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Tilt Angles

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Tilt angles, in contrast to Euler angles, tilt the body frame first about the line of nodes which is defined in the equator-plane of a reference frame. Thereafter only the proper rotation of the body frame takes place, through the angle φ about an axis perpendicular to its own equator-plane. Thus, the two motions are clearly separated, their order is exchangeable. The tilt is described either in polar form (χ,ψ), where χ determines the direction of the node line and ψ the magnitude of the tilt, or in axial form (ψ_1,ψ_2), where ψ_1,ψ_2 together determine node line and tilt by rotations about two perpendicular axes. When the tilt is visualized by geographical coordinates on a sphere, (χ,ψ,φ) are regular except for the N- and the S-pole, whereas the (ψ_1,ψ_2,φ) are regular, separately, on the northern and the southern hemisphere. Together the three charts provide an atlas for unlimited rotations about a fixed point. The basic relations for tilts are established for Cartesian and cylinder coordinates. Two examples demonstrate their versatility.

1 Introduction

It is difficult to select angular coordinates which are appropriate for the study of nonlinear oscillations, say, of a rotor with an axis which sways at a moderate swing. Euler angles are frequently applied but become singular at the nominal position of the rotor. Commonly, the singularities are avoided by consecutive rotations about three, respectively, orthogonal axes. In matrix notation (details below) they read $\underline{\underline{R}}_3\,\underline{\underline{R}}_2\,\underline{\underline{R}}_1$, whereas Euler's angles obey $\underline{\underline{R}}_3\,\underline{\underline{R}}_1\,\underline{\underline{R}}_3$, or $\underline{\underline{R}}_3\,\underline{\underline{R}}_2\,\underline{\underline{R}}_3$, see Goldstein (1980). The particular order of the rotations depends on the field of application, see e.g. Lur'é (1968). Widespread are the Cardan (or Tait-Bryan) angles, cf. Goldstein(1980), Appendix B. However, the order of the three orthogonal rotations affects the structure of the nonlinear equations: Series expansions with respect to small angles, for example, change their structure when the order of the rotations is permuted. (These problems vanish in the linear case.)

For a rotor suspended in gimbals, of course, the rotations are measured best by Cardan angles because they model the suspension directly. Such holds in general: Mathematical and kinematical characteristics of coordinates which model the type of the suspension directly mirror the characteristics of that suspension, thus, they are the appropriate ones. Consequently, for a suspension which is 'round' with respect to the axis of rotation the angles which measure the inclination should enter the model as neutral or symmetrical as possible. Several symmetry conditions are satisfied when the rotations are described in a formal way, for example via Rodrigues' formula, or quaternions, see Hamel (1949), and by 'Euler's parametric specification of rotations about a point' – a step in the direction to quaternions – briefly explained in Whittaker (1964). But the engineer, as a rule, prefers the geometrical way of describing rotations. A brief synopsis and comparison of various descriptions, also from the numerical point of view, is given in Géradin et al. (2001).

Tilt angles are introduced in Brommundt (1993) for Cartesian bases. They specify the inclination of the axis (of rotation) by its orthographic projections onto two planes as in a technical drawing. Then, the inclination can be understood as resulting from tilts about axes normal to the two planes. The rotation proper and the tilt are separated. This 'axial form' of the tilt angles is regular at the zero position.

Here we review the results from 1993 but put the tilt angles in a more general context and extend them to cover the general rotation of a body frame about a fixed point in the form of an atlas of three charts. The details of the angles and the corresponding angular velocities are outlined for Cartesian and for cylinder coordinates. Two examples help to get insight and show the versatility of the method.

The tilt angles are introduced in Section 2 which starts with some elementary facts needed for a consistent reasoning. Section 3 gives the details of the rotational transforms for Cartesian coordinates, Section 4 elaborates the transforms for cylinder coordinates. Section 5 presents the examples.

2 Frames and Tilt Angles

2.1 The Reference Frame

Let the rectangular basis $(\mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$ of unit vectors $\mathbf{e}_{0k}$ together with the fixed point O, the origin, define the reference frame $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$, see Figure 1. For some explanations we apply terms from geography: We look at the circular disk in the plane $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02})$ of Figure 1 as 'equator-plane' $\mathbf{E}_0$ (of a unit sphere around O), the unit vector $\mathbf{e}_{03}$ coincides with the 'polar axis', directed from S (south) to N (north), see Figure 4 below.

With respect to $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$ the vector $\mathbf{x}$ is given by its Cartesian coordinates (x_{01}, x_{02}, x_{03}):

$$\mathbf{x} = \mathbf{e}_{01}\, x_{01} + \mathbf{e}_{02}\, x_{02} + \mathbf{e}_{02}\, x_{03}\,, \tag{1}$$

in matrix notation

$$\mathbf{x} = \underline{\mathbf{e}}_0^T\, \underline{x}_0 = \underline{x}_0^T\, \underline{\mathbf{e}}_0\,, \tag{2}$$

where $\underline{\mathbf{e}}_0$ and $\underline{x}_0$ are the column matrices

$$\underline{\mathbf{e}}_0 := (\mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})^T,\ \underline{x}_0 := (x_{01}, x_{02}, x_{03})^T. \tag{3}$$

Remark 1 (notations): Physical vectors ('arrows') are denoted by bold-face letters, column matrices have a single underscore, square matrices a double underscores. (The elements of matrices can be scalars or physical vectors.)

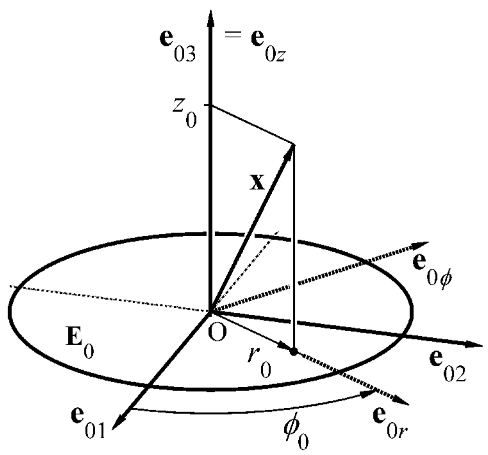

Figure 1. Rectangular frame with equator-plane $\mathbf{E}_0$, and cylinder coordinates (r_0, φ_0, z_0); $\varphi \equiv \phi$.

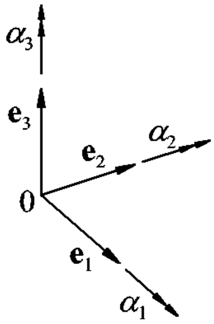

Figure 2. Triad for coordinate rotations, positive senses of the angles α_k.

The orthogonal coordinate or base rotations $\hat{\underline{\mathbf{e}}} = \underline{\underline{R}}_k\, \underline{\mathbf{e}},\ k = 1,2,3,$ facilitate rotations of a triad $\underline{\mathbf{e}} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^T$ about its unit vectors $\mathbf{e}_k$ through the angles α_k, cf. Figure 2, to $\hat{\underline{\mathbf{e}}} = (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)^T$. The rotation matrices $\underline{\underline{R}}_k$ are given by

$$\underline{\underline{R}}_1(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}, \underline{\underline{R}}_2(\alpha_2) = \begin{pmatrix} \cos\alpha_2 & 0 & -\sin\alpha_2 \\ 0 & 1 & 0 \\ \sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}, \underline{\underline{R}}_3(\alpha_3) = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Remark 2: The positive sense of all angles, indicated by the twin arrows in Figure 2, obeys the right-hand rule.

Cylinder coordinates are defined by the triad

$$\underline{\mathbf{e}}_{0C} = (\mathbf{e}_{0r}, \mathbf{e}_{0\varphi}, \mathbf{e}_{0z})^T, \tag{5}$$

which is rotated with respect to the triad $\underline{\mathbf{e}}_0$ through the angle φ_0 about $\mathbf{e}_{03}$, see Figure 1,

$$\underline{\mathbf{e}}_{0C} = \underline{\mathbf{e}}_{0C}(\varphi_0) = \underline{\underline{R}}_3(\varphi_0)\, \underline{\mathbf{e}}_0 . \tag{6}$$

Remark 3: In the plane $\mathbf{E}_0$ the origin O and the line $(\mathrm{O}, \mathbf{e}_{01})$ fix pole and reference direction of the polar coordinates (r_0, φ_0).

With respect to the cylinder coordinates (r_0, φ_0, z_0) the vector $\mathbf{x}$ of Figure 1 is given by

$$\mathbf{x} = \mathbf{e}_{0r}(\varphi_0)\, r_0 + \mathbf{e}_{0z}\, z_0, \tag{7}$$

in matrix notation

$$\mathbf{x} = \underline{\mathbf{e}}_{0C}^T\, \underline{x}_{0C}, \;\; \underline{x}_{0C} := (r_0, 0, z_0)^T. \tag{8}$$

The transitions from (1) to (7) and vice versa read

$$(x_{01}, x_{02}, x_{03}) = (r_0 \cos\varphi_0, r_0 \sin\varphi_0, z_0), \tag{9}$$

$$(r_0, \varphi_0, z_0) = (\sqrt{x_{01}^2 + x_{02}^2}, \arctan(x_{02}/x_{01}), x_{03}). \tag{10}$$

In (10) the quadrant of $\arctan(x_{02}/x_{01})$ has to be chosen such that (9) is satisfied with correct signs.

2.2 Rotated Frame and Tilt Angles in Polar Form

For the frame $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, briefly called body frame, we apply the same notations as above but drop the subscript 0. (To the body frame its own equator-plane $\mathbf{E}$ is attached, similar as in Fig.1.)

Starting from initial coincidence of $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$ and $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, let the frame $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ rotate about the fixed point O. To specify its new position $(\mathrm{O}, \mathbf{e}_{P1}, \mathbf{e}_{P2}, \mathbf{e}_{P3})$ – or orientation – we proceed in three steps: First, we choose in the plane $\mathbf{E}_0$ of the reference frame a node line about which the plane $\mathbf{E}$ of the body frame is going to tilt. We identify the node line with the unit vector $\mathbf{e}_N$ which is determined by the node angle χ in the equator-plane $\mathbf{E}_0$ as shown in Figure 3:

$$\mathbf{e}_N = \mathbf{e}_{01} \cos\chi + \mathbf{e}_{02} \sin\chi. \tag{11}$$

The vector $\mathbf{e}_{0\perp}$, orthogonal to $\mathbf{e}_N$ in the plane $\mathbf{E}_0$, will be needed below:

$$\mathbf{e}_{0\perp} = \mathbf{e}_{03} \times \mathbf{e}_N = \mathbf{e}_{01} \cos(\chi + \pi/2) + \mathbf{e}_{02} \sin(\chi + \pi/2). \tag{12}$$

Secondly, the body frame is tilted from its initial position through the inclination angle ψ about the node line $\mathbf{e}_N$. During this rotation the node line remains fixed with respect to the plane $\mathbf{E} = \mathbf{E}_T$ of the tilted body frame:

$$\mathbf{e}_N = \mathbf{e}_{T1} \cos\chi + \mathbf{e}_{T2} \sin\chi, \tag{13}$$

see Figure 3. But the vector $\mathbf{e}_{T\perp}$, orthogonal to $\mathbf{e}_N$ in $\mathbf{E}$, $\mathbf{e}_{T\perp} = \mathbf{e}_{T3} \times \mathbf{e}_N$, that is

$$\mathbf{e}_{T\perp} = \mathbf{e}_{T1}\cos(\chi + \pi/2) + \mathbf{e}_{T2}\sin(\chi + \pi/2) = -\mathbf{e}_{T1}\sin\chi + \mathbf{e}_{T2}\cos\chi, \tag{14}$$

is tilted with the triad $\underline{\mathbf{e}}_T = (\mathbf{e}_{T1}, \mathbf{e}_{T2}, \mathbf{e}_{T3})^T$. Figure 3 points out the angles (χ, ψ) in different ways.

