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Local Vibration and Buckling Modes of a Conic Shell. Comparison of Numerical and Asymptotic Results

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Free vibrations and buckling under uniform external lateral pressure of a thin conic shell are analyzed. The asymptotic and finite element methods are used to obtain the vibration frequencies, critical loads, and vibrations and buckling modes.

1 Introduction

The hulls of many types of submarines and also aircrafts are systems of two connected shells. The forward part of the connected shell system has the form of an indirect conic shell. Therefore, the investigation of the vibrations and buckling under external pressure of such shells is very actual. In the paper free vibrations and buckling under uniform external lateral pressure of a thin conic shell are analyzed. The asymptotic results for the vibration frequency parameter and buckling parameter are compared with the numerical results (Finite Element Method) for different types of indirect conic shells. The specific feature of the vibration and buckling modes of the shell is their location in the neighbourhood of the longest shell generatrix. The figures presented in the paper show the deformation process on the shell surface.

2 Basic Definitions. Geometry of an Indirect Conus

Consider an indirect circular conic shell. Further we use the terms ”direct conus” and ”indirect conus”. The circular conus is called ”direct” if the conus top projection in the base plane (point C) coincides with the circle center (point O). The circular conus is called ”indirect” if the points O and C are not coincident as shown in Fig. 1. We introduce the orthogonal Cartesian coordinate system (x, y, z). The radius R of the shell base is taken as unit length. The orthogonal dimensionless coordinate system (s, φ) is introduced on the middle surface of the shell (s is the length of a meridian arc on the shell surface, φ is the angle in the circumferential direction). Later we prove that coordinates s and φ are orthogonal. In Figure 1 the longitudinal section (left) and cross-section (right) of the conic shell are shown.

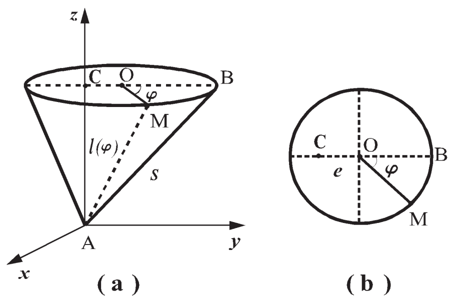

Figure 1. An indirect circular conic shell surface (a) - side view, (b) - top view

The distance OC between the circle center (point O) and the conus top projection on the base plane (point C) is denoted by e. The value $\varphi = 0$ corresponds to the longest generatrix AB of the shell, the distance OB = R is a circle radius. The shell height H = AC. The point M is an arbitrary circumferencial point.

3 Formula for the Length of the Shell Generatrix

The length of the generatrix $AM = l$ of the indirect conic shell is not constant, but depends on the angle φ. To obtain this dependence, we consider ΔCOM in Figure 1(b). We denote the angle OCM further by ψ. According to the sinus theorem and the Pythagoras theorem

$$CM = \frac{\sin\varphi}{\sin\psi}, \tag{1}$$

$$l^2 = H^2 + CM^2, \tag{2}$$

$$\operatorname{tg}\psi = \frac{\sin\varphi}{e + \cos\varphi}, \tag{3}$$

$$\sin^2\psi = \frac{\operatorname{tg}^2\psi}{1 + \operatorname{tg}^2\psi} = \frac{\sin^2\varphi}{e^2 + 2e\cos\varphi + 1}. \tag{4}$$

Substituting (1), (3), (4) into (2) we obtain the dependence between the generartix length of the shell and angle φ in the following form $l^2(\varphi) = H^2 + e^2 + 2e\cos\varphi + 1$.

4 Formula for a Curvature Radius of the Conic Surface

The cartesian coordinates ($x,\ y,\ z$) of point M of the conic surface can be expressed by curvilinear ones (s, φ) as

$$x = \frac{s}{l(\varphi)}\cdot\sin\varphi, \quad y = \frac{s}{l(\varphi)}\cdot(\cos\varphi + e), \quad z = \frac{s}{l(\varphi)}\cdot H. \tag{5}$$

The equation for an indirect conic shell surface in vectorial form can be expressed as

$$\begin{array}{l} \mathbf{r}(s,\varphi) = s\mathbf{p}(\varphi), \quad l\mathbf{p} = \mathbf{a} = (\sin\varphi,\ \cos\varphi + e,\ H) \\ l^2 = H^2 + e^2 + 2e\cos\varphi + 1, \quad \mathbf{pp} = 1, \quad \mathbf{aa} = l^2. \end{array} \tag{6}$$

To obtain the formula for the curvature radius of a conic surface it is necessary to find the coefficients of the first quadratic form

$$\begin{array}{c} d\mathbf{r}^2 = A^2 ds^2 + 2AB\cos\gamma ds d\varphi + B^2 d\varphi^2, \quad d\mathbf{r} = \mathbf{r}_s ds + \mathbf{r}_\varphi d\varphi, \\ A^2 = \mathbf{r}_s\mathbf{r}_s, \quad AB\cos\gamma = \mathbf{r}_s\mathbf{r}_\varphi, \quad B^2 = \mathbf{r}_\varphi\mathbf{r}_\varphi \end{array}. \tag{7}$$

