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Free Vibrations and Buckling of a Thin Cylindrical Shell of Variable Thickness with Curvelinear Edge

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Low-frequency vibrations and buckling under an uniform external lateral pressure of a thin cylindrical shell of variable thickness with curvelinear edge are analyzed. The asymptotic and finite element methods are used to get the vibration frequencies and critical loads. The vibration and buckling modes are also presented. The comparison of numerical and asymptotic results is performed.

1 Introduction

Interest in studying of the shell arises from the fifties of the twentieth century. The assemblies, containing thin shells, find wide use in the modern engineering, especially in aircraft and spacecraft industry. In many papers the shell vibrations and buckling are analyzed by means of numerical methods (Bathe, 1984; Rikards and Chate, 2001; Kulikov and Plotnikova, 2002). Asymptotic integration methods developed in Bauer et al. (1993) and Tovstik (2001) clarify qualitatively allocation of vibration frequencies, critical loads and behavior of vibration and buckling modes.

The vibration and buckling modes of thin elastic shells essentially depend on some determining functions such as the radii of the curvature of the neutral surface, the shell thickness, the shape of the shell edges, etc. In simple cases when these functions are constant, the vibration and buckling pits occupy the entire shell surface. This case takes place for low-frequency vibration and buckling of a circular cylindrical shell under uniform external pressure. If the determining functions vary from point to point of the neutral surface then localization of the vibration and buckling modes near some weakest lines on the shell surface is possible. Vibrations and buckling of cylindrical shell under external pressure can be accompanied by the appearance of concavities which are stretched along the shell generatrix from one shell edge to another. The depth of the concavities is maximum near the weakest generatrix and decreases fast away from the generatrix.

The asymptotic expansions for critical pressure and buckling modes localized near the weakest generatrix of the cylindrical shell are constructed in Tovstik (2001). In particular, the buckling of the cylindrical shell of variable thickness with straight edges and the cylindrical shell of constant thickness with slanted edge are analyzed. In the first problem the thickness changes only in circular direction and the weakest generatrix corresponds to a minimum value of the thickness. In the second case the buckling mode is localized near the longest generatrix of cylindrical shell.

In Eliseeva and Filippov (2003) the effect of both variable thickness and slanted edge on the vibration and buckling modes of the free supported cylindrical shell is studied by means of asymptotic method. It is shown that depending on the values of parameters one weakest line or two such lines may be appear on the shell surface. In this paper the same problem for clamped cylindrical shell is solved using the asymptotic and numerical methods.

2 Basic Equations

We take the radius, R, of the cylindrical shell as unit of measurement and introduce on the middle surface of the shell the orthogonal dimensionless coordinate system (s, φ):

$$s \in [0, l(\varphi)] \qquad \varphi \in (-\pi, \pi],$$

where $l(\varphi)$ is the length of the shell generatrix. The shell edge $s = 0$ is straight and the edge $s = l(\varphi)$ is curvelinear.

If $l(\varphi) = l_c - \operatorname{tg}\beta\cos\varphi$ then the edge $s = l(\varphi)$ is slanted by the angle β. In case $\beta = 0$ the both shell edges are straight. In Figure 1 are shown two cross-sections of the cylindrical shell with slanted edge.

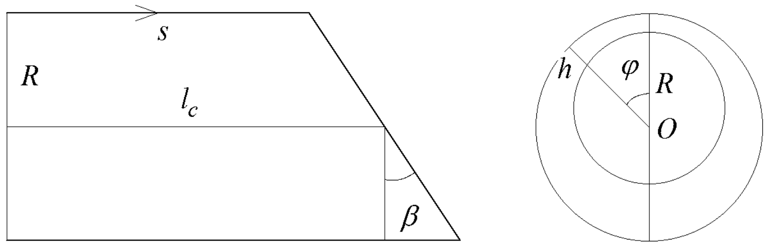

Figure 1. Cylindrical shell with slanted edge.

We suppose that dimensionless shell thickness h is small and depends on the coordinate φ:

$$h(\varphi) = h_0\, g(\varphi)$$

where h_0 is a small parameter, $g \sim 1$.

