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Initial Penetration of an Elastic Axially Symmetric Indenter into a Rigid-Perfectly-Plastic Half-Space

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This paper is concerned with the axially symmetric plastic flow of a rigid perfectly-plastic nonhardening half-space. The initial penetration of the elastic indenter is studied based on Haar and von Karman hypothesis. The analytical distribution of contact stress and the approximate penetration depth of the indenter are obtained.

1 Introduction

Many problems of plastic flow for rigid-perfectly-plastic nonhardening materials under conditions of axial symmetry have been solved (Ishlinsky, 1944; Shield, 1955; Richmond et al., 1974; Johnson, 1985; Ishlinsky and Ivlev, 2001). The basic equations of axially symmetric plastic fields are well known. It was shown that these equations are statically determined when the Haar and v. Karman hypothesis is satisfied (Ishlinsky, 1944; Shield, 1955; Ishlinsky and Ivlev, 2001). However, there are no analytical investigations for the contact problem of an elastic indenter and a plastic half-space. It represents a basic theoretical problem for tests on Meier or Brinell hardness. Its solution can be used for constructing models of mechanical interaction for an elastic rough indenter and a plastic substrate.

In the present paper we investigate approximate equations for the maximal penetration depth and the distribution of contact stresses. It is assumed that the elastic indenter has a curvilinear smooth surface, the free bound of the ideally rigid-perfectly-plastic half-space is plain and the contact area has a small size compared to the indenter size (Johnson, 1985).

2 Definition of Contact Stress

The contact problem can be conveniently studied with the help of the cylindrical polar co-ordinates (r, φ, z), where 0z is axis of symmetry of bodies (Figure 1). The surface of the half-space at the plain z0r after indenter penetration is defined by the equation (Figure 1):

$$z = \mathrm{f}(r)\,,$$

where

$$\mathrm{f}(r) = \begin{cases} \delta(r) + \Delta - \left(\mathrm{u}_z(0) - \mathrm{u}_z(r)\right), r \in [0, a), \\ 0, r \in [a, +\infty), \end{cases} \tag{1}$$

$\delta(r), (\,\delta(0) = 0)$ is the initial equation of indenter surface; Δ is depth of indenter penetration; $\mathrm{u}_z(r)$ is elastic displacement of indenter surface along z-axis; a is the radius of contact area. It is necessary to note that $\mathrm{f}'(r) < 0$ when $r \in [0, a)$. It is supposed that the $(\mathrm{f}'(r))^2$ is negligible in the case of initial penetration. Therefore the deformation of the indenter is similar to the deformation of an elastic half-space. The condition of smoothness of indenter at point $(0, \mathrm{f}(0))$ and the smallness of variation of the derivative can be described by following inequality:

$$\left|\delta'(r) - \mathrm{u}'_z(r)\right| \le M \cdot r, r \in [0, a]\,, \tag{2}$$

where $M\,(0 < M << 1)$ is a constant such that $(M \cdot a)^2$ is a negligible value.

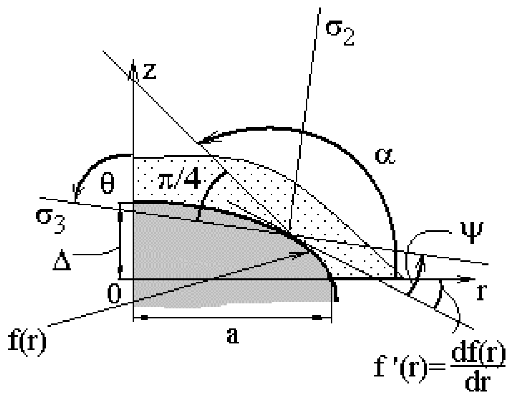

Figure 1. Plane section of contact of axially symmetric bodies

The stress distribution in the half-space involves the four stress components $\sigma_r, \sigma_\varphi, \sigma_z, \tau_{rz}$. The circumferential stress σ_φ is a principal stress. These components satisfy the equation of equilibrium (Ishlinsky, 1944; Shield, 1955; Ishlinsky and Ivlev, 2001):

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\varphi}{r} = 0,$$
$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0. \tag{3}$$