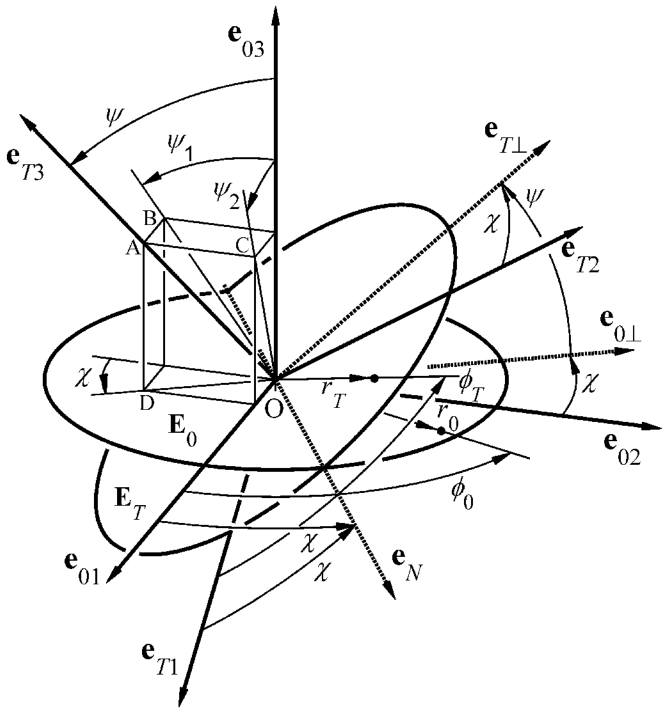

Figure 3. Reference frame $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$, tilted frame $(\mathrm{O}, \mathbf{e}_{T1}, \mathbf{e}_{T2}, \mathbf{e}_{T3})$, node line, tilt angles etc.

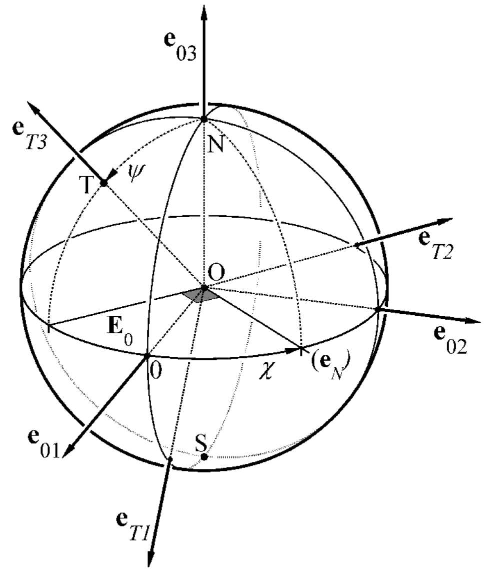

Figure 4. Geographical representation of the tilt on a sphere, centred at O, by the trace point T: geographical length (χ-π/2), pole distance ψ.

The (overall) rotation is completed, thirdly, by $\underline{\underline{R}}_R$, the rotation proper, of the triad $\underline{\mathbf{e}}_T$ about its unit vector $\mathbf{e}_{T3}$ through the angle φ. There results the position $\underline{\mathbf{e}}_P$:

$$\underline{\mathbf{e}}_P = \underline{\underline{R}}_R(\varphi)\,\underline{\mathbf{e}}_T, \text{ where } \underline{\underline{R}}_R(\varphi) = \underline{\underline{R}}_3(\varphi). \tag{15}$$

To render the transitions $\underline{\mathbf{e}}_0 \leftrightarrow \underline{\mathbf{e}}_P$ precisely, the inclination angle ψ is restricted to

$$0 < \psi < \pi, \tag{16}$$

whereas the node angle, χ, and the rotation proper, φ, are not limited but enter the transform via 2π-periodic functions. Therefore it is convenient to assume

$$0 \le \chi, \varphi < 2\pi, \tag{17}$$

or to select some other intervals of that length. Descriptively, we call the triple (χ, ψ, φ) position or orientation angles. The details of the transformations follow in Section 3.1.

Remark 4: Figure 4 illustrates the tilt (without the rotation proper) on a sphere with centre O in a geographical way: The unit vector $\mathbf{e}_{T3}$ of the tilted body frame intersects the sphere at the trace point T, it has the geographical length $(\chi - \pi/2)$ and the pole distance ψ. (The node angle χ measures the geographical length of the intersection point of $\mathbf{e}_N$ with the sphere, indicated by $(\mathbf{e}_N)$ in Figure 4.) Following Bullo et al. (2005), the transition from $(\mathrm{O},\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03})$ to $(\mathrm{O},\mathbf{e}_{P1},\mathbf{e}_{P2},\mathbf{e}_{P3})$ by the orientation angles (χ,ψ,φ) provides a mapping, a chart, which is smooth on the sphere of Figure 4 except for the poles, i.e. the neighbourhoods of the polar axis of the reference frame. Precisely, we will call (χ,ψ,φ) tilt angles in polar form.

Several conclusions (enumerated jointly with the Remarks) follow readily:

Conclusion 5: For the tilt (χ,ψ) as described above, the tilted triad $\underline{\mathbf{e}}_T$ shown in Figure 3 reads in matrix notation:

$$\underline{\mathbf{e}}_T = \underline{\underline{R}}_T(\chi,\psi)\,\underline{\mathbf{e}}_0. \tag{18}$$

Expressed by the base rotations (4), the tilt matrix $\underline{\underline{R}}_T(\chi,\psi)$ gets the form, cf. Figure 3,

$$\underline{\underline{R}}_T(\chi,\psi) = \underline{\underline{R}}_3(-\chi)\,\underline{\underline{R}}_1(\psi)\,\underline{\underline{R}}_3(\chi). \tag{19}$$

The detailed form of the tilt matrix is given in Section 3.1.

Conclusion 6: With (18), (19) the position given by (15) reads

$$\underline{\mathbf{e}}_P = \underline{\underline{R}}_P(\chi,\psi,\varphi)\,\underline{\mathbf{e}}_0, \tag{20}$$

where

$$\begin{aligned} &\underline{\underline{R}}_P(\chi,\psi,\varphi) = \underline{\underline{R}}_R(\varphi)\,\underline{\underline{R}}_T(\chi,\psi), \text{ or} \\ &\underline{\underline{R}}_P(\chi,\psi,\varphi) = \underline{\underline{R}}_3(\varphi)\,\underline{\underline{R}}_3(-\chi)\,\underline{\underline{R}}_1(\psi)\,\underline{\underline{R}}_3(\chi). \end{aligned} \tag{21}$$

Remark 7: Comparison of $(21)_2$ with Euler's angles in matrix form, indicated in the Introduction, shows that $(21)_2$ can be understood as an Euler sequence $\underline{\underline{R}}_3\,\underline{\underline{R}}_1\,\underline{\underline{R}}_3$ with $(\varphi-\chi)$ as angle of the third rotation.

Conclusion 8: The relation $(21)_2$ shows: The succession of the tilt $\underline{\underline{R}}_T(\chi,\psi)$ with respect to the fixed frame $(\mathrm{O},\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03})$ and of the rotation proper $\underline{\underline{R}}_R(\varphi)$ are exchangeable in the following way:

$$\underline{\underline{R}}_P(\chi,\psi,\varphi) = \underline{\underline{R}}_R(\varphi)\,\underline{\underline{R}}_T(\chi,\psi) = \underline{\underline{R}}_T(\chi-\varphi,\psi)\,\underline{\underline{R}}_R(\varphi), \tag{22}$$

cf. $(15)_2$ and (21). This justifies the denomination 'rotation proper' above.

Remark 9: Since the rotation proper, given by (15), is trivial in many cases, we often abbreviate and – instead of (χ,ψ,φ) – call (χ,ψ) tilt angles in polar form or simply 'tilt in polar form'.

2.3 Tilt Angles in Axial Form

The description of the tilt by its polar form (χ,ψ) is convenient for intermediate calculations, see sections 3 to 5. But in the neighbourhood of the poles N and S (cf. Figure 4) it becomes singular and has to be altered:
We start from a geometric construction, the thin-lined box shown in Figure 3: Choose point A arbitrarily on the inclined unit vector $\mathbf{e}_{T3}$. Draw the orthographic projections of the line $\overline{\mathrm{OA}}$ onto the coordinate planes $x_{01}=0$, $x_{02}=0$, $x_{03}=0$, respectively, with box-edges parallel to the unit vectors of the reference triad. This leads to the face-diagonals $\overline{\mathrm{OB}}$, $\overline{\mathrm{OC}}$ and $\overline{\mathrm{OD}}$. On the faces of the box, the tilt angles (χ,ψ) are shown as

$$\chi = \angle(\mathbf{e}_{02},\overline{\mathrm{DO}}), \psi = \angle(\mathbf{e}_{03},\overline{\mathrm{OA}}). \tag{23}$$

But the angles

$$\psi_1 = \angle(\mathbf{e}_{03}, \overline{\mathrm{OB}}), \psi_2 = \angle(\mathbf{e}_{03}, \overline{\mathrm{OC}}) \tag{24}$$

can serve likewise to describe the inclination of $\mathbf{e}_{T3}$, i.e. the position of the tilted (only) triad $\underline{\mathbf{e}}_T$.

These 'axial tilt angles' (ψ_1, ψ_2), as we will call them, have several nice characteristics:

i) they are depicted like in a technical drawing, see Figure 3,
ii) they can be interpreted as consecutive separate or as simultaneous tilts of the body frame $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ about the unit vectors $\mathbf{e}_{01}, \mathbf{e}_{02}$ as axes yielding $(\mathrm{O}, \mathbf{e}_{T1}, \mathbf{e}_{T2}, \mathbf{e}_{T3})$,
iii) the tilts are regular at $(\psi_1, \psi_2) = (0,0)$, see below,
iv) the tilt is interchangeable with the rotation proper (as before).