In particular, for the cylindrical shell which cross-section is obtained by the cutaway the circle of the radius r from the circle of the radius $R = 1$ (see Figure 1) function $g(\varphi)$ and parameter h_0 have the following form

$$g(\varphi) = 1 + \gamma(1 - \cos\varphi) \qquad h_0 = h(0) = 1 - r - e \qquad \gamma = e/h_0 \tag{1}$$

where e is the distance between the circles centers. In case $e = 0$ the shell has the constant thickness h_0. In general case $h_0 = h(0)$ is the least value of $h(\varphi)$ whereas $h_m = h(\pi) = 1 - r + e$ is the maximum of $h(\varphi)$. The dependence of the thickness ratio $\eta = h_m/h_0$ on γ has the form $\eta = 1 + 2\gamma$.

The dimensionless equations describing the small free vibrations and buckling of a thin elastic cylindrical shell (see Eliseeva and Filippov (2003)), can be written as

$$\varepsilon^4\Delta(g^3\Delta w) - \frac{\partial^2\Phi}{\partial s^2} + \lambda Z = 0 \qquad \varepsilon^4\Delta(g^{-1}\Delta\Phi) + \frac{\partial^2 w}{\partial s^2} = 0 \tag{2}$$

Here $w(s, \varphi)$ is the normal deflection, $\Phi(s, \varphi)$ is the force function,

$$\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial\varphi^2} \qquad \varepsilon^8 = \frac{h_0^2}{12(1-\nu^2)}$$

ν is Poisson's ratio. In case of the buckling under an uniform external lateral pressure p

$$Z = \varepsilon^2\frac{\partial^2 w}{\partial\varphi^2} \qquad \lambda = \frac{p}{Eh_0\varepsilon^6},$$

where E is Young's modulus. For the problem of the shell vibrations

$$Z = -gw \qquad \lambda = \frac{\rho R^2\omega^2}{\varepsilon^4 E},$$

where ω is the vibration frequency, ρ is the mass density.

To find the parameter λ we have to take into account the boundary conditions on the shell edges $s = 0$ and $s = l(\varphi)$.

3 Asymptotic Analysis

According to a procedure proposed by Tovstik (2001), the asymptotic solution of the boundary value problem for equations (2), can be expressed as

$$w(s,\varphi,\varepsilon) = \exp\left(\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi} q(\varphi)\,d\varphi\right)\sum_{k=0}^{\infty}\varepsilon^k w_k(s,\varphi) \qquad \lambda = \sum_{k=0}^{\infty}\varepsilon^k\lambda_k \tag{3}$$

where

$$\operatorname{Im} q(\varphi_0) = 0 \qquad \operatorname{Im}\left\{\frac{dq}{d\varphi}(\varphi_0)\right\} > 0 \tag{4}$$

Function Φ has similar asymptotic expansion. It follows from conditions (4) that the functions w and Φ are localized near the line $\varphi = \varphi_0$. By substituting expressions (3) into (2) and boundary conditions we get the equations for $q(\varphi), w_k(s,\varphi), \Phi_k(s,\varphi)$ and λ_k. In the zeroth-order approximation we obtain

$$\frac{d^2\Phi_0}{ds^2} - q^4g^3w_0 + \lambda_0 N w_0 = 0 \qquad \frac{d^2w_0}{ds^2} + \frac{q^4}{g}\Phi_0 = 0 \tag{5}$$

where $N = q^2$ for the buckling problem, $N = g$ for the vibration one. The elimination of the function Φ_0 from system (5) gives the following equation

$$\frac{d^4w_0}{ds^4} - \alpha^4 w_0 = 0 \qquad \alpha^4 = \lambda_0\frac{q^4N}{g} - g^2q^8 \tag{6}$$

We suppose that the shell edges are clamped. Then the boundary conditions for equation (6) are

$$w_0 = \frac{dw_0}{ds} = 0 \qquad s = 0 \qquad s = l(\varphi) \tag{7}$$

The asymptotic solution in case of freely supported shell edges is analyzed in Eliseeva and Filippov (2003). The choice of boundary conditions for equation (6) is discussed in Tovstik (2001).