For the conic surface considered

$$\begin{array}{c} \mathbf{r}_s = \mathbf{p}, \quad \mathbf{r}_\varphi = s\mathbf{p}_\varphi, \\ A^2 = \mathbf{pp} = 1, \quad \mathbf{r}_s\mathbf{r}_\varphi = s\mathbf{p}\mathbf{p}_\varphi = \frac{s}{2}(\mathbf{pp})_\varphi = \mathbf{0}. \end{array} \tag{8}$$

Hence, $\gamma = \pi/2$ and curvilinear coordinates $s,\ \varphi$ are orthogonal.
Let us differentiate the equality $l\mathbf{p} = \mathbf{a}$ with respect to φ

$$l_\varphi\mathbf{p} + l\mathbf{p}_\varphi = \mathbf{a}_\varphi = (\cos\varphi,\ -\sin\varphi,\ 0), \quad \mathbf{a}_\varphi\mathbf{a}_\varphi = 1. \tag{9}$$

Let us differentiate the equality $\mathbf{aa} = l^2$ with respect to φ, hence: $\mathbf{a}\mathbf{a}_\varphi = ll_\varphi$

$$\mathbf{p}\mathbf{a}_\varphi = l_\varphi. \tag{10}$$

Formulae (9) and (10) can be used to find the scalar product $\mathbf{p}_\varphi\mathbf{p}_\varphi$:

$$l^2\mathbf{p}_\varphi\mathbf{p}_\varphi = (\mathbf{a}_\varphi - l_\varphi\mathbf{p})(\mathbf{a}_\varphi - l_\varphi\mathbf{p}) = 1 - l_\varphi^2. \tag{11}$$

Then

$$B^2 = s^2\mathbf{p}_\varphi\mathbf{p}_\varphi = \frac{s^2}{l^2}(1 - l_\varphi^2), \quad B = \frac{s}{l}\sqrt{1 - l_\varphi^2}, \quad l_\varphi = -\frac{e}{l}\sin\varphi. \tag{12}$$

The main curvature radius of the conic surface can be defined through the second quadratic surface form $Lds^2 + 2Mdsd\varphi + Nd\varphi^2$, where

$$L = \mathbf{r}_{ss}\mathbf{n}, \quad M = \mathbf{r}_{s\varphi}\mathbf{n}, \quad N = \mathbf{r}_{\varphi\varphi}\mathbf{n}, \quad \mathbf{n} = \mathbf{m}/|\mathbf{m}|, \quad \mathbf{m} = \mathbf{r}_s \times \mathbf{r}_\varphi. \tag{13}$$

For the conic surface $L = M = 0$, because of

$$\mathbf{r}_{ss} = \mathbf{p}_s = 0, \quad \mathbf{r}_{s\varphi}\mathbf{n} = \mathbf{p}_\varphi(\mathbf{p} \times s\mathbf{p}_\varphi)/|\mathbf{m}| = 0. \tag{14}$$

To obtain the main curvature radius of the conic surface $R_2 = -B^2/N$ we have to find the coefficients N. If we differentiate formula (9) with respect to φ

$$l_{\varphi\varphi}\mathbf{p} + 2l_\varphi\mathbf{p}_\varphi + l\mathbf{p}_{\varphi\varphi} = \mathbf{a}_{\varphi\varphi} \tag{15}$$

and multiply it by vector $\mathbf{m} = s\mathbf{p} \times \mathbf{p}_\varphi$

$$\mathbf{p}_{\varphi\varphi}\mathbf{m} = \frac{1}{l}\mathbf{a}_{\varphi\varphi}\mathbf{m} \tag{16}$$

taking into account the orthogonality of vectors $\mathbf{p}$ and $\mathbf{p}_\varphi$ we get the length of the vector $\mathbf{m}$

$$|\mathbf{m}| = s|\mathbf{p} \times \mathbf{p}_\varphi| = s|\mathbf{p}||\mathbf{p}_\varphi| = s|\mathbf{p}_\varphi| = B. \tag{17}$$

According to formulae (9)–(18) we obtain

$$\mathbf{m} = \frac{s}{l^2}(l\mathbf{p} \times l\mathbf{p}_\varphi) = \frac{s}{l^2}[\mathbf{a} \times (\mathbf{a}_\varphi - l_\varphi\mathbf{p})] = \frac{s}{l^2}(\mathbf{a} \times \mathbf{a}_\varphi), \tag{18}$$

$$N = \mathbf{r}_{\varphi\varphi}\mathbf{n} = \frac{s}{B}\mathbf{p}_{\varphi\varphi}\mathbf{m} = \frac{s^2}{Bl^3}(\mathbf{a} \times \mathbf{a}_\varphi)\mathbf{a}_{\varphi\varphi}. \tag{19}$$