The transform between the two descriptions of the tilt, its polar form (χ, ψ) and its axial form (ψ_1, ψ_2), are read from the box in Figure 3: With

$$t_1 := \tan\psi_1,\ t_2 := \tan\psi_2 \tag{25}$$

hold

$$t_1 = \tan\psi\ \cos\chi,\ t_2 = \tan\psi\ \sin\chi, \tag{26}$$

and

$$\tan\psi = \sqrt{t_1^2 + t_2^2},\ (\cos\chi, \sin\chi) = \left(t_1 \Big/ \sqrt{t_1^2 + t_2^2}, t_2 \Big/ \sqrt{t_1^2 + t_2^2}\right). \tag{27}$$

From $(27)_1$ and (16) follow

$$\sin\psi = \sqrt{t_1^2 + t_2^2} \Big/ \sqrt{1 + t_1^2 + t_2^2},\ \cos\psi = 1 \Big/ \sqrt{1 + t_1^2 + t_2^2} \tag{28}$$

which will be used frequently as abbreviations for the right hand sides.

The form of the relations (25) to (27) requires the restriction

$$-\pi/2 < \psi_1, \psi_2 < \pi/2. \tag{29}$$

Remark 10: It is not difficult to show that in the overlap region, where (16), (17) and (29) are satisfied, the equations (25) to (27) can be applied to transform the tilt matrix (18), (19):

$$\underline{\underline{R}}_T(\chi, \psi) \xrightleftharpoons[\psi_1 = \psi_1(\chi,\psi),\, \psi_2 = \psi_2(\chi,\psi)]{\chi = \chi(\psi_1,\psi_2),\, \psi = \psi(\psi_1,\psi_2)} \underline{\underline{R}}_T(\psi_1, \psi_2). \tag{30}$$

These transforms as well as the corresponding mappings are smooth (see Ch. 3.2 in Bullo et al. (2005) for the strict criteria). Thus, in the geographical interpretation of Remark 4, Figure 4, the axial tilt angles (ψ_1, ψ_2) provide a chart for the northern hemisphere. (The regularity at $(\psi_1, \psi_2) = (0,0)$ will be seen in Section 3.1.)

In the southern hemisphere we have $\pi/2 < \psi < \pi$, the relations (25) to (27) do no longer hold, axial tilt angles $(\breve{\psi}_1, \breve{\psi}_2)$ for that region have to be defined separately: For $(\breve{\psi}_1, \breve{\psi}_2)$ we want to retain the right-hand rule (see Remark 2) and to place the south pole S at $(\breve{\psi}_1, \breve{\psi}_2) = (0,0)$. A look at Figure 3 and the equations (25) to (27) suggests

$$\psi_1 = \pi + \breve{\psi}_1, \psi_2 = \pi + \breve{\psi}_2, \tag{31}$$

and, instead of (25) to (27), there hold

$$
\begin{aligned}
&\breve{t}_1 := \tan\breve{\psi}_1,\ \breve{t}_2 := \tan\breve{\psi}_2,\\
&\breve{t}_1 = \tan\psi\,\cos\chi,\ \breve{t}_2 = \tan\psi\,\sin\chi,\\
&\tan\psi = -\sqrt{\breve{t}_1^{\,2} + \breve{t}_2^{\,2}}\,,\ (\cos\chi, \sin\chi) = \left(-\breve{t}_1 \Big/ \sqrt{\breve{t}_1^{\,2} + \breve{t}_2^{\,2}}\,, -\breve{t}_2 \Big/ \sqrt{\breve{t}_1^{\,2} + \breve{t}_2^{\,2}}\right).
\end{aligned}
\tag{32}
$$

To the restriction (29) corresponds

$$
-\pi/2 < \breve{\psi}_1, \breve{\psi}_1 < \pi/2. \tag{33}
$$

Conclusion 11: When the inclination ψ passes through $\pi/2$, because of (31), (32) the angles (ψ_1, ψ_2) will jump from values near $(\pm\pi/2, \pm\pi/2)$ at the rim of the northern hemisphere to values near $(\mp\pi/2, \mp\pi/2)$ of the angles $(\breve{\psi}_1, \breve{\psi}_2)$ at the rim of the southern hemisphere, cf. Figure 3 and Figure 4. (There are special cases when one of the axial tilt angles vanishes.)

Remark 12: Similar as in Remark 10, the axial tilt angles $(\breve{\psi}_1, \breve{\psi}_2)$ enter the matrix $\underline{\underline{R}}_T(\breve{\psi}_1, \breve{\psi}_2)$ for the southern hemisphere and provide a chart of it. This chart, together with the charts of Remark 4 and Remark 10, provide an atlas for the general rotation about O, cf. Remark 20.

Remark 13: The above preference of the base vector $\mathbf{e}_{03}$ can removed by an altered enumeration of the $\mathbf{e}_{0k}$.

3 Cartesian Rotations and Angular Velocities

Tilts in axial form and for Cartesian coordinates are developed in Brommundt (1993). Here, we present results from that paper in view of the above considerations. Rotations and angular velocities are treated separately. In general, we will list the equations only for (χ, ψ) and (ψ_1, ψ_2), the northern hemisphere. Section 3.1 deals with the geometric transforms, Section 3.2 with the angular velocities.

3.1 Tilts and Rotations of Cartesian Systems

The equations (20), (21) express the rotation of the body frame $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ from its initial position, coincident with the reference frame $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$, to its final position $(\mathrm{O}, \mathbf{e}_{P1}, \mathbf{e}_{P2}, \mathbf{e}_{P3})$. The tilt part of (21), the matrix $\underline{\underline{R}}_T$, is of main interest since the rotation proper, $\underline{\underline{R}}_R = \underline{\underline{R}}_3$, is simple.

The polar form $\underline{\underline{R}}_T(\chi, \psi)$ follows from (19) by $(4)_1$, $(4)_3$ and some trigonometric manipulation:

$$
\begin{aligned}
\underline{\underline{R}}_T &= \begin{pmatrix} \cos^2\chi + \sin^2\chi\,\cos\psi & \sin\chi\,\cos\chi\,(1-\cos\psi) & -\sin\chi\,\sin\psi \\ \sin\chi\,\cos\chi\,(1-\cos\psi) & \cos^2\chi\,\cos\psi + \sin^2\chi & \cos\chi\,\sin\psi \\ \sin\chi\,\sin\psi & -\cos\chi\,\sin\psi & \cos\psi \end{pmatrix}\\
&= \begin{pmatrix} \cos^2(\psi/2) + \cos(2\chi)\,\sin^2(\psi/2) & \sin(2\chi)\,\sin^2(\psi/2) & -\sin\chi\,\sin\psi \\ \sin(2\chi)\,\sin^2(\psi/2) & \cos^2(\psi/2) - \cos(2\chi)\,\sin^2(\psi/2) & \cos\chi\,\sin\psi \\ \sin\chi\,\sin\psi & -\cos\chi\,\sin\psi & \cos\psi \end{pmatrix}.
\end{aligned}
\tag{34}
$$

The transition (30), together with (27), leads from the polar (34) to the axial form $\underline{\underline{R}}_T(\psi_1, \psi_2)$. There are two equivalent formulae, differing by trigonometric transforms, cf. Brommundt (1993):

$$\underline{\underline{R}}_T = \begin{pmatrix} 1 & 0 & -t_2\cos\psi \\ 0 & 1 & t_1\cos\psi \\ t_2\cos\psi & -t_1\cos\psi & \cos\psi \end{pmatrix} + \frac{\cos^2\psi}{1+\cos\psi}\begin{pmatrix} -t_2^2 & t_1 t_2 & 0 \\ t_1 t_2 & -t_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{35a}$$

$$\underline{\underline{R}}_T = \cos\psi \begin{pmatrix} 1 & 0 & -t_2 \\ 0 & 1 & t_1 \\ t_2 & -t_1 & 1 \end{pmatrix} + \frac{\cos^2\psi}{1+\cos\psi}\begin{pmatrix} t_1^2 & t_1 t_2 & 0 \\ t_1 t_2 & t_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{35b}$$

see (25) and $(28)_2$ for the abbreviations t_1, t_2 and $\cos\psi$.

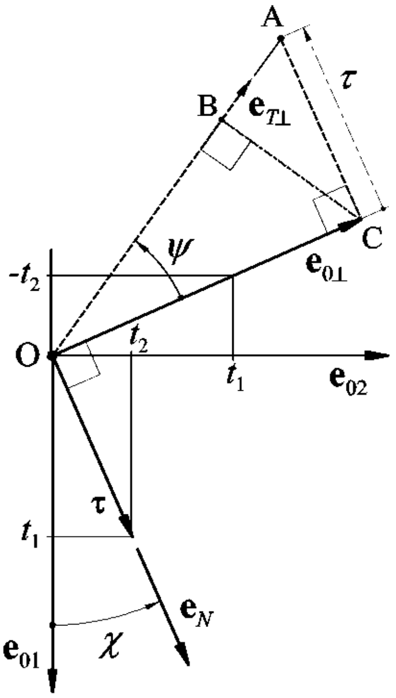

Figure 5. Presentation of (t_1, t_2), $\boldsymbol{\tau}$.

Remark 14: The forms (35) of $\underline{\underline{R}}_T$ suggest to measure the angles (ψ_1, ψ_2) not directly by their magnitude but by their tan-functions (t_1, t_2). This means, the (t_1, t_2), as generalized coordinates, become immediate functions of the time $t: (t_1, t_2) = (t_1(t), t_2(t))$. Then, by (35) and $(28)_2$, the tilt matrix becomes a function of (t_1, t_2): $\underline{\underline{R}}_T = \underline{\underline{R}}_T(t_1, t_2)$.

Conclusion 15: The tilted position of the body frame is easily outlined by means of the coordinates (t_1, t_2). Figure 5 shows the plane $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02})$ of the reference system, drawn true to scale. The tilt vector $\boldsymbol{\tau}$,

$$\boldsymbol{\tau} := t_1\,\mathbf{e}_{01} + t_2\,\mathbf{e}_{02} + 0\cdot\mathbf{e}_{03}, \tag{36}$$

points in the direction of the node line $\mathbf{e}_N$, in matrix notation it reads

$$\boldsymbol{\tau} = \underline{\mathbf{e}}_0^T\,\underline{\tau}, \text{ where } \underline{\tau} := (t_1, t_2, 0)^T. \tag{37}$$

The tilt (column) vector $\underline{\tau}$ is an eigen-vector of $\underline{\underline{R}}_T$, eigen-value = 1, see (35a). The tilt does not affect the distances in the direction of $\mathbf{e}_N$. In Figure 5 indicates

$$\tau = |\boldsymbol{\tau}| = \tan\psi = \sqrt{t_1^2 + t_2^2} = \overline{\mathrm{CA}}, \tag{38}$$

the height of the dashed triangle (folded from the vertical plane), the ascent of $\mathbf{e}_{T\perp}$ with respect to $\mathbf{e}_{0\perp}$, the angle ψ. The ratio

$$\overline{\mathrm{OB}}/1 = \cos\psi = 1\Big/\sqrt{1+\tau^2} \tag{39}$$

represents the constriction of the distances in the direction of $\mathbf{e}_{T\perp}$.