The solutions of boundary value problem (6), (7) have the form

$$w_{0n} = U(\alpha_n s)T(\alpha_n l) - T(\alpha_n s)U(\alpha_n l) \qquad n = 1, 2, \dots$$

where

$$T(z) = \sinh z - \sin z \qquad U(z) = \cosh z - \cos z$$

and α_n is a root of the equation

$$\cosh(\alpha l)\cos(\alpha l) = 1 \tag{8}$$

The values of α_n depend on the boundary conditions. If the shell edges are freely supported then $\alpha_n = \pi n/l$.

Taking into account second formula (6) we get that λ_0 is the function of the parameters q and φ:

$$\lambda_0 = f(q,\varphi) = \frac{\alpha^4 g}{q^4N} + \frac{q^4g^3}{N}. \tag{9}$$

As the zeroth-order approximation for the eigenvalue λ we select

$$\lambda_0 = \min_{q,\varphi} f(q,\varphi) = f(q_0,\varphi_0) \tag{10}$$

Then

$$\lambda_q = \frac{\partial f}{\partial q} = 0 \qquad \lambda_\varphi = \frac{\partial f}{\partial \varphi} = 0 \qquad \text{for} \qquad q = q_0 \qquad \varphi = \varphi_0 \tag{11}$$

The first-order correction λ_1 for the eigenvalue λ can be found by the following formula

$$\lambda_1 = \frac{m+1}{2}\sqrt{\lambda_{qq}\lambda_{\varphi\varphi} - \lambda_{q\varphi}^2} \qquad m = 0, 1, 2\ldots \tag{12}$$

where the partial derivatives

$$\lambda_{qq} = \frac{\partial^2 f}{\partial q^2} \qquad \lambda_{\varphi\varphi} = \frac{\partial^2 f}{\partial \varphi^2} \qquad \lambda_{q\varphi} = \frac{\partial^2 f}{\partial q \partial \varphi}$$

are calculated for $q = q_0$, $\varphi = \varphi_0$.

The asymptotic expansion exists on conditions that

$$\lambda_{qq}\lambda_{\varphi\varphi} - \lambda_{q\varphi}^2 > 0 \tag{13}$$

since λ_1 is a real number. If the inequality (13) is true then the vibration and buckling modes are localized near the generatrix of cylinder $\varphi = \varphi_0$, where φ_0 satisfies equations (11). The generatrix $\varphi = \varphi_0$ is called the weakest generatrix.

4 Buckling of the Cylindrical Shell

Consider the buckling of a thin cylindrical shell of a variable thickness with a curvelinear edge under an uniform external lateral pressure. To obtain the critical pressure we will search the minimal value of the parameter λ. In the buckling problem formula (9) takes the form

$$f(q, \varphi) = \frac{\alpha^4 g}{q^6} + q^2 g^3$$

where $\alpha \simeq 4.73/l$ is the least positive root of equation (8). Equations (11) are equivalent to the following equations

$$q^8 = \frac{3\alpha^4}{g^2} \qquad \frac{2l'}{l} = \frac{5g'}{g} \tag{14}$$

where $l' = dl/d\varphi$, $g' = dg/d\varphi$. The parameter φ_0 is a root of the second equation (14). If φ_0 is known then the number q_0 can be found by the first equation (14).

The first-order correction λ_1 can be express as

$$\lambda_1 = 2qg^3\sqrt{\frac{\Lambda_1}{3}} \qquad \Lambda_1 = \frac{10g''}{g} - \frac{4l''}{l} + \frac{15g'^2}{g^2}.$$

The generatrix $\varphi = \varphi_0$ is the weakest one if $\Lambda_1 > 0$.

As an example we will consider the buckling problem for the cylindrical shell with slanted edge. We assume that the thickness of the shell $h = h_0 g(\varphi)$, where h_0 and g are given by formulas (1).

In this case the second equation (14) has the form

$$A(\varphi)\sin\varphi = 0 \tag{15}$$

where $A(\varphi) = 5\gamma/g - 2t/l$, $l = l_c - t\cos\varphi > 0$, $t = \operatorname{tg}\beta$, β is the edge slope angle, l_c is the length of the shell axis (see Figure 1). It follows from the inequality $l > 0$ that $l_c > t$.