Taking into account the equality

$$(\mathbf{a} \times \mathbf{a}_\varphi)\mathbf{a}_{\varphi\varphi} = \begin{vmatrix} \sin\varphi & \cos\varphi + e & H \\ \cos\varphi & -\sin\varphi & 0 \\ -\sin\varphi & -\cos\varphi & 0 \end{vmatrix} = -H, \tag{20}$$

we obtain

$$N = -\frac{Hs^2}{Bl^3}, \quad R_2 = -\frac{B^2}{N} = \frac{B^3l^3}{Hs^2} = \frac{s}{H}(1 - l_\varphi^2)^{3/2}. \tag{21}$$

In case of the direct conus $e = 0$, $l_\varphi = 0$, $B = s/l$, $R_2 = s/H$.

5 Buckling of the Conic Shell

The dimensionless equations describing the buckling of a thin elastic conic shell under uniform lateral external pressure p Bauer et al. (1993), can be written as

$$\varepsilon^4\Delta^2 w + \lambda\varepsilon^2\Delta_t w - \Delta_k\Phi = 0, \tag{22}$$

$$\varepsilon^4\Delta^2\Phi + \Delta_k w = 0, \tag{23}$$

where

$$\Delta_t = \frac{1}{B}\frac{\partial}{\partial\varphi}\left(\frac{t_2}{B}\frac{\partial w}{\partial\varphi}\right), \quad t_2 = R_2, \quad \lambda = \frac{p}{Eh\varepsilon^6}, \tag{24}$$

$$\Delta w = \frac{1}{s^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial w}{\partial s}\right), \quad \Delta_k w = \frac{k}{s}\frac{\partial^2 w}{\partial s^2}, \quad \varepsilon^8 = \frac{h^2}{12(1-\nu^2)R^2}. \tag{25}$$

Here $w(s,\varphi)$ is a component of normal displacement, $\Phi(s,\varphi)$ is a stress function, k is a curvature of the shell surface, h is a dimensionless shell thickness, R is a radius of a circumference in the base of conus, t_2 is a dimensionless tangent stress function, B is a distance between a point of a middle surface and the shell height, R_2 is a radius of curvature, λ is a buckling parameter (in the next section - a frequency parameter), ν is Poisson's ratio, E is Young's modulus, ρ is the mass density, p is the value of the external pressure. The dimensionless thickness h is a small parameter.

$$k(\varphi) = \frac{s}{R_2(\varphi)}, \quad R_2(\varphi) = \frac{s}{H}\left(1 - l_\varphi^2\right)^{\frac{3}{2}}, \quad B(\varphi) = \frac{s}{l}\left(1 - l_\varphi^2\right)^{\frac{1}{2}}, \tag{26}$$

$$l^2(\varphi) = H^2 + e^2 + 2e\cos\varphi + 1. \tag{27}$$

The boundary conditions for system (22) – (23) can be written as

$$u_n = u_t = w = \theta_n = 0, \tag{28}$$

when the shell edges are clamped. In case when the shell edges are simply supported

$$T_n = u_t = w = M_n = 0 \tag{29}$$

and when the shell edges are free

$$T_n = S_{nt} = M_n = N_n = 0. \tag{30}$$

In conditions (28) – (30) we use the variables proposed by (Filippov (1999))

$$u_n = u\cos\gamma + v\sin\gamma, \quad u_t = v\cos\gamma - u\sin\gamma, \tag{31}$$

$$T_n = T_1\cos^2\gamma + 2S\sin\gamma\cos\gamma + T_2\sin^2\gamma, \tag{32}$$

$$M_n = M_1\cos^2\gamma + 2H\sin\gamma\cos\gamma + M_2\sin^2\gamma, \tag{33}$$

$$\theta_n = \theta_1\cos\gamma + \theta_2\sin\gamma, \quad N_n = N_1\cos\gamma + N_2\sin\gamma, \tag{34}$$

$$S_{nt} = (T_2 - T_1)\sin\gamma\cos\gamma + S\left(\cos^2\gamma - \sin^2\gamma\right), \quad \sin\gamma = \frac{e}{l}\sin\varphi. \tag{35}$$

The angle γ in the expressions for the functions u_n, u_t, T_n, M_n, θ_n, N_n, S_{nt}, is the angle between the shell edge and a coordinate line s = const. According to a procedure proposed by P.E.Tovsik (1995), the asymptotic solution of the boundary value problem for equations (22)–(23), can be expressed as

$$w(s,\varphi,\varepsilon) = w^0 \exp\left\{\frac{i}{\varepsilon}\int\limits_{\varphi_0}^{\varphi} q(\varphi)d\varphi\right\}, \tag{36}$$

$$w^0 = \sum_{n=0}^{\infty} \varepsilon^n w_n^0(s,\varphi), \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \tag{37}$$

where

$$\operatorname{Im} q(\varphi_0) = 0, \quad \operatorname{Im}\left\{\frac{dq}{d\varphi}(\varphi_0)\right\} > 0. \tag{38}$$