Remark 16: Now we have five ways to view the tilt matrix $\underline{\underline{R}}_T$:

$$\underline{\underline{R}}_T(\chi,\psi) \leftrightarrow \underline{\underline{R}}_T(\psi_1,\psi_2) \leftrightarrow \underline{\underline{R}}_T(t_1,t_2) \leftrightarrow \underline{\underline{R}}_T(\boldsymbol{\tau}) \leftrightarrow \underline{\underline{R}}_T(\underline{\tau}). \tag{40}$$

These matrices are equivalent where their regions of validity overlap on the sphere of Figure 4.

Conclusion 17: The last form of (40) suits well to express the structure of the exchangeability of $\underline{\underline{R}}_R$ and $\underline{\underline{R}}_T$, cf. Conclusion 8:

$$\underline{\underline{R}}_R(\varphi)\,\underline{\underline{R}}_T(\underline{\tau}) = \underline{\underline{R}}_T\left(\underline{\underline{R}}_R(\varphi)\cdot\underline{\tau}\right)\underline{\underline{R}}_R(\varphi). \tag{41}$$

Finally, the rotation proper concludes the transforms, cf. (21):

$$\underline{\underline{R}}_P(\chi,\psi,\varphi) = \underline{\underline{R}}_R(\varphi)\,\underline{\underline{R}}_T(\chi,\psi),\ \underline{\underline{R}}_P(\psi_1,\psi_2,\varphi) = \underline{\underline{R}}_R(\varphi)\,\underline{\underline{R}}_T(\psi_1,\psi_2). \tag{42}$$

With $(15)_2$, $(4)_3$ and

$$c := \cos\varphi,\ s := \sin\varphi, \tag{43}$$

follows from $(42)_1$ and (34) the polar form of the position $\underline{\underline{R}}_P$,

$$\underline{\underline{R}}_P = \begin{pmatrix} c\cdot\cos^2(\psi/2)+\cos(2\chi-\varphi)\sin^2(\psi/2) & s\cdot\cos^2(\psi/2)+\sin(2\chi-\varphi)\sin^2(\psi/2) & -\sin(\chi-\varphi)\sin\psi \\ -s\cdot\cos^2(\psi/2)+\sin(2\chi-\varphi)\sin^2(\psi/2) & c\cdot\cos^2(\psi/2)-\cos(2\chi-\varphi)\sin^2(\psi/2) & \cos(\chi-\varphi)\sin\psi \\ \sin\chi\,\sin\psi & -\cos\chi\,\sin\psi & \cos\psi \end{pmatrix}, \tag{44}$$

and $(42)_2$, together with (35a/b) and $(28)_2$ lead to its axial form

$$\underline{\underline{R}}_P = \begin{pmatrix} c & s & (s\cdot t_1 - c\cdot t_2)\cos\psi \\ -s & c & (c\cdot t_1 + s\cdot t_2)\cos\psi \\ t_2\cos\psi & -t_1\cos\psi & \cos\psi \end{pmatrix} + \frac{\cos^2\psi}{1+\cos\psi}\begin{pmatrix} t_2\,(s\cdot t_1 - c\cdot t_2) & -t_1\,(s\cdot t_1 - c\cdot t_2) & 0 \\ t_2\,(c\cdot t_1 + s\cdot t_2) & -t_1\,(c\cdot t_1 + s\cdot t_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{45a}$$

$$\underline{\underline{R}}_P = \cos\psi\begin{pmatrix} c & s & (s\cdot t_1 - c\cdot t_2) \\ -s & c & (c\cdot t_1 + s\cdot t_2) \\ t_2 & -t_1 & 1 \end{pmatrix} + \frac{\cos^2\psi}{1+\cos\psi}\begin{pmatrix} t_1\,(c\cdot t_1 + s\cdot t_2) & t_2\,(c\cdot t_1 + s\cdot t_2) & 0 \\ -t_1\,(s\cdot t_1 - c\cdot t_2) & -t_2\,(s\cdot t_1 - c\cdot t_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{45b}$$

Remark 18: When an arbitrary position $\underline{\underline{R}}_P$ is given numerically, e.g. in the form

$$\underline{\underline{R}}_P = \begin{pmatrix} r_{P11} & r_{P12} & r_{P13} \\ r_{P21} & r_{P22} & r_{P23} \\ r_{P31} & r_{P32} & r_{P33} \end{pmatrix}, \tag{46}$$

it can be identified as result of a tilt and a rotation proper of the base $\underline{\mathbf{e}}_0$ by a numerical solution of (44), (45), or the corresponding equation for (χ,ψ,φ), for (t_1,t_2,φ), or $(\breve{t}_1,\breve{t}_2,\varphi)$, respectively, depending on the region of the hemisphere of Figure 4. With respect to (χ,φ) this inverse transform is unique only up to integer multiples of 2π, see (17).

Remark 19: For an arbitrary rotation matrix $\underline{\underline{R}}$ follows with $\hat{\underline{\mathbf{e}}} = \underline{\underline{R}}\,\underline{\mathbf{e}}$ from the dyadic scalar product $\underline{\mathbf{e}}\cdot\underline{\mathbf{e}}^T = \underline{\underline{I}}$, $\underline{\underline{I}}$ – identity matrix, the ortho-normality

$$\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}},\ \text{or}\ \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T. \tag{47}$$

3.2 Angular Velocities at the Cartesian System

Let the orientation angles given by one of the above coordinate triples, (χ,ψ,φ), (ψ_1,ψ_2,φ), or (t_1,t_2,φ), depend on the time *t*. From (20) and (36) follow for time dependent orientations by differentiation

$$\dot{\underline{\mathbf{e}}}_P = \dot{\underline{\underline{R}}}_P\,\underline{\mathbf{e}}_0, \tag{48}$$

where

$$\dot{\underline{\underline{R}}}_P = \dot{\underline{\underline{R}}}_R\, \underline{\underline{R}}_T + \underline{\underline{R}}_R\, \dot{\underline{\underline{R}}}_T. \tag{49}$$

From (48), (49) follow by (20), (21), (47) stepwise

$$\dot{\underline{\mathbf{e}}}_P = \dot{\underline{\underline{R}}}_P\, \underline{\mathbf{e}}_0 = \dot{\underline{\underline{R}}}_P\, \underline{\underline{R}}_P^T\, \underline{\mathbf{e}}_P =: \underline{\underline{\Omega}}_P\, \underline{\mathbf{e}}_P, \tag{50}$$

and

$$\begin{aligned} \underline{\underline{\Omega}}_P &= (\dot{\underline{\underline{R}}}_R\, \underline{\underline{R}}_T + \underline{\underline{R}}_R\, \dot{\underline{\underline{R}}}_T)\, \underline{\underline{R}}_T^T\, \underline{\underline{R}}_R^T \\ &= (\dot{\underline{\underline{R}}}_R\, \underline{\underline{R}}_R^T + \underline{\underline{R}}_R\, \dot{\underline{\underline{R}}}_T\, \underline{\underline{R}}_T^T\, \underline{\underline{R}}_R^T) = (\underline{\underline{\Omega}}_R + \underline{\underline{R}}_R\, \underline{\underline{\Omega}}_T\, \underline{\underline{R}}_R^T), \end{aligned} \tag{51}$$

where

$$\underline{\underline{\Omega}}_T := \dot{\underline{\underline{R}}}_T\, \underline{\underline{R}}_T^T,\ \underline{\underline{\Omega}}_R := \dot{\underline{\underline{R}}}_R\, \underline{\underline{R}}_R^T. \tag{52}$$

The, because of $(47)_1$, skew symmetric matrices $\underline{\underline{\Omega}}_P, \underline{\underline{\Omega}}_T, \underline{\underline{\Omega}}_R$ represent the angular velocities

$$\underline{\underline{\Omega}}_P = \begin{pmatrix} 0 & \omega_{P3} & -\omega_{P2} \\ -\omega_{P3} & 0 & \omega_{P1} \\ \omega_{P2} & -\omega_{P1} & 0 \end{pmatrix},\ \underline{\underline{\Omega}}_T = \begin{pmatrix} 0 & \omega_{T3} & -\omega_{T2} \\ -\omega_{T3} & 0 & \omega_{T1} \\ \omega_{T2} & -\omega_{T1} & 0 \end{pmatrix},\ \underline{\underline{\Omega}}_R = \begin{pmatrix} 0 & \dot{\varphi} & 0 \\ -\dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

In vector notation the angular velocities read (note $\mathbf{e}_{R3} = \mathbf{e}_{T3}$):

$$\begin{aligned} \boldsymbol{\omega}_P &= \mathbf{e}_{P1}\,\omega_{P1} + \mathbf{e}_{P2}\,\omega_{P2} + \mathbf{e}_{P3}\,\omega_{P3} = \underline{\mathbf{e}}_P^T \underline{\omega}_P, \quad \text{where } \underline{\omega}_P := (\omega_{P1}, \omega_{P2}, \omega_{P3})^T, \\ \boldsymbol{\omega}_T &= \mathbf{e}_{T1}\,\omega_{T1} + \mathbf{e}_{T2}\,\omega_{T2} + \mathbf{e}_{T3}\,\omega_{T3} = \underline{\mathbf{e}}_T^T \underline{\omega}_T, \quad \text{where } \underline{\omega}_T := (\omega_{T1}, \omega_{T2}, \omega_{T3})^T, \\ \boldsymbol{\omega}_R &= \mathbf{e}_{R3}\,\dot{\varphi}. \end{aligned} \tag{54}$$

By means of the angular-velocity vector $\boldsymbol{\omega}_P$, e.g., the velocity $\dot{\underline{\mathbf{e}}}_P$, cf. (48), can be expressed as cross-product:

$$\dot{\underline{\mathbf{e}}}_P = \boldsymbol{\omega}_P \times \underline{\mathbf{e}}_P, \tag{55}$$

where the factor $(\boldsymbol{\omega}_P \times)$ applies row-wise to the elements of $\underline{\mathbf{e}}_P$.
To express the tilt angular velocities by time derivatives of the tilt angles we look at the composition (19) as three consecutive rotations about $\mathbf{e}_{03}, \mathbf{e}_N, \mathbf{e}_{T3}$ by the angles $\chi, \psi, -\chi$, respectively, in that order. During the infinitesimal time interval Δt the angular velocity $\boldsymbol{\omega}_T$ produces the rotation

$$\boldsymbol{\omega}_T\, \Delta t = \left(\mathbf{e}_{03}\, \dot{\chi} + \mathbf{e}_N\, \dot{\psi} - \mathbf{e}_{T3}\, \dot{\chi}\right) \Delta t. \tag{56}$$