Equation (15) has two roots 0 and π in the interval $(-\pi, \pi]$ if $\gamma \le \gamma_1$ or $\gamma \ge \gamma_2$, where

$$\gamma_1 = \frac{2t}{5l_c + t} \qquad \gamma_2 = \frac{2t}{5l_c - 5t}$$

In case $\gamma_1 < \gamma < \gamma_2$ equation (15) has four roots 0, π, φ_* and $-\varphi_*$ in the same interval, where

$$\varphi_* = \arccos\frac{5\gamma l_c - 2t(1+\gamma)}{3\gamma t}$$

The root φ_0 of equation (15) corresponds to the weakest generatrix if

$$\Lambda_1 = 2A(\varphi_0)\cos\varphi_0 + 15\gamma^2 \sin^2\varphi_0/g^2(\varphi_0) > 0$$

It is easy seen that for $\gamma < \gamma_1$ exists only one weakest generatrix $\varphi = \pi$, for $\gamma_1 < \gamma < \gamma_2$ we have two weakest lines $\varphi = \pm\varphi_*$ and in case $\gamma > \gamma_2$ on the shell is again only one weakest generatrix $\varphi = 0$. The location of weakest lines does not depend on the boundary conditions.

Assume that the parameters l_c and $\beta > 0$ are fixed whereas the thickness ratio $\eta = h_m/h_0 = 1 + 2\gamma$ increases. The case $\eta = 1$ corresponding to a shell of a constant thickness was analyzed in Tovstik (2001). If $\eta = 1$ then the longest bottom generatrix $\varphi = \pi$ of the cylindrical shell is its weakest generatrix. The generatrix $\varphi = \pi$ remains the weakest one while $\eta \leq \eta_1 = 1 + 2\gamma_1$. The further increase in η leads to appearance of two weakest lines $\varphi = \pm\varphi_*$ near the generatrix $\varphi = \pi$. These lines disperse, go up and for $\eta = \eta_2 = 1 + 2\gamma_2$ join on the top of the shell turning into the generatrix $\varphi = 0$. By the subsequent increase in η the generatrix $\varphi = 0$ stands the weakest one. In Tovstik (2001) is shown that for the shell with straight edges the generatrix $\varphi = 0$ is the weakest one if $\eta > 1$. Hence, for $\eta > \eta_2$ the change of the shell thickness makes larger effect on the location of the weakest generatrix than the slanted edge.

In Table 1 are shown the results of calculations of the critical pressure for the cylindrical shell of variable thickness with slanted edge. The following values of parameters are used: $R = 1$ m, $l_c = 3$, $h_0 = 0.001$, $\beta = 45^o$, $\nu = 0.3$, $E = 1.93\,10^{11}$ Pa. For such parameters $\eta_1 = 1.25$, $\eta_2 = 1.4$.

η	Critical pressure (Pa)	
	Asymptotic	FEM result
1.10	2858	2857
1.30	4143	4084
1.45	4354	4351
3.00	4988	4989
4.00	5238	5201
5.00	5454	5375

Table 1. The values of the critical external pressure vs. ratio η.

The critical pressure obtained for various values of ratio η with the help of the asymptotic formulas (3), (10) and (12) is presented in the second column. In the third column one can see the numerical results computed by finite elements method. About 5000 four-node shell elements defined by four thicknesses was used in calculation. The computation time of a value of pressure by FEM is a few minutes. The calculations by means of the asymptotic formulas execute in a trice. The maximal relative error in the asymptotic results compared with the numerical ones is 1.5%.

The buckling modes plotted by FEM for various values of η are shown in Figures 2–4. While $\eta < 1.25$ (Figure 2) the buckling mode decrease away from the longest bottom generatrix of the cylindrical shell.

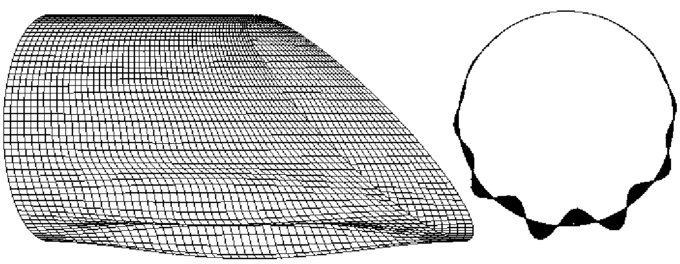

Figure 2. Buckling mode of the cylindrical shell for $\eta = 1.1$

If $1.25 < \eta < 1.4$ then in accordance with the asymptotic analysis the buckling mode must be localized near two weakest lines $\varphi = \pm\varphi_*$ on the shell surface. The buckling mode shape presented in Figure 3 shown that the areas of localization are covered. For more thin shell the localization near lines $\varphi = \pm\varphi_*$ will be more apparent.

