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On Potential Energy Shifts in Hyperelastic Energy-Momentum Tensors

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The invariance of the so-called energy-momentum format of the (spatial motion) Cauchy and the (material motion) Eshelby stress tensors of hyperelasticity is discussed with respect to shifts of the potential energy density. As an noteworthy result it turns out that the duality of the spatial and the material motion problem renders the corresponding energy-momentum tensors either invariant or equipped with an additional pressure-like contribution, respectively. This additional pressure like contribution captures the different total potential energy content due to the shift in potential energy density.

1 Introduction

The usual spatial motion formulation of continuum mechanics considers variations of the spatial placements of the continuum 'particles' with respect to the ambient space as power conjugate to what we shall call spatial forces. Seeking for equilibrium of these forces in the deformational problem allows one to determine from the knowledge of the (undeformed) material configuration and the applied loading the (deformed) spatial configuration. Nevertheless the reversal of this problem is possible and indeed renders additional meaningful information: based on the knowledge of the (deformed) spatial configuration variations of the material placement of the continuum 'particles' with respect to the ambient material are considered as power conjugate to what we shall call material forces. Conceptually, seeking for equilibrium of these forces in the configurational problem would allow one to determine a (undeformed) material configuration that possesses no energetic benefit from configurational changes. In reality though, continuum bodies are rarely in configurational equilibrium but rather display the tendency for configurational changes in order to release energy. A paradigmatic example is a specimen with a crack, a possible change of the crack length potentially releases energy that could then be reinvested into other physical processes like e.g. the creation of new surfaces. Material forces are thus a measure for the energetic changes that go along with potential configurational variations, they are driving forces acting on all kind of defects like e.g. cracks, phase boundaries, inclusions, dislocations, vacancies and thelike. Thereby material forces indicate the tendencies of these kind of defects to move relative to the ambient material.

In a continuum the equilibrium or absense of equilibrium is expressed by the quasi-static balance or unbalance of momentum. The fluxes participating in the corresponding relations of the spatial and the material motion problem are the Cauchy and the Eshelby stress. These stresses allow for a particular representation that is known as the energy-momentum format due to its formal similarity with the Maxwell stress in electro/magnetodynamics. Thereby the energy density contributes without any derivatives applied to it to the spherical part of these stress tensors. It is due to this particular format of the energy-momentum tensor that the question arises whether or not a shift of the potential energy density by a constant leaves the Cauchy or the Eshelby stress invariant? This question is of course motivated by the usual understanding that the zero level-set of a potential is completely irrelevant. In the present case it turns out that a shift of the potential energy density is not necessarily irrelevant depending on the type of motion problem, spatial or material, that is actually considered. It will be demonstrated that for a particular choice of potential energy per unit volume in either the material or the spatial configuration that is shifted by a constant, stresses of the one problem remain invariant whereas the stresses of the corresponding dual problem transform in a meaningful way displaying an additional pressure like contribution.

Publications on configurational mechanics are numerous by now, here we shall merely mention the comprehensive treatments in the books by Maugin (1993), Gurtin (2000) and Kienzler and Herrmann (2000) that display various viewpoints regarding the nature of configurational forces. Moreover the topic has been popularised in the overview papers by Maugin (1995) and Gurtin (1995). Early works that deal with the duality of the spatial (the direct) and the material (the inverse) motion problem together with the detailed consideration of the properties of the energy-momentum tensor are given by Shield (1967), Chadwick (1975), Eshelby (1975) and Hill (1986). A recent overview on the usefulness of configurational mechanics is presented by Gross et al. (2003). A computational evaluation of discrete material node point forces within the finite element setting, the so-called material force method, has been proposed in Steinmann et al. (2001). The intriguing duality of the hyperelastic spatial and the material motion problem has been the key focus in many of our own works, see e.g. Steinmann (2000), Steinmann (2002a) and Steinmann (2002b) from where the notation and terminology is taken.

The manuscript is organised as follows: section 2 reiterates the basic relations pertaining to the spatial and the material motion problem for hyperelasticity. In this context in particular the format of the various stress measures participating in the appropriate quasi-static balance of momentum statements is highlighted. Section 3 then investigates the effect of adding arbitrary constants to the potential energy density per unit volume in either the material or spatial configuration, respectively, rendering another example of the intriguing duality of the spatial and the material motion problem. Finally section 4 concludes the manuscript.