Therefore holds

$$\boldsymbol{\omega}_T = \mathbf{e}_{03}\, \dot{\chi} + \mathbf{e}_N\, \dot{\psi} - \mathbf{e}_{T3}\, \dot{\chi}. \tag{57}$$

Projections of $\mathbf{e}_N, \mathbf{e}_{03}$ onto $\underline{\mathbf{e}}_T$, see (13), (18), lead to the polar form of $(54)_2$:

$$\begin{aligned}\omega_{T1} &= \ \ \dot\psi\cos\chi - \dot\chi\sin\chi\sin\psi,\\ \omega_{T2} &= \ \ \dot\psi\sin\chi + \dot\chi\cos\chi\sin\psi,\\ \omega_{T3} &= -2\,\dot\chi\sin^2(\psi/2) \qquad = -\dot\chi\,(1-\cos\psi).\end{aligned}\tag{58}$$

The transitions between the velocities $(\dot\chi,\dot\psi)$ of the polar tilt, the (generalized) velocities $(\dot t_1,\dot t_2)$ and the velocities $(\dot\psi_1,\dot\psi_2)$ of the axial tilt follow from (25) to (27) by differentiation:

$$\dot t_1 = (1+t_1^2)\,\dot\psi_1\,,\ \dot t_2 = (1+t_2^2)\,\dot\psi_2\,;\tag{59}$$

$$\begin{aligned}\dot\psi &= \cos^2\psi\,(t_1\dot t_1 + t_2\dot t_2)\Big/\sqrt{t_1^2+t_2^2} = \cos^2\psi\,(\ \dot t_1\cos\chi + \dot t_2\sin\chi),\\ \dot\chi &= (t_1\dot t_2 - \dot t_1 t_2)\Big/\left(t_1^2+t_2^2\right) \qquad = \cot\psi\ (-\dot t_1\sin\chi + \dot t_2\cos\chi);\end{aligned}\tag{60}$$

$$\begin{aligned}\dot t_1\cos^2\psi &= \dot\psi\cos\chi - \dot\chi\sin\chi\sin\psi\cos\psi\,,\\ \dot t_2\cos^2\psi &= \dot\psi\sin\chi + \dot\chi\cos\chi\sin\psi\cos\psi\,.\end{aligned}\tag{61}$$

By (27) and (60) follows from (58) the axial form of the angular velocities (cf. Brommundt (1993)):

$$\begin{aligned}\omega_{T1} &= \cos\psi\left(1-\frac{t_1^2\cos^2\psi}{1+\cos\psi}\right)\dot t_1 - \frac{t_1t_2\cos^3\psi}{1+\cos\psi}\,\dot t_2 = \cos^2\psi\left(1+\frac{t_2^2\cos\psi}{1+\cos\psi}\right)\dot t_1 - \frac{t_1t_2\cos^3\psi}{1+\cos\psi}\,\dot t_2\,,\\ \omega_{T2} &= \cos\psi\left(1-\frac{t_2^2\cos^2\psi}{1+\cos\psi}\right)\dot t_2 - \frac{t_1t_2\cos^3\psi}{1+\cos\psi}\,\dot t_1 = \cos^2\psi\left(1+\frac{t_1^2\cos\psi}{1+\cos\psi}\right)\dot t_2 - \frac{t_1t_2\cos^3\psi}{1+\cos\psi}\,\dot t_1\,,\\ \omega_{T3} &= \cos^2\psi\,\frac{t_2\dot t_1 - t_1\dot t_2}{1+\cos\psi}\,.\end{aligned}\tag{62}$$

For $\boldsymbol{\omega}_P$, cf. $(53)_1$, $(54)_1$, (55), follow from (51), (52) by $(53)_{2,3}$, (15) and $(4)_3$

$$\begin{aligned}\omega_{P1} &= \ \ \omega_{T1}\cos\varphi + \omega_{T2}\sin\varphi\,,\\ \omega_{P2} &= -\,\omega_{T1}\sin\varphi + \omega_{T2}\cos\varphi\,,\\ \omega_{P3} &= \ \ \omega_{T3} + \dot\varphi\,.\end{aligned}\tag{63}$$

When the polar form (58) of $\boldsymbol{\omega}_T$ is put into (63) we obtain

$$\begin{aligned}\omega_{P1} &= \ \ \dot\psi\cos(\chi-\varphi) - \dot\chi\sin(\chi-\varphi)\sin\psi,\\ \omega_{P2} &= \ \ \dot\psi\sin(\chi-\varphi) + \dot\chi\cos(\chi-\varphi)\sin\psi,\\ \omega_{P3} &= -2\,\dot\chi\sin^2(\psi/2)+\dot\varphi \qquad = -\dot\chi\,(1-\cos\psi)+\dot\varphi\,.\end{aligned}\tag{64}$$

Not much insight results from putting the axial form (62) of $\boldsymbol{\omega}_T$ into (63).

Remark 20: In Remark 12 we stated that the three forms (χ,ψ,φ), (ψ_1,ψ_2,φ), and $(\breve\psi_1,\breve\psi_2,\varphi)$ of the mapping constitute a (smooth) atlas for the general rotation. (The axial forms (t_1,t_2,φ), and $(\breve t_1,\breve t_2,\varphi)$ are just special variants.) From (35) it is easily seen that (ψ_1,ψ_2,φ) cease to hold at the borders of the regions (29). The matrix (34), as function of (χ,ψ,φ), looks smooth at $\psi = 0$ and $\psi = \pi$, but $(60)_2$ shows the lack of differentiability there. On the other hand it is easily seen that the polar and the axial forms of the tilt are complementary: where one form fails the other takes over.

Remark 21 (historical): Ishlinskii studied the coordinates (x,y) of the 'representative point of a gyroscope', see Ishlinskii (1960), reprinted in Ishlinskii (1986), pp. 233-244. Mutatis mutandis, (x,y) can be identified with the generalized coordinates (t_1,t_2), compare Figure 3 and Figure 5 with Fig.1 of Ishlinskii (1960). But Ishlinskii identifies (x, y) with direction cosines: $(x, y) = (\sin(\angle(\overline{\mathrm{OC}}, \overline{\mathrm{OA}})), \sin(\angle(\overline{\mathrm{OB}}, \overline{\mathrm{OA}})))$, cf. Figure 3. He is not able to solve his equations for the nonlinear case, therefore, he restricts the magnitudes of the acting forces and linearizes.

4 Rotations and Velocities at Cylinder Coordinates

A cylindrical rotor running in a cylindrical casing with some clearance might tilt and rub. Of course, when studying this problem, cylinder coordinates will be most convenient to formulate contact conditions and relative velocities. To solve this and many related problems we need to transfer the ideas of the preceding sections to two frames, tilting with respect to each other, each carrying its own system of cylinder coordinates. What do the – now nonlinear – transforms look like? Section 4.1 deals with the geometrical relations, Section 4.2 with the velocities.

4.1 Tilts and Rotations at Cylinder Coordinates

Let the vector $\mathbf{x}$ of Figure 1 be represented with respect to the reference frame $(\mathrm{O}, \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03})$ of Figure 3 by the cylinder coordinates (r_0, φ_0, z_0) as given by (7) and, simultaneously, with respect to the tilted body frame $(\mathrm{O}, \mathbf{e}_{T1}, \mathbf{e}_{T2}, \mathbf{e}_{T3})$ – shown in the same figure – by the cylinder coordinates (r_T, φ_T, z_T):

$$\mathbf{x} = \mathbf{e}_{0r}(\varphi_0)\, r_0 + \mathbf{e}_{0z}\, z_0 \,, \text{ and } \mathbf{x} = \mathbf{e}_{Tr}(\varphi_T)\, r_T + \mathbf{e}_{Tz}\, z_T \,, \text{ where } \mathbf{e}_{0z} \equiv \mathbf{e}_{03}, \mathbf{e}_{Tz} \equiv \mathbf{e}_{T3}. \tag{65}$$

Let the tilt be given by its polar form (χ, ψ), and let the rotation(s) proper be incorporated into the polar angles φ_0, φ_T, respectively. We want to express (r_T, φ_T, z_T) as functions of (r_0, φ_0, z_0) and (χ, ψ).

The two forms of the vector $\mathbf{x}$ in (65) are equal:

$$\mathbf{e}_{Tr}(\varphi_T)\, r_T + \mathbf{e}_{Tz}\, z_T = \mathbf{e}_{0r}(\varphi_0)\, r_0 + \mathbf{e}_{0z}\, z_0, \tag{66}$$

their projections onto the orthogonal triad $(\mathbf{e}_N, \mathbf{e}_{T\perp}, \mathbf{e}_{T3})$ read, see Figure 3:

$$\begin{aligned}
\mathbf{e}_N &: \quad r_T \cos(\varphi_T - \chi) = \quad r_0 \cos(\varphi_0 - \chi), \\
\mathbf{e}_{T\perp} &: \quad r_T \sin(\varphi_T - \chi) = \quad r_0 \sin(\varphi_0 - \chi) \cos\psi + z_0 \sin\psi, \\
\mathbf{e}_{T3} &: \quad z_T \qquad\qquad\quad = - r_0 \sin(\varphi_0 - \chi) \sin\psi + z_0 \cos\psi.
\end{aligned} \tag{67}$$

The projections hold for inclinations $0 < \psi < \pi$, cf. (16), arbitrary angles χ, φ_0 and coordinate z_0. Both radii, r_0 and r_T, must be non-negative, cf. (10):

$$0 < r_0, r_T. \tag{68}$$

The restriction (68) may lead to discontinuities of $\varphi_T = \varphi_T(r_0, \varphi_0, z_0, \chi, \psi)$ when $r_T = r_T(r_0, \varphi_0, z_0, \chi, \psi)$ approaches $r_T = 0$, e.g. along z_0 for $\sin(\varphi_0 - \chi) = \pm 1$, see below. This must be taken into account in the solutions to (67).