Function Φ has similar asymptotic expansion. It follows from conditions (38) that the functions w, Φ have location near the line $\varphi = \varphi_0$. By substituting expressions (36) and (37) into (22), (23) and boundary conditions, we get the equations for $q(\varphi), w_n(s,\varphi), \Phi_n(s,\varphi)$ and values λ_n. In the zeroth-order approximation we obtain

$$k\frac{\partial^2\Phi_0}{\partial s^2} - \left(\lambda_0 - \frac{q^4}{B^4}\right) w_0 = 0, \tag{39}$$

$$k\frac{\partial^2 w_0}{\partial s^2} + \frac{q^4}{B^4}\Phi_0 = 0 \tag{40}$$

with the boundary conditions

$$w_0 = \Phi_0 = 0 \text{ if } s = s_i(\varphi) \tag{41}$$

when the shell edges are simply supported,

$$w_0 = \frac{\partial w_0}{\partial s} = 0 \text{ if } s = s_i(\varphi), \quad i = 1, 2. \tag{42}$$

when the shell edges are clamped.

The boundary conditions (41)–(42) were obtained separating all boundary conditions (28)–(29) into the main and additional ones. The choice of boundary conditions for equations (39)–(40) is discussed in Tovstik (1995). The elimination of the function Φ_0 from the system (39)–(40) gives the equation of fourth-order concerning a function w_0 and value λ_0. The magnitude of a critical load can be deduced from λ_0. Due to the fact that we can not define the analytical exact solution of the equation system (39)–(40) we use a numerical method of a matrix orthogonalization Godunov (1961). To apply this method we introduce a vector $\mathbf{y}$ which contains the unknown functions

$$\mathbf{y} = (w_0, w_0', \Phi_0, \Phi_0'). \tag{43}$$

Then the system (39)–(40) of two differential equations of the second order can be reduced to the system of four equations of the first order

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -\frac{q^4}{kB^4} y_3, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = \left(\frac{q^4}{B^4} - \lambda_0 \frac{q^2 R_2}{B}\right) \cdot \frac{1}{k} y_1 \end{cases} \tag{44}$$

The system (44) can be presented in a vector form

$$\dot{\mathbf{y}} = A \cdot \mathbf{y}. \tag{45}$$

The boundary conditions (41)–(42) in the new variables take the form for simply supported shell edges

$$y_4 = y_3 = 0, \tag{46}$$

for the clamped shell edges

$$y_1 = y_2 = 0. \tag{47}$$

The system (44) can be solved by using numerical methods. Consider the buckling of a thin indirect conic shell under uniform external lateral pressure. The shell generatrix $\varphi = \varphi_0$ is the weakest one. For given examples $\varphi_0 = 0$ (we prove it in the next section) corresponds to the longest generatrix. As an example, a direct conic shell under external lateral pressure is presented in Figure 2. The buckling modes plotted by FEM (side view (left) and top view (right)) cover the shell surface uniformly .

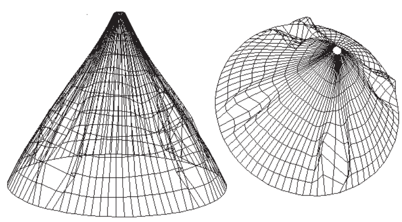

Figure 2. Buckling mode of a direct circle conic shell ($e = 0$)

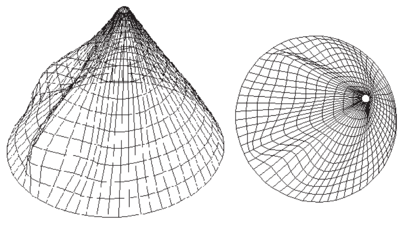

Figure 3. Buckling mode of an indirect conic shell ($e = 0.3$)

When $e \neq 0$ (Fig. 3–5), the buckling mode has a location near the longest generatrix of the conic shell.

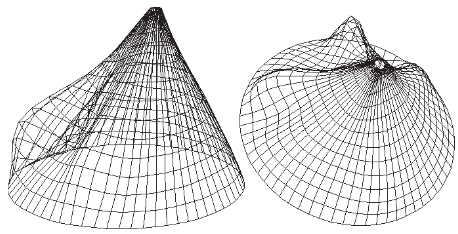

Figure 4. Buckling mode of an indirect conic shell ($e = 0.5$)

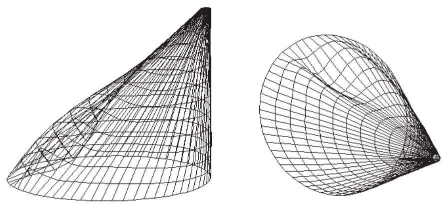

Figure 5. Buckling mode of an indirect conic shell ($e = 1$)