Let us use the conditions of "full plasticity" in the form (Ishlinsky, 1944; Shield, 1955; Ishlinsky and Ivlev, 2001):

$$\sigma_1 = \sigma_3 - 2K, \ \sigma_2 = \sigma_1,$$

where $\sigma_{i, i \in \overline{1,3}}$ is a component of principal stress, $K = \sigma_s / 2$, σ_s is yield stress. The following equations are valid (Ishlinsky, 1944; Shield, 1955; Ishlinsky and Ivlev, 2001) (Figure 1):

$$\sigma_r = \sigma - K\sin(2\alpha), \qquad \sigma_z = \sigma + K\sin(2\alpha),$$
$$\sigma_\varphi = \sigma - K, \qquad \tau_{rz} = K\cos(2\alpha), \tag{4}$$
$$\alpha = \theta + \pi/4,$$

where $\sigma = \frac{1}{2}(\sigma_1 + \sigma_3)$, θ is angle between positive direction 0z and third principle stress (Figure 1) (Ishlinsky and Ivlev, 2001; Sokolovsky, 1969).

Let us consider the function $\alpha(r,z)$. It is defined on the contact surface by the boundary condition (Ishlinsky and Ivlev, 2001; Sokolovsky, 1969):

$$\alpha(r, \mathrm{f}(r)) = \mathrm{f}'(r) + \omega, \tag{5}$$

where

$$\omega = \begin{cases} \omega^- = \dfrac{3\pi}{4} + \psi, \ r \in [0, a), \\ \omega^+ = \dfrac{\pi}{4}, \ r \in [a, \infty), \end{cases} \tag{6}$$

$\psi \in [0, \pi/4]$ is a constant angle which is defined by the direction of plastic shear.

Making substitution (4) and (5) into (3), we obtain the following system at the surface of half-space:

$$\begin{gathered}
\frac{\partial \sigma}{\partial r} - 2K\{\cos(2\omega) - 2\,\mathrm{f}'(r)\sin(2\omega)\}\frac{\partial \alpha}{\partial r} - 2K\{\sin(2\omega) + 2\,\mathrm{f}'(r)\cos(2\omega)\}\frac{\partial \alpha}{\partial z} + \\
+\frac{K}{r}\left(1 - \{\sin(2\omega) + 2\,\mathrm{f}'(r)\cos(2\omega)\}\right) = 0 \\
\\
\frac{\partial \sigma}{\partial z} - 2K\{\sin(2\omega) + 2\,\mathrm{f}'(r)\cos(2\omega)\}\frac{\partial \alpha}{\partial r} + 2K\{\cos(2\omega) - 2\,\mathrm{f}'(r)\sin(2\omega)\}\frac{\partial \alpha}{\partial z} + \\
+\frac{K}{r}\{\cos(2\omega) - 2\,\mathrm{f}'(r)\sin(2\omega)\} = 0
\end{gathered} \tag{7}$$

Making some transformation of system (7) after elimination of terms which contain $(\mathrm{f}'(r))^2$ we get the equation:

$$\begin{gathered}
\left\{\frac{\partial \sigma}{\partial r} + \frac{\partial \sigma}{\partial z}\mathrm{f}'(r)\right\} - \\
-2K\left[\cos(2\omega)\left\{\frac{\partial \alpha}{\partial r} + \frac{\partial \alpha}{\partial z}\mathrm{f}'(r)\right\} + \sin(2\omega)\left\{\frac{\partial \alpha}{\partial r}\left(-\mathrm{f}'(r)\right) + \frac{\partial \alpha}{\partial z}\right\}\right] + \\
+\frac{K}{r}\left(1 - \sin(2\omega) - \mathrm{f}'(r)\cos(2\omega)\right) = 0.
\end{gathered} \tag{8}$$