Multiplication of $(67)_1$ by $\sin(\varphi_T - \chi)$ and of $(67)_2$ by $\cos(\varphi_T - \chi)$ leads after subtraction and some trigonometric manipulations to

$$r_0 \sin(\varphi_T - \varphi_0) = q_0 \cos(\varphi_T - \chi), \tag{69}$$

where $q_0 = q_0(r_0, \varphi_0, z_0, \chi, \psi)$ is the auxiliary function

$$q_0 := -2\, r_0 \sin^2(\psi/2) \sin(\varphi_0 - \chi) + z_0 \sin\psi. \tag{70}$$

With

$$\Delta := \varphi_T - \varphi_0, \tag{71}$$

and $(\varphi_T - \chi) = (\varphi_0 - \chi + \Delta)$, equation (69) can be solved for the angle Δ (criteria for the quadrant see (76)):

$$0 = -p_{01} \sin\Delta + p_{02} \cos\Delta, \tag{72}$$

where

$$p_{01} := \left(r_0 + q_0 \sin(\varphi_0 - \chi)\right), \; p_{02} := \left(q_0 \cos(\varphi_0 - \chi)\right). \tag{73}$$

Multiplication of $(67)_1$ by $\cos(\varphi_T - \chi)$ and of $(67)_2$ by $\sin(\varphi_T - \chi)$ leads after addition and some trigonometric manipulations to

$$r_T = r_0 \cos(\varphi_T - \varphi_0) + q_0 \sin(\varphi_T - \chi). \tag{74}$$

In a similar way as above we obtain

$$r_T = p_{01} \cos\Delta + p_{02} \sin\Delta. \tag{75}$$

The condition (68), applied to (75), yields the criteria for the quadrant of Δ in (72):

Quadrant for Δ:		
	$p_{01} > 0$:	$p_{01} < 0$:
$p_{02} > 0$:	1. quadrant	2. quadrant
$p_{02} < 0$:	4. quadrant	3. quadrant.

(76)

The angle Δ can be eliminated from (72), (75) without divisions by possibly vanishing $\cos\Delta$ or $\sin\Delta$: From (72) follows of necessity

$$(\cos\Delta, \sin\Delta) = c \cdot (p_{01}, p_{02}), \; c \text{ – some constant.} \tag{77}$$

Put into (75) this leads to

$$r_T = c\left(p_{01}^2 + p_{02}^2\right). \tag{78}$$

But $(\cos\Delta, \sin\Delta)$ from (77) has to satisfy $\sin^2\Delta + \cos^2\Delta = 1$. Thus,

$$c = \pm 1 \Big/ \sqrt{p_{01}^2 + p_{02}^2}\,. \tag{79}$$

Because of (68) and (78) holds the positive sign, and from (78), (79) and (73) follow

$$r_T = \sqrt{p_{01}^2 + p_{02}^2} = \sqrt{r_0^2 + 2\, r_0\, q_0 \sin(\varphi_0 - \chi) + q_0^2}\,. \tag{80}$$

Obviously, the radius r_T vanishes only for $q_0 = \pm r_0$ if $\sin(\varphi_0 - \chi) = \mp 1$.

Summarized, equation (80), the equation (71) with (72) and (76), and equation $(67)_3$ are the sought transforms

$$r_T = r_T(r_0, \varphi_0, z_0; \chi, \psi), \; \varphi_T = \varphi_T(r_0, \varphi_0, z_0; \chi, \psi), \; z_T = z_T(r_0, \varphi_0, z_0; \chi, \psi). \tag{81}$$

Remark 22: From (71) and (77) to (80) follows for $(81)_2$ the sometimes convenient form

$$(\cos\varphi_T, \sin\varphi_T) = \big((p_{01}\cos\varphi_0 - p_{02}\sin\varphi_0)/r_0\, ,\, (p_{01}\sin\varphi_0 + p_{02}\cos\varphi_0)/r_0\big). \tag{82}$$

Remark 23: For simple calculations we assumed right from the beginning of this Section 4.1 the tilt to be given in the polar form (χ,ψ). To transfer the relations (81) to the axial form it is necessary to apply the expansions

$$\begin{aligned}\cos(\varphi_0-\chi) &= \cos\varphi_0\,\cos\chi + \sin\varphi_0\,\sin\chi,\\ \sin(\varphi_0-\chi) &= \sin\varphi_0\,\cos\chi - \cos\varphi_0\,\sin\chi.\end{aligned} \tag{83}$$

Thereafter, $\sin\chi, \cos\chi$ and $\sin(\psi/2), \sin\psi, \cos\psi$ can be expressed via the formulae (25) to (28) by the axial tilts (ψ_1,ψ_2) or (t_1,t_2). (Taylor expansions with respect to small tilts, $|\psi_1|, |\psi_2| << 1$, are easily done.)

4.2 Velocities at Tilted Cylinder Coordinates

We start from the form (7), (8) of the vector $\mathbf{x}$,

$$\mathbf{x} = \underline{\mathbf{e}}_{0C}^T\,\underline{x}_{0C} = \big(\underline{\underline{R}}_3(\varphi_0)\,\underline{\mathbf{e}}_0\big)^T\,\underline{x}_{0C} = \underline{\mathbf{e}}_0^T\,\underline{\underline{R}}_{30}^T\,\underline{x}_{0C}, \tag{84}$$

where $\underline{\underline{R}}_{30} := \underline{\underline{R}}_3(\varphi_0)$ serves as abbreviation. Correspondingly, $(65)_2$ reads in matrix notation

$$\mathbf{x} = \underline{\mathbf{e}}_{TC}^T\,\underline{x}_{TC} = \big(\underline{\underline{R}}_3(\varphi_T)\,\underline{\underline{R}}_T(\chi,\psi)\,\underline{\mathbf{e}}_0\big)^T\,\underline{x}_{TC} = \underline{\mathbf{e}}_0^T\,\underline{\underline{R}}_T^T\,\underline{\underline{R}}_{3T}^T\,\underline{x}_{TC}, \tag{85}$$

where

$$\underline{x}_{TC} = (r_T, 0, z_T)^T. \tag{86}$$

and $\underline{\underline{R}}_T := \underline{\underline{R}}_T(\chi,\psi)$, $\underline{\underline{R}}_{3T} := \underline{\underline{R}}_3(\varphi_T)$ serve as abbreviations.

Now, let $\mathbf{x}$ as well as its coordinates (r_0,φ_0,z_0), (r_T,φ_T,z_T) and the tilt angles (χ,ψ) depend on the time *t*. Then follow by differentiation from (84) and (85) for the velocity $\mathbf{v} = \dot{\mathbf{x}}$ two forms.

The first form, the velocity written with respect to $\underline{\mathbf{e}}_{0C} = (\mathbf{e}_{0r}, \mathbf{e}_{0\varphi}, \mathbf{e}_{0z})^T$, we obtain from (84):

$$\begin{aligned}\mathbf{v} &= \underline{\mathbf{e}}_0^T\,\big(\dot{\underline{\underline{R}}}_{30}^T\underline{x}_{0C} + \underline{\underline{R}}_{30}^T\dot{\underline{x}}_{0C}\big) = \underline{\mathbf{e}}_0^T\,\underline{\underline{R}}_{30}^T\big(\underline{\underline{R}}_{30}\dot{\underline{\underline{R}}}_{30}^T\underline{x}_{0C} + \dot{\underline{x}}_{0C}\big)\\ &= \underline{\mathbf{e}}_{0C}^T\,\big(\underline{\underline{\Omega}}_{30}^T\underline{x}_{0C} + \dot{\underline{x}}_{0C}\big) \quad = \big(\underline{\underline{\Omega}}_{30}\underline{\mathbf{e}}_{0C}\big)^T\,\underline{x}_{0C} + \underline{\mathbf{e}}_{0C}^T\dot{\underline{x}}_{0C},\end{aligned} \tag{87}$$

where, cf. Section 3.2,

$$\underline{\underline{\Omega}}_{30} = \dot{\varphi}_0\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{88}$$

The last term of (87) can be read in two ways: Direct evaluation by (8), (86) and (88) leads to

$$\mathbf{v} = \underline{\mathbf{e}}_{0C}^T\,(\dot{r}_0, r_0\,\dot{\varphi}_0, \dot{z}_0)^T. \tag{89}$$

Kept apart, the two summands are read as vehicle velocity $\mathbf{v}_{0veh}$ and relative velocity $\mathbf{v}_{0rel}$, respectively,

$$\mathbf{v}_{0veh} := \left(\underline{\underline{\Omega}}_{30}\,\underline{\mathbf{e}}_{0C}\right)^T \underline{x}_{0C} = \underline{\mathbf{e}}_{0C}^T\,\underline{\underline{\Omega}}_{30}^T\,\underline{x}_{0C},\ \mathbf{v}_{0rel} := \underline{\mathbf{e}}_{0C}^T\,\underline{\dot{x}}_{0C}; \tag{90}$$

$\mathbf{v}_{0veh}$ contributes the velocity at the fixed point $\underline{x}_{0C}$, $\mathbf{v}_{0rel}$ adds the velocity due to $\underline{\dot{x}}_{0C}$ when φ_0 is fixed:

$$\mathbf{v} = \mathbf{v}_{0veh} + \mathbf{v}_{0rel}. \tag{91}$$

The second form of the velocity $\mathbf{v} = \dot{\mathbf{x}}$, written with respect to the tilted – and rotated through the angle φ_T – instantaneous basis $\underline{\mathbf{e}}_{TC} = (\mathbf{e}_{Tr}, \mathbf{e}_{T\varphi}, \mathbf{e}_{Tz})^T$, we obtain from (85) by differentiation (manipulation in (92) base mostly on (52) with (15), and the ortho-normality $(47)_1$):

$$\begin{aligned}
\mathbf{v} &= \underline{\mathbf{e}}_0^T\,\left(\underline{\underline{\dot{R}}}_T^T\,\underline{\underline{R}}_{3T}^T\,\underline{x}_{TC} + \underline{\underline{R}}_T^T\,\underline{\underline{\dot{R}}}_{3T}^T\,\underline{x}_{TC} + \underline{\underline{R}}_T^T\,\underline{\underline{R}}_{3T}^T\,\underline{\dot{x}}_{TC}\right)\\
&= \underline{\mathbf{e}}_0^T\,\underline{\underline{R}}_T^T\,\underline{\underline{R}}_{3T}^T\left(\underline{\underline{R}}_{3T}\,\underline{\underline{R}}_T\,\underline{\underline{\dot{R}}}_T^T\,\underline{\underline{R}}_{3T}^T\,\underline{x}_{TC} + \underline{\underline{R}}_{3T}\,\underline{\underline{\dot{R}}}_{3T}^T\,\underline{x}_{TC} + \underline{\dot{x}}_{TC}\right)\\
&= \left(\underline{\underline{R}}_{3T}\,\underline{\underline{R}}_T\,\underline{\mathbf{e}}_0\right)^T\left(\underline{\underline{R}}_{3T}\,\underline{\underline{\Omega}}_T^T\,\underline{\underline{R}}_{3T}^T\,\underline{x}_{TC} + \underline{\underline{\Omega}}_{3T}^T\,\underline{x}_{TC} + \underline{\dot{x}}_{TC}\right)\\
&= \underline{x}_{TC}^T\left(\underline{\underline{R}}_{3T}\,\underline{\underline{\Omega}}_T\,\underline{\underline{R}}_{3T}^T + \underline{\underline{\Omega}}_{3T}\right)\underline{\mathbf{e}}_{TC} + \underline{\mathbf{e}}_{TC}^T\,\underline{\dot{x}}_{TC}.
\end{aligned} \tag{92}$$