Assuming that the both shell edges are clamped and bounded by parallels $s = s_i,\ i = 1, 2$, numerical and asymptotic calculations were performed.

e	λ_0	λ, $\frac{h}{R} = 0.001$ (FEM)	λ, $\frac{h}{R} = 0.01$ (FEM)
0.	11.256	11.359	10.107
0.1	9.496	9.550	9.550
0.2	7.839	8.884	8.888
0.3	6.697	7.718	8.367
0.4	5.736	7.364	7.405
0.5	4.484	6.878	6.923

Table 1. The values of a critical external pressure in dependence on a shell eccentricity e.

The results of the asymptotic (formulae (37)—(55)) and numerical calculations (FEM) of the pressure parameter are presented in a dimensionless form. In the third column for $\frac{h}{R} = 0.001$ and in the fourth column for $\frac{h}{R} = 0.01$ one can see the numerical results obtained by using of the Finite Elements Method. The following values of shell parameters were used: $R = 1\ m$, $H = 1.6\ m$, $\nu = 0.3$, $E = 1.93 \cdot 10^{11} Pa$. The data of Table 1 show that the increase of the distance e leads to decrease of the external pressure. In case $e = 0.3$ (Fig. 3) we can see the beginning of the location of a buckling mode near the longest shell generatrix. With a further increase of the distance e (Fig. 4–5), the buckling modes have a location only near the longest shell generatrix. In particular, when $e = 0.5$ one of the shell generatrix has the length 1.9 for $\varphi = \pi$, when $\varphi = 0$ the longest shell generatrix has the length 2.3, the zeroth-order approximation of the critical pressure and magnitude λ obtained by FEM are very different. When the shell thickness increases, the relative errors in the asymptotic results compared with the numerical ones increase respectively. When $\frac{h}{R} = 0.001$, the distinction between λ_0 and λ (FEM) is equal to 0.9% for $e = 0$, and 34% for $e = 0.5$. When $\frac{h}{R} = 0.01$, the distinction between λ_0 and λ obtained by FEM is equal to 11.3% for $e = 0$, and 35% for $e = 0.5$. To correct the magnitudes of the zeroth-order approximation λ_0, we search the first-order approximation λ_1 for the buckling parameter in the next section .

6 The First-Order Approximation for the Buckling Parameter

As was mentioned above, the system (39 - 40) of the second order differential equations can be reduced to one differential equation of the forth order

$$-k\frac{\partial^2}{\partial s^2}\left(\frac{B^4}{q^4}k\frac{\partial^2 w_0}{\partial s^2}\right) - \frac{q^4}{B^4}w_0 + \lambda q^2\frac{R_2}{B^2}w_0 = 0. \tag{48}$$

We divide the equation by q^2 and multiply by $-\frac{B^2}{R_2}$. Taking into account the connection between the curvature k and the radius of the curvature R_2: $k = \frac{1}{R_2}$, the equation (48) will be

$$\frac{B^2}{q^6 R_2^2}\frac{\partial^2}{\partial s^2}\left(\frac{B^4}{R_2}\frac{\partial^2 w_0}{\partial s^2}\right) + \frac{q^2}{B^2 R_2}w_0 = \lambda w_0. \tag{49}$$

The differential operator situated at the left part of the equation (49) is denoted as

$$D^4 w_0 = \frac{B^2}{R_2^2}\frac{\partial^2}{\partial s^2}\left(\frac{B^4}{R_2}\frac{\partial^2 w_0}{\partial s^2}\right). \tag{50}$$

Then (49) can be presented as

$$\frac{1}{q^6}D^4 w_0 = \alpha^4 w_0, \quad \alpha^4 = \lambda - \frac{q^2}{B^2 R_2}. \tag{51}$$

We differentiate the equality (49) with respect to q

$$-\frac{6B^2}{q^7 R_2^2}\frac{\partial^2}{\partial s^2}\left(\frac{B^4}{R_2}\frac{\partial^2 w_0}{\partial s^2}\right) + \frac{2q}{B^2 R_2}w_0 = \lambda_q w_0 \tag{52}$$

and multiply (52) by Bw_0 and take the integral on the interval $[s_1, s_2]$, taking into account (51), we obtain

$$\lambda_q = -\frac{6\lambda_0}{q} + 8q\frac{I_1}{I_2}, \tag{53}$$

where

$$I_1 = \int_{s_1}^{s_2}\frac{w_0^2}{BR_2}ds, \quad I_2 = \int_{s_1}^{s_2}Bw_0^2 ds. \tag{54}$$