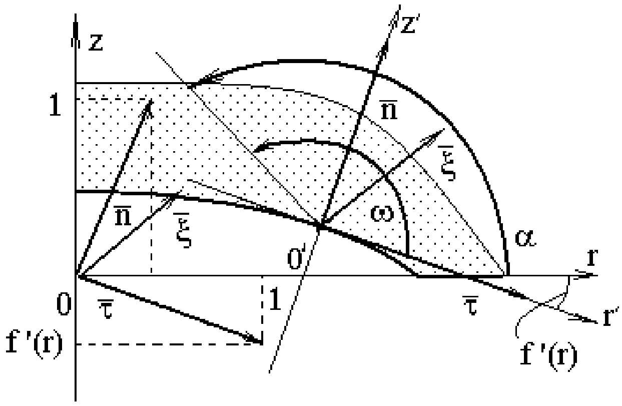

Figure 2. Scheme of additional coordinate system

The vectors $\vec{\boldsymbol{\tau}}$ and $\vec{\mathbf{n}}$ with coordinates $\{1, \mathrm{f}'(r)\}$ and $\{-\mathrm{f}'(r), 1\}$ make the orthonormal basis at any point of contact line $(r, \mathrm{f}(r)), r \in [0, a) \cup (a, +\infty)$ (Figure 2). Let us consider the unit vector $\bar{\xi}$, which is orthogonal to the direction α. Then vector $\left\{\frac{\partial \alpha}{\partial r}, \frac{\partial \alpha}{\partial z}\right\}$ can be represented as $\pm\lambda\bar{\xi}$, where λ is its length. Therefore we have that the following equation is valid (Figure 2):

$$\left(\vec{\mathbf{n}} \cdot \bar{\xi}\right) = \mathrm{tg}(\omega - \pi/2)\left(\vec{\boldsymbol{\tau}} \cdot \bar{\xi}\right).$$

Hence we hold the additional equation:

$$\left\{\frac{\partial \alpha}{\partial r}\left(-\mathrm{f}'(r)\right) + \frac{\partial \alpha}{\partial z}\right\} = -\mathrm{ctg}(\omega)\left\{\frac{\partial \alpha}{\partial r} + \frac{\partial \alpha}{\partial z}\mathrm{f}'(r)\right\}. \tag{9}$$

Using (8), (9), we obtain that the function σ satisfy the differential equation at the bound of half-space:

$$\frac{d\,\sigma}{dr}+2K\frac{d\,\alpha}{dr}=-\frac{K}{r}\left(1+\cos(2\psi)-\mathrm{f}'(r)\sin(2\psi)\right), \tag{10}$$
$$r\in[0,a)\cup(a,\infty)\,.$$

It is known that $\sigma_z=0$ when $r\in[a,\infty)$. Therefore we obtain from (4) the boundary condition:

$$\sigma=-K\ ,\ r\in[a,\infty)\,. \tag{11}$$

The solution of the equation (10) which satisfies the boundary condition (11) when $r\in[0,a)$ is found to be

$$\sigma=-2K\,\mathrm{f}'(r)-2K\left(\frac{\pi}{2}+\psi\right)-K+K\left(1+\cos(2\psi)\right)\ln\left(\frac{r}{a}\right)-K\sin(2\psi)\int_a^r\frac{\mathrm{f}'(\rho)}{\rho}d\rho, r\in[0,a) \tag{12}$$

Hence

$$\sigma_z\approx\sigma-K\cos(2\psi)+2K\,\mathrm{f}'(r)\sin(2\psi)\,. \tag{13}$$

Taking into account (2), (4)-(6), we obtain that

$$\max_{r\in[0,a]}\left\{\left|\frac{\tau_{rz}}{K}-\sin(2\psi)\right|\right\}\leqslant 2\,M\,a\,.$$

Thus

$$\tau_{rz}\approx K\sin(2\psi). \tag{14}$$

The contact friction does not essentially depend on the shape of the indenter in the case of initial penetration.