In case $\eta > \eta_2 = 1.4$ (Figure 4) the weakest line is the shortest top generatrix.

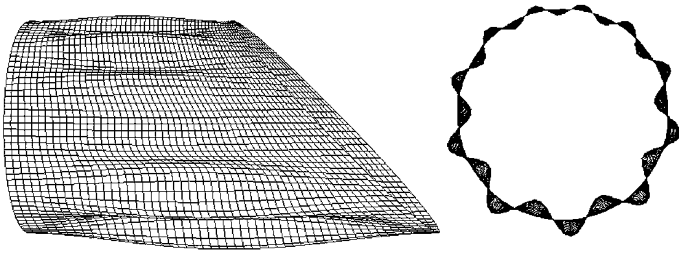

Figure 3. Buckling form of the cylindrical shell ($\eta = 1.3$)

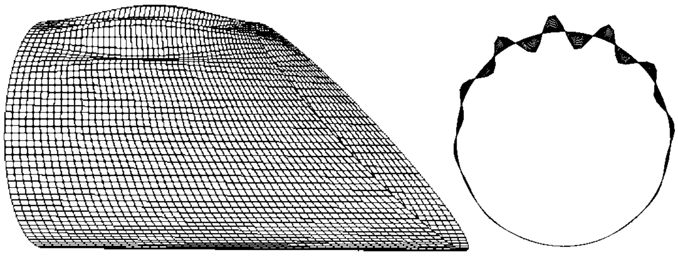

Figure 4. Buckling form of the cylindrical shell ($\eta = 1.45$)

5 Vibrations of the Cylindrical Shell

Consider free low-frequency vibrations of a thin cylindrical shell of a variable thickness with a curvelinear edge. Then formula (9) and equations (11) take the following forms

$$f(q,\varphi) = \frac{\alpha^4}{q^4} + q^4 g^2$$

$$q^4 = \frac{\alpha^2}{g} \qquad \frac{2l'}{l} = \frac{g'}{g} \tag{16}$$

We find the first-order correction

$$\lambda_1 = 2(m+1)q^3 g^2 \sqrt{2\Lambda_1} \qquad \Lambda_1 = \frac{2g''}{g} - \frac{4l''}{l} - \frac{g'^2}{g^2}$$

using formula (12).

As an example consider the vibrations of the cylindrical shell with slanted edge assuming that the function $h(\varphi)$ is given by (1). The generatrix $\varphi = \varphi_0$ is the weakest one if φ_0 satisfies equation

$$A(\varphi)\sin\varphi = 0 \qquad A(\varphi) = \gamma/g - 2t/l. \tag{17}$$

and

$$\Lambda_1 = 2A(\varphi_0)\cos\varphi_0 - \gamma^2 \sin^2\varphi_0/g^2(\varphi_0) > 0$$

The roots of equation (17) in the interval $(-\pi, \pi]$ are 0, π and roots of equation $A(\varphi) = 0$. If φ_0 is the root of equation $A(\varphi) = 0$ then

$$\Lambda_1 = -\gamma^2 \sin^2\varphi_0/g^2(\varphi_0) \le 0$$

and the generatrix $\varphi = \varphi_0$ is not the weakest line. The vibration mode is localized near the top generatrix $\varphi = 0$ if $A(0) > 0$. The last inequality is fulfilled if $\gamma > \gamma_1 = 2t/(l_c - t)$. The bottom generatrix $\varphi = \pi$ is the weakest one if $A(\pi) < 0$. The inequality $A(\pi) < 0$ is true for all $\gamma > 0$ if $t \le l_c \le 3t$. If $l_c > 3t$ then inequality $A(\pi) < 0$ is fulfilled only for $0 < \gamma < \gamma_2 = 2t/(l_c - 3t)$.