According to (51) buckling parameter λ is the function of the parameters q and φ: $\lambda = \frac{q^2}{B^2(\varphi)R_2(\varphi)} + \alpha^4$ As the zeroth-order approximation for the eigenvalue λ we select

$$\lambda_0 = \min_{q,\varphi} f(q,\ \varphi) = f(q_0,\ \varphi_0). \tag{55}$$

Then $\lambda_q = \frac{\partial f}{\partial q} = 0, \quad \lambda_\varphi = \frac{\partial f}{\partial \varphi} = 0$ for $q = q_0$, $\varphi = \varphi_0$. Solve equation $\lambda_q = 0$ and define the magnitude q_0 that gives the minimum of $\lambda(q, \varphi)$. Further we assume q_0 instead of q

$$q = q_0 = \left(\frac{3I_2\lambda}{4I_1}\right)^{1/2}. \tag{56}$$

We differentiate equality (52) with respect to q, multiply it by Bw_0, and take the integral on interval $[s_1, s_2]$ then

$$\lambda_{qq} = \frac{42\lambda_0}{q^2} - 40\frac{I_1}{I_2}. \tag{57}$$

To find λ_φ and $\lambda_{\varphi\varphi}$ we consider the zeroth-order approximation system (39)–(40). The system can be written by the following differential operator

$$L(w_0, \Phi_0) = 0. \tag{58}$$

The elements of the matrix L can be written as

$$l_{11} = -\left(\lambda_0 - \frac{q^4}{B^4}\right), \quad l_{12} = k\frac{\partial^2}{\partial s^2}, \tag{59}$$

$$l_{21} = k\frac{\partial^2}{\partial s^2}, \quad l_{22} = \frac{q^4}{B^4}. \tag{60}$$

The first derivation of equality (58) with respect to φ can be expressed as

$$\frac{\partial}{\partial\varphi}L(w_0, \Phi_0) = \frac{\partial L}{\partial\varphi}(w_0, \Phi_0) + L(w_\varphi, \Phi_\varphi) = 0. \tag{61}$$

The generatrix $\varphi = \varphi_0$ is the weakest one if φ_0 satisfies equation $\lambda_\varphi = 0$. In the case $\varphi_0 = 0$. The second derivative of equality (58) with respect to φ can be expressed as

$$\frac{\partial^2}{\partial\varphi^2}L(w_0, \Phi_0) = \frac{\partial^2 L}{\partial\varphi^2}(w_0, \Phi_0) + 2\frac{\partial L}{\partial\varphi}(w_\varphi, \Phi_\varphi) + L(w_{\varphi\varphi}, \Phi_{\varphi\varphi}) = 0. \tag{62}$$

In equation (62) we use the following denotations:

$$\frac{\partial^2 L}{\partial\varphi^2} = \begin{pmatrix} -\lambda_{\varphi\varphi} + \frac{\partial^2}{\partial\varphi^2}\left(\frac{q^2}{B^2 R_2}\right), & \frac{\partial^2}{\partial\varphi^2}\left(\frac{-B^2}{q^2 R_2^2}\right)\cdot\frac{\partial^2}{\partial s^2} \\ \frac{\partial^2 k}{\partial\varphi^2}\cdot\frac{\partial^2}{\partial s^2}, & \frac{\partial^2}{\partial\varphi^2}\left(\frac{q^4}{B^4}\right) \end{pmatrix} \tag{63}$$

$$\frac{\partial L}{\partial\varphi} = \begin{pmatrix} -\lambda_\varphi + \frac{\partial}{\partial\varphi}\left(\frac{q^2}{B^2 R_2}\right), & \frac{\partial}{\partial\varphi}\left(\frac{-B^2}{q^2 R_2^2}\right)\cdot\frac{\partial^2}{\partial s^2} \\ \frac{\partial k}{\partial\varphi}\cdot\frac{\partial^2}{\partial s^2}, & \frac{\partial}{\partial\varphi}\left(\frac{q^4}{B^4}\right) \end{pmatrix} \tag{64}$$

The coefficients in functions w_φ, Φ_φ and their derivatives with respect to s in formula (62) are equal to zero when $\varphi = 0$, hence, to find $\lambda_{\varphi\varphi}$ we take into account only the first and the third components of the sum in (62). Then

$$\lambda_{\varphi\varphi} = \frac{1}{I_2}\left(J_1 + J_2 + J_3 + J_4 + J_5\right) \tag{65}$$

where

$$J_1 = -\frac{1}{q^2}\int_{s_1}^{s_2}\frac{\partial^2}{\partial\varphi^2}\left(\frac{-B^2}{R_2^2}\right)Bw_0\cdot\frac{\partial^2\Phi_0}{\partial s^2}ds, \tag{66}$$

$$J_2 = q^2\int_{s_1}^{s_2}\frac{\partial^2}{\partial\varphi^2}\left(\frac{1}{B^2 R_2}\right)Bw_0^2 ds, \; J_3 = -q^2\int_{s_1}^{s_2}\frac{\partial^2}{\partial\varphi^2}\left(\frac{1}{B^4}\right)B\Phi_0^2 ds, \tag{67}$$

$$J_4 = -\int_{s_1}^{s_2}\frac{\partial^2 k}{\partial\varphi^2}B\Phi_0\frac{\partial^2 w_0}{\partial s^2}ds, \; J_5 = -\left[kB\Phi_0\frac{\partial w_{\varphi\varphi}}{\partial s}\right]_{s_1}^{s_2}. \tag{68}$$