3 Meyers Hardness for Initial Penetration of Elastic Indenter

Taking into account (12) and (13), Meyers hardness (*HM*) is defined by the following equation:

$$HM=\frac{F}{\pi a^2}=\frac{1}{\pi a^2}\left(2\pi\int_0^a\left(-\sigma_z(r)\right)r dr\right)=\mathrm{C}(a)\sigma_s\ , \tag{15}$$

where

$$\mathrm{C}(a)=\left\{\left(\frac{\pi}{2}+\frac{3}{4}+\psi+\frac{3}{4}\cos(2\psi)\right)-\frac{5}{2}\frac{\sin(2\psi)}{a^2}\int_0^a\mathrm{f}'(r)r dr\right\}\ .$$

Using (2), we obtain the following inequality:

$$\left|\frac{\mathrm{C}(a)}{\left(\frac{\pi}{2}+\frac{3}{4}+\psi+\frac{3}{4}\cos(2\psi)\right)}-1\right|\le\frac{5}{6}\frac{\sin(2\psi)}{\left(\frac{\pi}{2}+\frac{3}{4}+\psi+\frac{3}{4}\cos(2\psi)\right)}M\cdot a\ .$$

Hence the value of Meyers hardness for a half-space does not essentially depend on shape of indenter and its deformability at initial penetration. In addition we can obtain the following approximate equation with sufficient accuracy:

$$C(a) \approx \left(\frac{\pi}{2} + \frac{3}{4} + \psi + \frac{3}{4}\cos(2\psi) \right) .$$

4 Comparison with Other Numerical Results

We can compare analytical solutions (12), (13), (15) with numerical results of other investigations when $u_z(r) = 0, r \in [0, a)$ in the eq. (1). They correspond to the case of a rigid indenter with a smooth surface. An interesting stress distribution is obtained by assuming that the indenter is a flat rigid cylindrical punch. Since it follows that from (12), (13):

$$\sigma_z \big/ K = -(\pi + 2\psi) - 1 - \cos(2\psi) + (1 + \cos(2\psi)) \ln\left(\frac{r}{a} \right) . \quad (16)$$

The calculations of $(-\sigma_z / K)$ with help of (16) and *HM* (15) is similar to the results of numerical investigations done by A.J. Ishlinsky and R.T. Shield when the contact friction is absent ($\psi = 0$) (Figure 3).

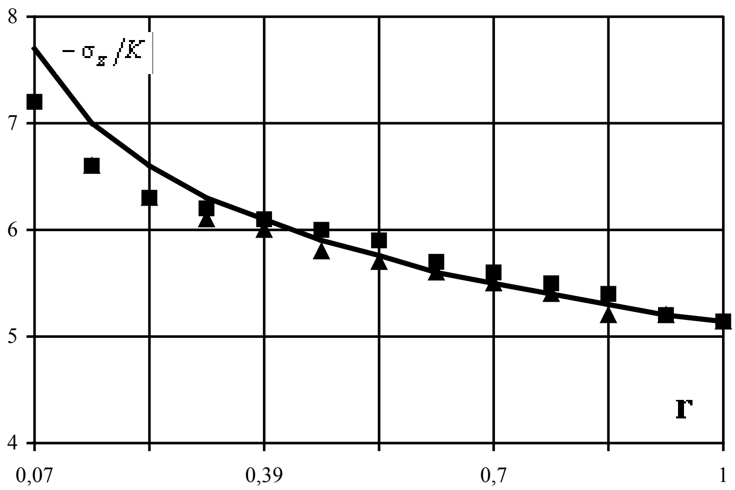

Figure 3. The distribution of $(-\sigma_z / K)$ for a flat rigid cylindrical punch: The line (—) is calculated by (16); The sign (▲) is for numerical results of A.J. Ishlinsky (Ishlinsky, 1944; Ishlinsky and Ivlev, 2001); The sign (■) is for numerical results of R.T. Shield (Shield, 1955);

The value HM / σ_s is close to the numerical results of O. Richmond et al., (Richmond et al., 1974) when $a / R < 0.12$ for the case of no-slip contact between the surfaces of rigid sphere and plastic half-space ($\psi = \pi / 4$).

The difference between results of the present paper and other calculations is explained by the logarithmic contribution in function (12). This term has a singular point and provides significant influence on the accuracy of a numerical calculations reported in the other papers.