Therefore, in case of vibrations, in contrast to buckling, the location of the weakest lines on the shell surface does not depend on ratio $\eta = 1 + 2\gamma$. On η depends only the number of such lines. If $\eta < \eta_1 = 1 + 2\gamma_1$ then bottom generatrix $\varphi = \pi$ is weakest one. In case $\eta_1 < \eta < \eta_2$, where $\eta_2 = 1 + 2\gamma_2$, both lines $\varphi = 0$ and $\varphi = \pi$ are weak. The number of the weakest lines for $\eta > \eta_2$ is a function of geometric shell parameters. If $t \le l_c \le 3t$, then two weakest lines are on the shell surface, otherwise for $\eta > \eta_2$ only generatrix $\varphi = 0$ is the weakest line.

The values of the fundamental frequency for the cylindrical shell of variable thickness with slanted edge are shown in Table 2. The mass density of the shell material ρ=7860 kg/m^3. Others shell parameters have the same values as in the buckling problem.

η	Fundamental frequency (s^{-1})	
	Asymptotic	FEM result
1.0	24.833	24.144
2.0	34.715	33.123
4.0	48.585	45.082
5.0	49.666	47.625

Table 2. The values of the fundamental frequency vs. ratio η.

The values of the fundamental frequency obtained by the asymptotic formulas and with the help of FEM are placed in the second column and third column respectively. The relative discrepancy in asymptotic and numerical results is less than 8%. The computation time of a value of frequency is approximately the same as of the value of pressure.

The vibration mode shapes computed by FEM and plotted in Figures 5 and 6 amplify asymptotic results. In case $\eta < \eta_1 = 3$ (Figure 5) the vibration mode, corresponding to the fundamental frequency is localized near the longest bottom generatrix $\varphi = \pi$ of the cylindrical shell.

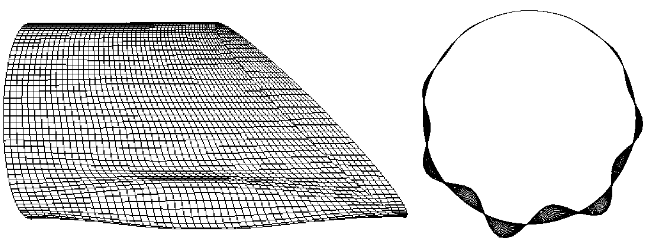

Figure 5. Vibration form of the cylindrical shell ($\eta = 2$)

If $\eta > \eta_1 = 3$ then on the shell surface are two weakest lines $\varphi = \pi$ and $\varphi = 0$ corresponding in general to different frequencies. In Figure 6 is shown the vibration mode corresponding to the fundamental frequency. This mode is localized near the shortest top generatrix $\varphi = 0$.

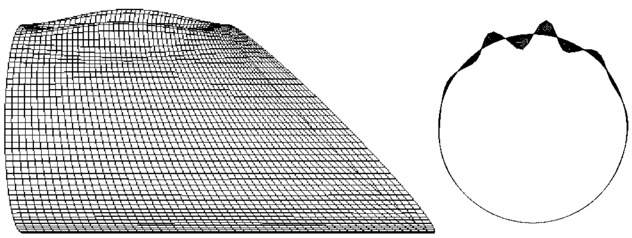

Figure 6. Vibration form of the cylindrical shell ($\eta = 5$)

6 Conclusion

By means of both asymptotic solution and numerical analysis it is shown that vibrations and buckling of thin cylindrical shell of variable thickness with curvelinear edge may be accompanied by the appearance of concavities which are stretched along the shell surface. Near a weakest generatrix the concavities have the maximum depth. The depth of the concavities decreases fast away from the weakest lines.

In contrast to previously studied problems, the number and the location of the weakest lines on the shell surface depend on the shell parameters. This dependence for buckling is different from one for vibration. In particular, for the symmetric cylindrical shell with the slanted edge and circular inner and outer surfaces the vibration modes are localized only near the longest and the shortest generatrixes. The buckling modes of such shell may be localized near two any generatrixes, centered at the symmetry plane.

The simple approximate asymptotic formulas for the lowest frequencies and critical external pressure are derived. The comparison of asymptotic and FEM results shows that for the thin shell the error of asymptotic formulas is small.

Acknowledgements

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