According to Tovstik (1995), the first-order correction for the eigenvalue λ can be found by the following formula

$$\lambda_1 = \frac{1}{2}\left(\lambda_{qq}\cdot\lambda_{\varphi\varphi} - \lambda_{q\varphi}\right)^{\frac{1}{2}} \tag{69}$$

where the partial derivatives (57) and (65) are calculated for $q = q_0$, $\varphi = \varphi_0$ and $\lambda_{q\varphi} = 0$.

As the example we consider an indirect conic shell with $e = 0.5$.

λ_0	$\lambda_0 + \varepsilon\lambda_1$	λ (FEM)
4.484	5.775	6.878

Table 2. The values of the critical external pressure.

The critical pressure obtained with the help of the asymptotic formulas (37), (69) is presented in the second column. In the third column one can see the numerical results computed by the finite element method. About 1600 four-node shell elements were used in calculation. The computation time for one pressure by FEM is a few minutes. The maximal relative error in the asymptotic results compared with the numerical ones is 6%.

7 The Vibrations of an Indirect Conic Shell. The Zeroth-Order Approximation

For the problem of the shell vibrations, the component $\varepsilon^2\Delta_t w$ in formula (22) should be changed into $-w$. The dimensionless equations describing the small free vibration of a thin elastic conic shell, can be written as

$$\varepsilon^4\Delta^2 w - \lambda w - \Delta_k\Phi = 0, \quad \varepsilon^4\Delta^2\Phi + \Delta_k w = 0, \tag{70}$$

where $\lambda = \frac{\rho R^2\omega^2}{E\varepsilon^4}$ and ω is the vibration frequency. Further we apply the asymptotic method proposed by Tovstik (see the formulae (36) – (37)). In the zeroth-order approximation we obtain

$$\frac{q^4}{B^4}w_0 - \lambda_0 w_0 - k\frac{\partial^2\Phi_0}{\partial s^2} = 0, \; \frac{q^4}{B^4}\Phi_0 + k\frac{\partial^2 w_0}{\partial s^2} = 0. \tag{71}$$

We can find the magnitudes for the frequency parameter λ_0 by numerical integration of the differential equation system (71) with boundary conditions (41) or (42). The magnitudes of λ_0 and λ are presented in Table 3 for an example where the shell edges are clamped

$$w_0 = \frac{\partial w_0}{\partial s} = 0 \text{ for } s = s_i(\varphi), \quad i = 1, 2. \tag{72}$$

In the third and in the forth columns one can see the numerical results computed by the finite element method. In the third column the relation of shell thickness and the radius of the shell base is equal to $\frac{h}{R} = 0.001$ and in the forth column – $\frac{h}{R} = 0.01$.

e	λ_0	λ, $\frac{h}{R} = 0.001$ (FEM)	λ, $\frac{h}{R} = 0.01$ (FEM)
0.	29.1	29.7	23.5
0.1	28.8	29.2	22.8
0.2	25.8	27.6	21.2
0.3	22.5	25.8	19.5
0.4	19.6	23.8	18.4
0.5	16.8	22.2	17.2

Table 3. The values of the frequency parameters vs. shell eccentricity e.

In case $e = 0$ the conic shell is a direct one. The data of Table 3 show that with increasing of the distance e the zeroth-order approximations λ_0 is noticeable distinguished from the magnitudes λ obtained by using the finite element method. When the shell thickness increases, the relative errors in the asymptotic results compared with the numerical ones increase respectively. When $\frac{h}{R} = 0.001$, the distinction between λ_0 and λ (FEM) is equal to 2% for $e = 0$, and 19.5% for $e = 0.5$. When $\frac{h}{R} = 0.01$, the distinction between λ_0 and λ obtained by FEM is equal to 19.2% for $e = 0$, and 2.3% for $e = 0.5$. To correct the magnitudes of the zeroth-order approximation λ_0 we obtain the correction of the first-order approximation λ_1 for the frequency parameter.

For a direct circular conic shell (Fig. 6) the vibration modes are uniformly distributed over the shell surface. If $e = 0.3$ (Fig. 7) one can see the beginning of the location of the vibration mode near the longest shell generatrix.

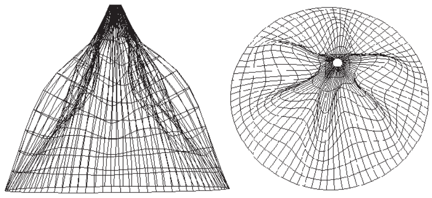

Figure 6. Vibration mode of a direct conic shell ($e = 0$)

With further increasing of the distance e (Fig. 8–9), the vibration modes have a location near the longest shell generatrix.