We obtain from (15) that force F and contact radius a are related by the following equation:

$$a \approx \sqrt{F \Big/ \left(\pi \cdot \sigma_s \left(\frac{\pi}{2} + \frac{3}{4} + \psi + \frac{3}{4}\cos(2\psi) \right) \right)} . \quad (17)$$

5 Estimation of Penetration Depth for a Deformable Indenter

We obtain from (1) that the depth of plastic penetration (Figure 1) is defined by the following equation:

$$\Delta = -\left(\delta(a) - \left(\mathrm{u}_z(0) - \mathrm{u}_z(a)\right)\right). \tag{18}$$

The elastic displacements $\mathrm{u}_z(r)$ of the indenter may be defined with the help of the well-known Timoshenko – Goodier equations for an elastic half-space (Johnson, 1985; Timoshenko and Goodier, 1979) and (13), (14):

$$\mathrm{u}_z(r) = -\left[4\frac{1-\nu^2}{\pi E}\int_0^a \rho\, \mathrm{p}_n(\rho) \int_0^{\pi/2} \frac{d\psi}{\sqrt{(\rho+r)^2 - 4\rho\cdot r\cdot \sin(\psi)^2}} d\rho - \frac{(1-2\nu)(1+\nu)}{\pi E}\int_r^a \tau_n(\rho)d\rho\right]$$

where

$$\begin{aligned}
\mathrm{p}_n(r) &= -\left(\sigma - K\cos(2\psi) + 2K\,\mathrm{f}'(r)\sin(2\psi)\right) = \\
&= K\left(2\left(1-\sin(2\psi)\right)\mathrm{f}'(r) + \left(\pi + 2\psi + 1 + \cos(2\psi)\right) - \left(1+\cos(2\psi)\right)\ln\left(\frac{r}{a}\right) + \right. \\
&\quad \left. + \sin\left(2\psi\right)\int_a^r \frac{\mathrm{f}'(\rho)}{\rho}d\rho\right), r\in[0,a], \\
\tau_n &\approx K\sin(2\psi), r \in [0,a].
\end{aligned}$$

Last equation, in the range $r\in[0,a]$, can be rewritten as:

$$\mathrm{u}_z(r) - \mathrm{v}(r) = -\left[4\frac{1-\nu^2}{\pi E}K\int_0^a \left(2\left(1-\sin\left(2\psi\right)\right)\mathrm{f}'(\rho) + \sin\left(2\psi\right)\int_a^\rho \frac{\mathrm{f}'(s)}{s}ds\right)\Omega(\rho,r)d\rho\right], \tag{19}$$

where

$$\begin{aligned}
\mathrm{v}(r) = -&\left[4\frac{1-\nu^2}{\pi E}K\left\{\left((\pi+2\psi)+1+\cos(2\psi)\right)\int_0^a \Omega(\rho,r)d\rho - \right.\right. \\
&\left.\left. -\left(1+\cos(2\psi)\right)\int_0^a \ln\left(\frac{\rho}{a}\right)\Omega(\rho,r)d\rho\right\} - \frac{(1-2\nu)(1+\nu)}{\pi E}K\sin(2\psi)(a-r)\right] \\
\Omega(\rho,r) = \rho &\int_0^{\pi/2} \frac{d\varsigma}{\sqrt{(\rho+r)^2 - 4\rho\cdot r\cdot\sin(\varsigma)^2}}.
\end{aligned}$$

Applying the second mean-value theorem for integrals to (19) (Bronshtein, and Semendiaev, 1986), we obtain, that for any point $r\in[0,a]$ there is a point $\gamma\in[0,a]$ for which the following equality is valid:

$$\mathrm{u}_z(r) - \mathrm{v}(r) = 4\frac{1-\nu^2}{\pi E}K\left(2(1-\sin(2\psi))\,\mathrm{f}'(\gamma) + \sin\left(2\psi\right)\int_a^\gamma \frac{\mathrm{f}'(s)}{s}ds\right)\int_0^a \Omega(\rho,r)d\rho.$$

Therefore for any $r\in[0,a]$ the following inequality is valid:

$$\left|u_z(r)-v(r)\right| \le 4\frac{1-\nu^2}{\pi E}K\max_{r\in[0,a]}\left(\left|2(1-\sin(2\psi))\,f'(r)+\sin(2\psi)\int_a^r\frac{f'(s)}{s}ds\right|\right)\cdot\left|\int_0^a\Omega(\rho,r)d\rho\right|$$