The last line means

$$\mathbf{v} = \underline{x}_{TC}^T\,\underline{\dot{\mathbf{e}}}_{TC} + \underline{\mathbf{e}}_{TC}^T\,\underline{\dot{x}}_{TC} = \underline{\dot{\mathbf{e}}}_{TC}^T\,\underline{x}_{TC} + \underline{\mathbf{e}}_{TC}^T\,\underline{\dot{x}}_{TC}. \tag{93}$$

The first terms on the right-hand side of (92) and (93) have essentially the same content as the equations (50) and (51). Especially hold for $\underline{\underline{\Omega}}_{PT} = \left(\underline{\underline{R}}_{3T}\,\underline{\underline{\Omega}}_T\,\underline{\underline{R}}_{3T}^T + \underline{\underline{\Omega}}_{3T}\right)$ the relations from (53) to (62), notations adapted. The three individual terms of the last line of (92) may be helpful to establish relative velocities in contact problems. We decompose the velocity $\mathbf{v} = \dot{\mathbf{x}}$ correspondingly:

$$\mathbf{v} = \mathbf{v}_{Tveh} + \mathbf{v}_{Cveh} + \mathbf{v}_{rel}, \tag{94}$$

where

$$\begin{aligned}
\mathbf{v}_{Tveh} &= \underline{x}_{TC}^T\left(\underline{\underline{R}}_{3T}\,\underline{\underline{\Omega}}_T\,\underline{\underline{R}}_{3T}^T\right)\underline{\mathbf{e}}_{TC} &&= \underline{\mathbf{e}}_{TC}^T\left(\underline{\underline{R}}_{3T}\,\underline{\underline{\Omega}}_T^T\,\underline{\underline{R}}_{3T}^T\right)\underline{x}_{TC},\\
\mathbf{v}_{Cveh} &= \underline{x}_{TC}^T\,\underline{\underline{\Omega}}_{3T}\,\underline{\mathbf{e}}_{TC} &&= \underline{\mathbf{e}}_{TC}^T\,\underline{\underline{\Omega}}_{3T}^T\,\underline{x}_{TC},\\
\mathbf{v}_{rel} &= \underline{\dot{x}}_{TC}^T\,\underline{\mathbf{e}}_{TC} &&= \underline{\mathbf{e}}_{TC}^T\,\underline{\dot{x}}_{TC},
\end{aligned} \tag{95}$$

are the vehicle velocities due to the tilt, due to the ordinary φ_T-rotation (as a cylinder coordinate), and the relative velocity. The last two can be contracted, cf. (89):

$$\mathbf{v}_{Cveh} + \mathbf{v}_{rel} = \underline{\mathbf{e}}_{TC}^T\,(\dot{r}_T, r_T\,\dot{\varphi}_T, \dot{z}_T)^T. \tag{96}$$

It would be very troublesome to try to calculate, say, $(\dot{r}_T, r_T\,\dot{\varphi}_T, \dot{z}_T)$ from $(\dot{\chi}, \dot{\psi})$ and $(\dot{r}_0, r_0\,\dot{\varphi}_0, \dot{z}_0)$ by a comparison of (87) and (93), which represent the same velocity. Instead we differentiate the relations (81) with respect to the time:

The function $\dot{q}_0$, obtained by differentiation from (70), serves as auxiliary function:

$$\begin{aligned}
\dot{q}_0 = \;& \dot{r}_0\,(\cos\psi - 1)\sin(\varphi_0 - \chi) + \dot{z}_0\sin\psi\\
&+ r_0\,(\dot{\varphi}_0 - \dot{\chi})\,(\cos\psi - 1)\cos(\varphi_0 - \chi)\\
&+ \dot{\psi}\,(z_0\cos\psi - r_0\sin\psi\,\sin(\varphi_0 - \chi)).
\end{aligned} \tag{97}$$

Then follow (see (80) for r_T):

$$\begin{aligned} \dot r_T &= \big(r_0\,\dot r_0 + (r_0\,\dot q_0 + \dot r_0\,q_0)\sin(\varphi_0-\chi) + r_0\,q_0\,(\dot\varphi_0-\dot\chi)\cos(\varphi_0-\chi) + q_0\,\dot q_0\big)\big/ r_T\,, \\ \dot\varphi_T &= \dot\varphi_0 + \big((r_0\,\dot q_0 - \dot r_0\,q_0)\cos(\varphi_0-\chi) - r_0\,q_0\,(\dot\varphi_0-\dot\chi)\sin(\varphi_0-\chi) - (\dot\varphi_0-\dot\chi)\,q_0\,\dot q_0\big)\big/ r_T^2\,, \\ \dot z_T &= -(\dot r_0\sin\psi + r_0\,\dot\psi\cos\psi)\sin(\varphi_0-\chi) - r_0\,(\dot\varphi_0-\dot\chi)\cos(\varphi_0-\chi)\sin\psi + \dot z_0\cos\psi - z_0\,\dot\psi\sin\psi\,. \end{aligned} \tag{98}$$

These velocities enter equation (96) at the right side.

5 Examples

The following two examples demonstrate the application of tilted cylinder coordinates.

5.1 Path of the Pin of a V-Belt Chain Drive for a Tilted Bevel Disk

Figure 6 displays schematically the two rigid bevel disks a), b) of the pulley of a V-belt drive rotating about the axis marked by the dash-dotted line (the shaft itself is not shown); the opening angle 2α of the cone and the two relevant distances a_1, a_2 are shown in the figure. The bar d) of the length L represents a pin of the chain, it is guided parallel to the axis and contacts the conic surfaces of the disks with its ends. Disk a) is fixed to the shaft, disk b) can tilt about the point O; cf. Srnik (1999). Given a fixed tilt (χ,ψ) of the disk b), cf. the side view and the projection c) of Figure 6: what is the path $r_0(\varphi_0)$ of the pin for $0 \le \varphi_0 \le 2\pi$?

For an answer, first, we introduce the stationary basis $(\mathrm{O},\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03})$ and the pertinent cylinder coordinates (r_0,φ_0,z_0) as shown in the front and the side view; the z_0-axis, and $\mathbf{e}_{03}$, coincide with the axis of rotation. To measure the tilt, secondly, the basis $(\mathrm{O},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$, with cylinder coordinates (r,φ,z), is attached with its z-axis, and $\mathbf{e}_3$, to the axis of the tilting disk; cf. the projection c) as seen from the side view. Initially, in the straight position, $(\mathrm{O},\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03})$ and $(\mathrm{O},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ coincide. (The basis $(\mathrm{O},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ tilts only – it doesn't rotate – , so we do not need the subscripts T.) Now, with the tilt (χ,ψ) given, the position of $(\mathrm{O},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is fixed, the relations from Section 4.1 hold, especially the transforms (81). We proceed in three steps:

1. For assumed values of the angle φ_0 and the radius r_0, the left end of the pin touching the fixed disk a), we read from the front view of Figure 6 for the right end of the pin with respect to (r_0,φ_0,z_0) the z_0-coordinate

$$z_0 = l - r_0\cot\alpha, \text{ where } l := L - a_1 - a_2 > 0. \tag{99}$$

2. By (81) we calculate the coordinates (r,z) of the pin's end point with respect to the tilted system (r,φ,z):

$$z = z(r_0,\varphi_0,l-r_0\cot\alpha;\chi,\psi),\ \ r = r(r_0,\varphi_0,l-r_0\cot\alpha;\chi,\psi). \tag{100}$$

3. To touch the tilted disk, the coordinates (r,z) of the right end of the pin with respect to (r,φ,z) have to satisfy (see Figure 6c):

$$r = (a_2 + z)\tan\alpha. \tag{101}$$

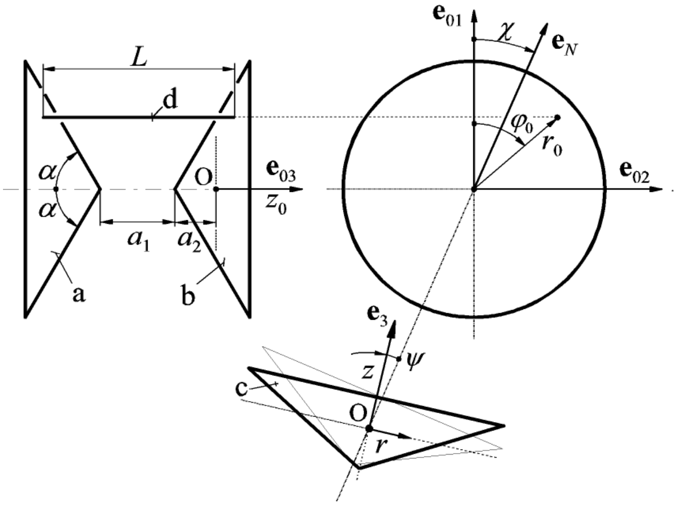

Figure 6: Bevel disks and representative pin of a chain drive.

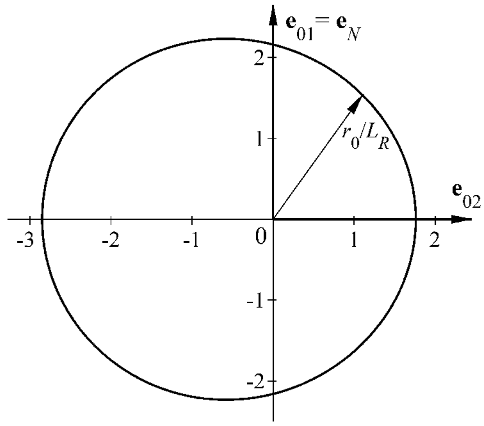

Figure 7: *Path for* $\chi = 0$, $\psi = 7°$, $L - a_1 = L_R$, $a_2 = 0.1L_R$, $\alpha = 77°$; L_R – a reference length.