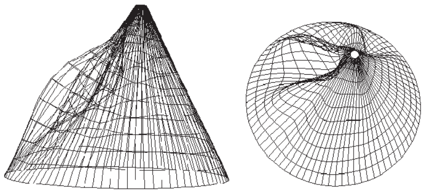

Figure 7. Vibration mode of an indirect conic shell ($e = 0.3$)

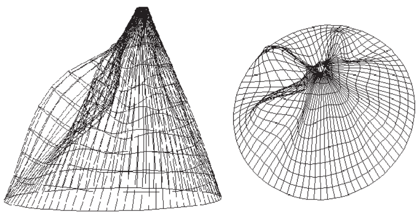

Figure 8. Vibration mode of an indirect conic shell ($e = 0.5$)

8 The First-Order Approximation for the Frequency Parameter

To define the correction of the first-order approximation λ_1 for the frequency parameter, we use the method described in previous section for the system (71). As a result we obtain

$$\lambda_q = -\frac{4\lambda_0}{q} + 8q^3\frac{I_1}{I_2}, \tag{73}$$

where

$$I_1 = \int_{s_1}^{s_2} \frac{w_0^2}{B^3}ds,\ I_2 = \int_{s_1}^{s_2} Bw_0^2 ds, \tag{74}$$

$$\lambda_q = 0, \tag{75}$$

$$q = q_{min} = \left(\frac{I_2\lambda}{2I_1}\right)^{1/4}, \tag{76}$$

$$\lambda_{qq} = \frac{20\lambda_0}{q^2} - 8q^2\frac{I_1}{I_2}, \tag{77}$$

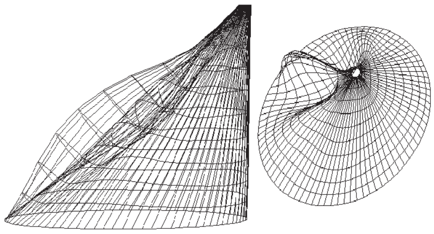

Figure 9. Vibration mode of an indirect conic shell ($e = 1$)

$$\lambda_{\varphi\varphi} = \frac{1}{I_2}\left(J_1 + J_2 + J_3 + J_4 + J_5\right). \tag{78}$$

where

$$J_1 = q^4 \int_{s_1}^{s_2} \frac{\partial^2}{\partial \varphi^2}\left(\frac{1}{B^4}\right) B w_0^2 ds, \tag{79}$$

$$J_2 = -\int_{s_1}^{s_2} \left(\frac{\partial^2 k}{\partial \varphi^2} B \frac{\partial^2 w_0}{\partial s^2} + 2\left[\frac{\partial}{\partial s}\frac{\partial^2 k}{\partial \varphi^2}\right] B \frac{\partial w_0}{\partial s}\right) ds, \tag{80}$$

$$J_3 = -q^4 \int_{s_1}^{s_2} \frac{\partial^2}{\partial \varphi^2}\left(\frac{1}{B^4}\right) B \Phi_0^2 ds, \; J_4 = -\int_{s_1}^{s_2} \frac{\partial^2 k}{\partial \varphi^2} B \Phi_0 \frac{\partial^2 w_0}{\partial s^2} ds, \tag{81}$$

$$J_5 = -\left[k B \Phi_0 \frac{\partial w_{\varphi\varphi}}{\partial s} - k B \frac{\partial \Phi_0}{\partial s} w_{\varphi\varphi}\right]_{s_1}^{s_2}. \tag{82}$$

The first-order correction for the eigenvalue λ can be found by formula (69), where the partial derivatives (77) and (78) are calculated for $q = q_0$, $\varphi = \varphi_0$ and $\lambda_{q\varphi} = 0$.

As an example we consider an indirect conic shell with $e = 0.5$.

λ_0	$\lambda_0 + \varepsilon\lambda_1$	λ (FEM)
16.8	21.3	22.2

Table 4. The values of the frequency parameters.

According to the data of Table 4. the zeroth-order approximation is $\lambda_0 = 16.8$, and if taking into account the correction of the first-order approximation $\lambda_0 + \varepsilon\lambda_1 = 21.3$. The relative discrepancy in asymptotic and numerical results is 4.1%.

The presented numerical calculations were performed for conic shells with the following material properties: $E = 7.3 \cdot 10^{10}\ N/m^2$ is Young's modulus, $\nu = 0.33$ is Poisson's ratio, $\rho = 2770\ kg/m^3$ is the mass density.

The simple approximation asymptotic formulas for the frequency and buckling parameters are derived. The comparison of asymptotic and FEM results shows the reliability of the presented formulae. However, the advantage of the asymptotic formulas is their relative simplicity and effective applications compared with the finite element method programs.

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