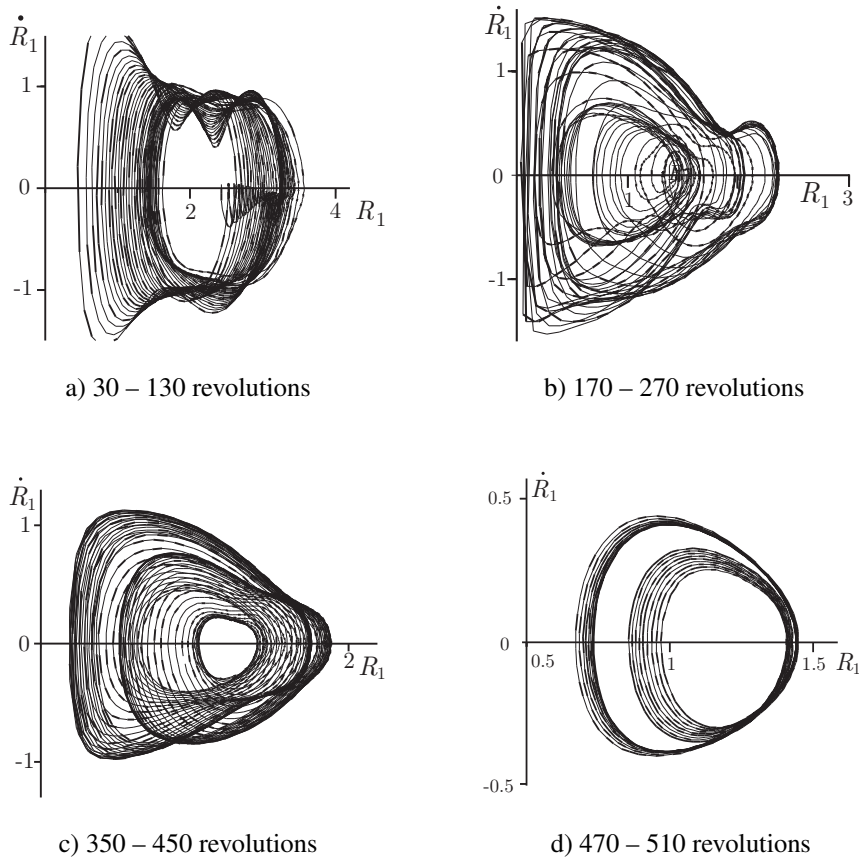


Figure 9: Phase trajectories for different periods of time.

The Lyapunov exponents are shown on Figure 10. They were computed as indicated in Parker and Chua (1989). The initial point corresponds to 100 revolutions of rotor, the step is equal to 15 revolutions, 300 iterations have been made. The Lyapunov exponents spectrum has a signature as  $(+, 0, 0, -, -, -, -)$ , which is typical for chaos. The values of  $\Lambda$  in the last point are

$$\Lambda = \{0.003, 0.000, -0.001, -0.028, -0.047, -0.048, -0.072, -0.072\}.$$

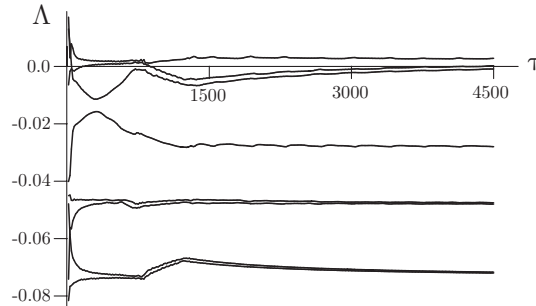


Figure 10: Lyapunov exponents.

Under chosen parameters a phenomenon of chaotization of Hopf's limit cycles was not revealed for  $X > 7$ . A special interest is to study the behavior of  $R_1$  while  $X$  is approaching to the bound  $X \approx 7$ . If initial conditions are symmetrical, i.e. close to a cylinder, one-loop limit cycle appears (see Figure 11, a). If initial conditions are non-symmetrical, no strange attractor appears but the double-loop limit cycle does (see Figure 11, b). The limit cycle in the  $(\varphi_1 - \varphi_2, \dot{\varphi}_1 - \dot{\varphi}_2)$ -plane is shown on Figure 12, a. On Figure 12, b a function  $R_1 = R_1(\tau)$  corresponding to the double-loop cycle is shown for three period time.