The equations (99) to (101) can be solved for r, r_0, z as functions of φ_0, χ, ψ and the geometrical parameters. After the elimination of r, z we get for r_0 the quadratic equation

$$A r_0^2 + 2B\, r_0 - C = 0, \tag{102}$$

where, with respect to the axial form (t_1, t_2) of the tilt (cf. (28) for $\sin\psi, \cos\psi$):

$$\begin{aligned} A =\ & (\cot^2\alpha - \tan^2\alpha)\sin^2\psi - 2(\cot\alpha + \tan\alpha)\cos^2\psi\,(t_1\sin\varphi_0 - t_2\cos\varphi_0) \\ & + (1+\tan^2\alpha)\cos^2\psi\,(t_1\cos\varphi_0 + t_2\sin\varphi_0), \\ B =\ & (l\cos\psi + a_2)\tan\alpha\cos\psi\,\big(1 + \tan\alpha\,(t_1\sin\varphi_0 - t_2\cos\varphi_0)\big) \\ & - l\cot\alpha\sin^2\psi - l\cos\psi\,(1-\cos\psi)\,(t_1\sin\varphi_0 - t_2\cos\varphi_0), \\ C =\ & \tan^2\alpha\,(l\cos\psi + a_2)^2 - l^2\sin^2\psi; \end{aligned} \tag{103}$$

For sufficiently small tilts t_1, t_2 hold B, $C > 0$, but A will change its sign along $0 \le \varphi_0 \le 2\pi$.

Because of (68), the positive root r_0 of the solution of (102) has to be chosen:

$$r_0 = \left(\sqrt{B^2 + AC} - B\right)\Big/A = C\Big/\left(B + \sqrt{B^2 + AC}\right). \tag{104}$$

Taylor expansions of (104) with respect to t_1, t_2 are easily obtained by a formula manipulator. Here is a second order expansion in axial and in polar form:

$$\begin{aligned} r_0 &= \frac{1}{2}(L - a_1)\tan\alpha + A_1(t_1\sin\varphi_0 - t_2\cos\varphi_0) - A_2\left((t_1^2 - t_2^2)\cos 2\varphi_0 + 2t_1t_2\sin 2\varphi_0\right) + A_3(t_1^2 + t_2^2) \\ &= \frac{1}{2}(L - a_1)\tan\alpha + A_1\tan\psi\sin(\varphi_0 - \chi) - A_2\tan^2\psi\cos 2(\varphi_0 - \chi) + A_3\tan^2\psi, \end{aligned} \tag{105}$$

where

$$\begin{aligned} & A_1 = \frac{1}{4}(L - a_1)(1 - \tan^2\alpha),\ A_2 = \frac{1}{16}(L - a_1)\tan\alpha\,(2\cot^2\alpha + \tan^2\alpha - 1), \\ & A_3 = \frac{1}{16}(L - a_1 - a_2)\tan\alpha\,(\tan^2\alpha - 3) + \frac{1}{16}a_2\left(8\cot\alpha + \tan\alpha(1+\tan^2\alpha)\right) - \frac{1}{2}a_2^2\cot\alpha/(L - a_1). \end{aligned} \tag{106}$$

Figure 7 shows a path $r_0(\varphi_0)$ obtained by numerical evaluation of (105) for the parameters listed in the legend.

5.2 Point of Contact of a Tilting Cylindrical Rotor Mounted Eccentrically in a Cylindrical Casing

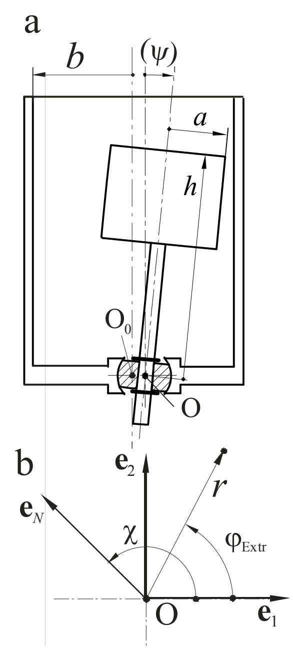

Figure 8: Rotor mounted eccentrically in a cylindrical casing.

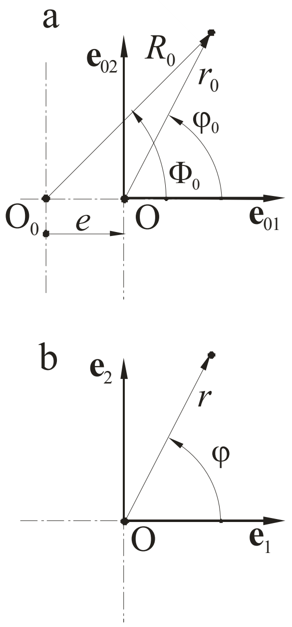

Figure 9: Cylinder coordinates: a fixed, b tilting with the axis of the rotor

A rigid cylindrical rotor, radius a, height h, is mounted at the bottom of a cylindrical casing, radius b, such that it can tilt, see Figure 9a. Its bearing point O deviates from the centre point O_0 of the bottom of the casing by the eccentricity $\overline{O_0O} = e$.

When the rotor tilts, where will it touch the wall of the casing?

For the solution of the problem, we need three sets of cylinder coordinates:

The bottom point O_0 of the casing is the origin of the first system, (R_0, Φ_0, Z_0), being stationary, whose Z_0-axis points upwards and coincides with the casing's centre line, its polar axis passes through $\overline{O_0O}$, see Figures 8a and 9a. The bearing point O of the rotor is the origin of the second system, (r_0, φ_0, z_0), again stationary, whose z_0-axis, and $\mathbf{e}_{03}$, point vertically upwards, the vector $\mathbf{e}_{01}$ has the direction of $\overrightarrow{O_0O}$, see Figure 9a. Thirdly, the system (r, φ, z) is attached with its z-axis to the axis of the rotor, it does not rotate but tilts with the rotor, see Figures 8b and 9b; in the upright rotor position coincide the systems (r, φ, z) and (r_0, φ_0, z_0). (Again, the subscripts T are omitted at the tilting system (r, φ, z).)

With respect to the rotor system (r, φ, z), for the upper rim of the rotor hold

$$(r, \varphi, z) = (a, \varphi, h). \tag{107}$$

Let the tilt of the rotor, of (r, φ, z) with respect to (r_0, φ_0, z_0), be given by the tilt angles (χ, ψ), or by (t_1, t_2), see Section 4.1. To find the point of contact of rim and casing we need the rim coordinates with respect to (R_0, Φ_0, Z_0):

$$R_0 = R_0(a, \varphi, h; \chi, \psi),\ \Phi_0 = \Phi_0(a, \varphi, h; \chi, \psi),\ Z_0 = Z_0(a, \varphi, h; \chi, \psi). \tag{108}$$

The transition from (107) to (108) requires two steps:

$$(r, \varphi, z) \xrightarrow{tilt(\chi,\psi)} (r_0, \varphi_0, z_0) \xrightarrow{eccentricity\ e} (R_0, \Phi_0, Z_0). \tag{109}$$

We start by the second step: Let $(r_0, \varphi_0, z_0) = (r_0(\varphi_0), \varphi_0, z_0(\varphi_0))$ describe the upper rim of the (tilted) rotor with respect to (r_0, φ_0, z_0); unessential parameters are omitted. From Figure 9a follow (by the cosine theorem)

$$R_0^2 = e^2 + 2\,e\,r_0 \cos\varphi_0 + r_0^2, \tag{110}$$

and

$$(\cos\Phi_0, \sin\Phi_0) = (e + r_0\cos\varphi_0, r_0\sin\varphi_0)/R_0,\ Z_0 = z_0. \tag{111}$$

For the first transition of (109) we must, in contrast to (81), (r_0, φ_0, z_0) express by (r, φ, z) and (χ, ψ):

$$r_0 = r_0(r, \varphi, z; \chi, \psi),\ \varphi_0 = \varphi_0(r, \varphi, z; \chi, \psi),\ z_0 = z_0(r, \varphi, z; \chi, \psi). \tag{112}$$

To get these relations, we have to project the relation (66) onto the orthogonal triad $(\mathbf{e}_N, \mathbf{e}_{0\perp}, \mathbf{e}_{03})$, cf. Figure 3. The subsequent calculations follow closely the proceeding in Section 4.1. With

$$q = -2\,r\sin^2(\psi/2)\sin(\varphi-\chi) - z\sin\psi,\ p_1 = r + q\sin(\varphi-\chi),\ p_2 = q\cos(\varphi-\chi), \tag{113}$$

cf. (70), (73), etc. we obtain

$$\begin{aligned} r_0 &= \sqrt{p_1^2 + p_2^2} = \sqrt{r^2 + 2\,r\,q\sin(\varphi-\chi) + q^2}, \\ (\cos\varphi_0, \sin\varphi_0) &= \big((p_1\cos\varphi - p_2\sin\varphi)/r, (p_1\sin\varphi + p_2\cos\varphi)/r\big), \\ z_0 &= r\sin(\varphi-\chi) + z\cos\psi. \end{aligned} \tag{114}$$

Now, we put $(r, \varphi, z) = (a, \varphi, h)$, see (107), into the right side of (114) and the resulting (r_0, φ_0, z_0) into (110) and get

$$\begin{aligned} &R_0^2 = e^2 + 2\,e\,a\cos\varphi + a^2 + 2\,q^*\big(a\sin(\varphi-\chi) - e\sin\chi\big) + q^{*2}, \\ &\text{where} \quad q^* = -2\,a\sin^2(\psi/2)\sin(\varphi-\chi) - h\sin\psi. \end{aligned} \tag{115}$$

The angle φ_{Extr} where R_0^2 assumes its extreme value follows by the condition $\partial R_0^2/\partial\varphi = 0$. That leads to

$$\cos(\chi - \varphi_{Extr}) = \frac{-2\,e\sin\varphi_{Extr}}{h\sin(2\psi) - 2a\sin^2\psi\ \sin(\chi-\varphi_{Extr}) - 4\,e\sin^2(\psi/2)\sin\chi}. \tag{116}$$

For vanishing eccentricity, $e = 0$, holds $(\chi - \varphi_{Extr}) = \pi/2$, see Figure 8b. For e > 0 we introduce

$$(\chi - \varphi_{Extr}) = \pi/2 - \Delta\varphi_E, \text{ or } \varphi_{Extr} = \chi - \pi/2 + \Delta\varphi_E \tag{117}$$

and obtain from (116)

$$\sin\Delta\varphi_E = \frac{2\,e\cos(\chi + \Delta\varphi_E)}{h\sin(2\psi) - 2a\sin^2\psi\ \cos\Delta\varphi_E - 4\,e\sin^2(\psi/2)\sin\chi}. \tag{118}$$

This equation for $\Delta\varphi_E$ contains the parameters (χ, ψ, e, a, h); if $h > a \gg e$ it can be solved, e.g., by recursion. The resulting φ_{Extr}, cf. $(117)_2$, is put into (115). Then holds the condition of contact: $R_0 = b$, see Figures 8a, 9a. This is an equation for the inclination angle ψ as function of (χ, e, a, h), again to be solved approximately by an expansion. At the end follow the coordinates (R_0, Φ_0, Z_0) of the contact point from (111) and (112). All these expansions are straightforward – also the transition to the tilts (t_1, t_2) – but clumsy. Their handling requires a formula manipulator, the details are too lengthy to be printed.

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