Let us consider the space $C[0,a]$ of continuous functions at segment **$[0,a]$** with norm $\|\chi(r)\| = \max_{r\in[0,a]}\left(\left|\chi(r)\right|\right)$.
Hence we obtain that

$$\left\|u_z(r)-v(r)\right\| \le 4\frac{1-\nu^2}{\pi E}K\left\|2(1-\sin(2\psi))\,f'(r)+\sin(2\psi)\int_a^r\frac{f'(s)}{s}ds\right\|\cdot\left\|\int_0^a\Omega(\rho,r)d\rho\right\|$$

where

$$\left\|\int_0^a\Omega(\rho,r)d\rho\right\| = \frac{\pi}{2}a$$

But additional inequalities are valid:

$$\left\|2(1-\sin(2\psi))\,f'(r)+\sin(2\psi)\int_a^r\frac{f'(s)}{s}ds\right\| \le 3M\cdot a\,,$$

$$\begin{aligned}\|v(r)\| &\ge 4\frac{1-\nu^2}{\pi E}K\big((\pi+2\psi)+1+\cos(2\psi)\big)\min_{r\in[0,a]}\left\{\int_0^a\Omega(\rho,r)d\rho\right\}-\frac{(1-2\nu)(1+\nu)}{\pi E}K\sin(2\psi)a = \\ &= 4\frac{1-\nu^2}{\pi E}Ka\left(\big((\pi+2\psi)+1+\cos(2\psi)\big)-\frac{(1-2\nu)}{(1-\nu)}\sin(2\psi)\right) \ge \\ &\ge 4\frac{1-\nu^2}{\pi E}Ka\left(\pi+2-\frac{(1-2\nu)}{(1-\nu)}\right)\end{aligned}$$

Therefore we obtain that:

$$\left\|u_z(r)-v(r)\right\| / \left\|v(r)\right\| \le \frac{3\pi}{2(\pi+1)}M\cdot a$$

Taking into account $(M\cdot a)$ to be small, we obtain the following approximate equality:

$$\begin{aligned}u_z(r) \approx v(r) = -\Bigg[&4\frac{1-\nu^2}{\pi E}K\Bigg\{\big((\pi+2\psi)+1+\cos(2\psi)\big)\int_0^a\Omega(\rho,r)d\rho - \\ &-\big(1+\cos(2\psi)\big)\int_0^a\ln\left(\frac{\rho}{a}\right)\Omega(\rho,r)d\rho\Bigg\}-\frac{(1-2\nu)(1+\nu)}{\pi E}K\sin(2\psi)(a-r)\Bigg]\end{aligned}$$

Thus we obtain that the elastic deformation of the surface of an axially symmetric indenter in the case of initial plastic flow and small penetration depth does not essentially depend on its surface shape. Our last conclusion is similar to the result of the investigation of Meyers hardness under the same conditions.

6 Penetration of an Elastic Sphere into a Rigid-Perfectly-Plastic Half-Space

On the other hand we obtain from equations (17), (18) in the case of elastic sphere that the penetration depth is approximately defined by expression:

$$\Delta_S \approx \frac{1}{2R} a^2 + (\mathrm{u}_z(0) - \mathrm{u}_z(a)) \approx$$
$$\approx \frac{1}{2R} a^2 - 2\frac{1-\nu^2}{\pi E}\sigma_s \cdot a \cdot \left\{ \left((\pi + 2\psi) + 1 + \cos(2\psi)\right)\left(\frac{\pi}{2} - 1\right) + \left(1 + \cos(2\psi)\right)\frac{3}{8}\pi - \frac{(1-2\nu)}{4(1-\nu)}\sin(2\psi) \right\}. \quad (20)$$