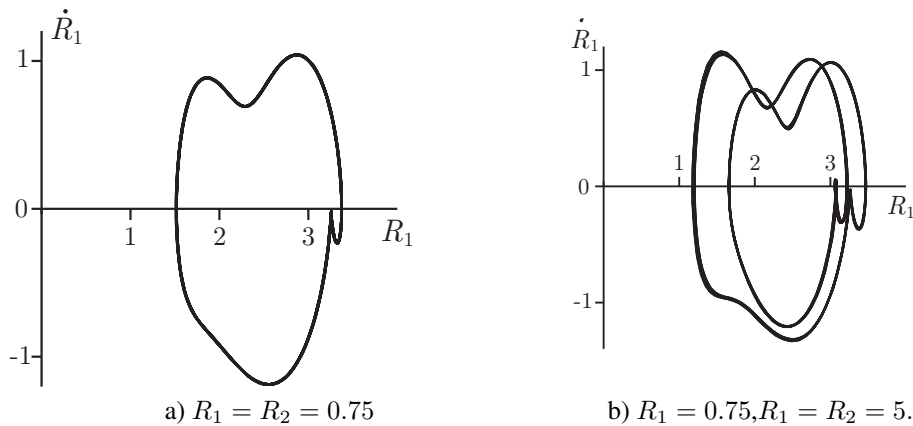


Figure 11: One-loop and double-loop Hopf's limit cycle under  $X = 6.9$ .

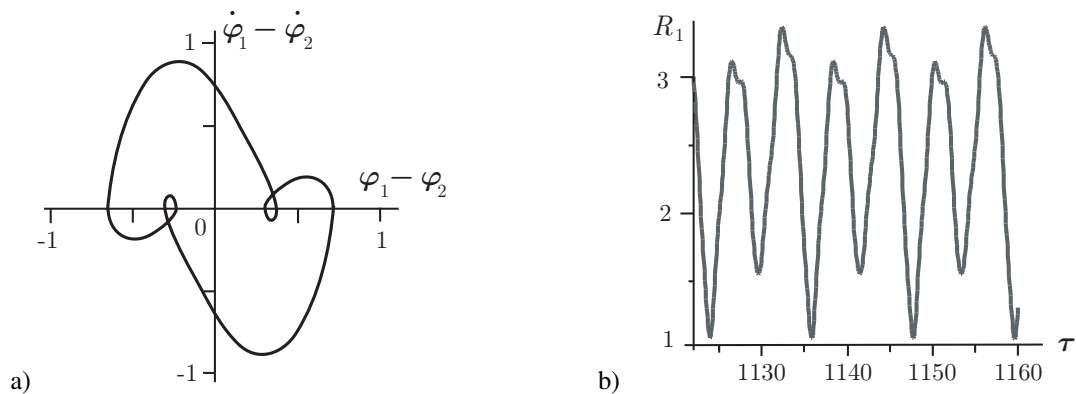


Figure 12: a) Limit cycle with respect to  $(\varphi_1 - \varphi_2, \dot{\varphi}_1 - \dot{\varphi}_2)$ ; b)  $R_1 = R_1(\tau)$  for three period time. Initial conditions  $R_1 = 0.75, R_1 = R_2 = 5.75; X = 6.9$ .

## 6 Conclusions

A model of the precessional motion of an unbalanced rotor has been developed. A notion of an hyperboloidal precession of Jeffcott rotor with 4 d.o.f. has been introduced. The hyperboloidal precession can be considered as geometrical sum of a cylindrical precession, which is a consequence of mass center motion, and a conic precession with respect to the center of mass. Actually, it is necessary to define eight parameters to describe a current state of the rotor. But a special choice of generalized coordinates allows to reduce a system of equations and make a clear interpretation of intricate rotation of the rotor using notions of cylindrical, conic or hyperboloidal precession.

Notion of a set of non-linear resonances has been determined as concept generalization of a resonance (or critical) angular velocity. It allows to investigate rotor dynamics problem with essentially non-linear characteristics of rotor support.

Precessions of a statically unbalanced rotor supported in bearings with Hertzian contact have been studied in details. Dynamic response and two non-linear resonances of a cylindrical precession have been obtained in analytical form and plotted in the plane. Stability conditions of a cylindrical precession have been investigated within a full range of angular velocity variation. In fact one-parameter bifurcation problem with the rotor angular velocity as a parameter has been solved. Instability occurs in close proximity to non-linear resonances. Different scenarios of stability loss were found out. While the parameter passing through the bifurcation value, stability loss could happen without any change of a precession type. In this case the classical phenomenon of "amplitude jump" and bi-stability was revealed. But in other situations stability loss could be accompanied by detachment of two new stable points of relative equilibrium, which are hyperboloidal. Consequently stable cylindrical precessions were converted into stable hyperboloidal ones.

By numerical integration it has been found that at the supercritical angular velocities Hopf's limit cycles could appear and in a narrow range of angular velocities close to the threshold value chaotization of stable limit cycles could take place. To confirm the chaotic character of rotation the Poincaré's map and the Lyapunov exponents spectrum have been determined. Limit cycles for parameters of complex amplitudes within different range of rotor angular velocities as strange attractors and double-loop limit cycles have been presented.

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