The equations (17) and (20) show that $\Delta_S \geq 0$ when the acting force F is greater than some force $\Im_0$, which corresponds to the beginning of plastic penetration. The value of $\Im_0$ is defined by the yield stress of plastic half-space, directions of share at the contact area, Young's modulus, Poisson's ratio for the indenter and its radius:

$$\Im_0 = \left(\pi \cdot \sigma_s \left(\frac{\pi}{2} + \frac{3}{4} + \psi + \frac{3}{4}\cos(2\psi) \right) \right) \cdot \left(4R \frac{1-\nu^2}{\pi E} \sigma_s \left\{ \left((\pi + 2\psi) + 1 + \cos(2\psi)\right)\left(\frac{\pi}{2} - 1\right) + \right. \right.$$
$$\left. \left. + \left(1 + \cos(2\psi)\right)\frac{3}{8}\pi - \frac{(1-2\nu)}{4(1-\nu)}\sin(2\psi) \right\} \right)^2 .$$

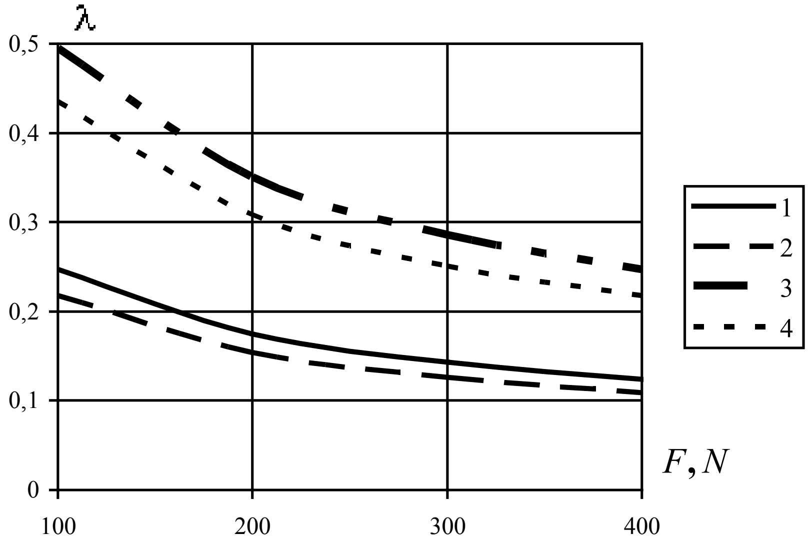

Figure 4. The dependence of dimensionless parameter λ in the case of the penetration of a deformable indenter ($R = 5 \cdot 10^{-3}$ m, $\nu = 0.3$) into an ideal rigid-perfectly-plastic half-space ($\sigma_s = 3.0 \cdot 10^8$ N/m^2): 1 – $\psi = 0$, $E = 2.0 \cdot 10^{11}$ N/m^2; 2 - $\psi = \pi/4$, $E = 2.0 \cdot 10^{11}$ N/m^2; 3 – $\psi = 0$, $E = 1.0 \cdot 10^{11}$ N/m^2; 4 - $\psi = \pi/4$, $E = 1.0 \cdot 10^{11}$ N/m^2;

Thus indenter can penetrate into an ideal rigid-perfectly-plastic half-space only when the inequality is valid:

$$F \geq \Im_0 .$$

Let us define the relative error λ for the penetration depth of an elastic indenter with respect to a rigid one:

$$\lambda = 1 - \Delta_S \frac{2R \cdot \left(\pi \cdot \sigma_s \left(\frac{\pi}{2} + \frac{3}{4} + \psi + \frac{3}{4}\cos(2\psi) \right) \right)}{F} \tag{21}$$

The equations (20), (21) show that the elastic deformations of a spherical indenter are significant in the case of a large indenter radius, large yield stress of half-space and small Young's modulus of the indenter. Increasing the force leads to decrease the influence of indenter deformation on penetration depth (Figure 4).

7 Conclusions

The deformation of the indenter has a significant influence on the penetration depth in the case of the incipient plastic flow for a half-space. The influence increases on increasing the ratio of the yield stress of the half-space to the Young's modulus of the indenter.
The Meier hardness does not depend on shape and deformability of the indenter in the case of initial plastic flow.

We propose to use the solution of the contact problem as an approximation for the penetration of plastic coatings under the assumptions of invariability of shape of the free surface of the plastic body before and after penetration.

Acknowledgments

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