2 Hyperelasticity

We shall first reiterate the essential relations pertaining to the spatial and the material motion problem and introduce by this way the notation and terminology.

2.1 Spatial Motion

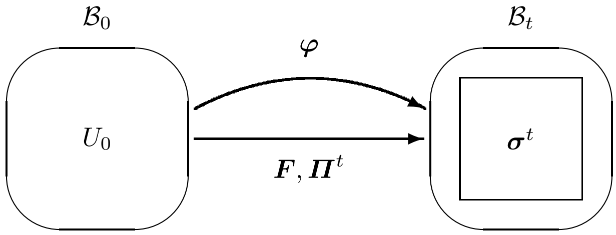

Figure 1. Kinematics, potential energy density and stress measures of the spatial motion problem: The material gradient of the deformation map $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$, i.e. the spatial motion deformation gradient $\boldsymbol{F} = \text{Grad}\boldsymbol{\varphi}$, and the stress measure $\boldsymbol{\Pi}^t$ are tensors in two-point description, the Cauchy stress $\boldsymbol{\sigma}^t$ is a tensor in fully spatial description, the potential energy density U_0 is measured per unit volume in the material configuration $\mathcal{B}_0$.

For the familiar spatial motion problem of continuum hyperelasticity as illustrated in Fig. 1 the total potential energy $I = I(\boldsymbol{\varphi})$ is a functional of the spatial deformation map $\boldsymbol{\varphi}(\boldsymbol{X}) = \boldsymbol{x}$ (assigning material placements of continuum 'particles' $\boldsymbol{X}$ to their spatial placements $\boldsymbol{x}$), whereby its density U_0 per unit volume in the material configuration $\mathcal{B}_0$ depends on $\boldsymbol{\varphi}(\boldsymbol{X})$ together with its material gradient $\boldsymbol{F} := \text{Grad}\boldsymbol{\varphi}(\boldsymbol{X})$ and is explicitly parametrised in $\boldsymbol{X}$, thus $I(\boldsymbol{\varphi})$ is expressed as

$$I(\boldsymbol{\varphi}(\boldsymbol{X})) = \int_{\mathcal{B}_0} U_0(\boldsymbol{\varphi}(\boldsymbol{X}), \boldsymbol{F}(\boldsymbol{X}); \boldsymbol{X})\,\text{d}V. \tag{1}$$

Here U_0 consists of the internal and external bulk potential energy density, i.e. we neglect for simplicity any potential energy contributions to the boundary. Thus we allow only for possible non-homogeneous boundary conditions on the Dirichlet boundary in the form of prescribed spatial deformations $\boldsymbol{\varphi} = \boldsymbol{\varphi}^p$ on $\partial\mathcal{B}_0^\varphi$ with $\partial\mathcal{B}_0 = \partial\mathcal{B}_0^\varphi \cup \partial\mathcal{B}_0^t$ and $\emptyset = \partial\mathcal{B}_0^\varphi \cap \partial\mathcal{B}_0^t$.

Taking a variation $\text{D}_\delta\{\bullet\}$ of the total potential energy functional at fixed material placement leads to a stationary point $\text{D}_\delta I = 0$ for the case of the familiar deformational equilibrium

$$\text{D}_\delta I(\boldsymbol{\varphi}) = 0. \tag{2}$$

Thereby the variation $\text{D}_\delta\{\bullet\}$ of the total potential energy functional $I(\boldsymbol{\varphi})$ is explicitly given by

$$\text{D}_\delta I(\boldsymbol{\varphi}) = \int_{\mathcal{B}_0} [\text{D}_{\boldsymbol{F}} U_0 : \text{Grad}\,\text{D}_\delta\boldsymbol{\varphi} + \partial_{\boldsymbol{\varphi}} U_0 \cdot \text{D}_\delta\boldsymbol{\varphi}]\,\text{d}V \tag{3}$$

and renders the primary option for the quasi-static balance of spatial momentum and the corresponding (homogeneous) Neumann boundary condition

$$\mathrm{Div}\boldsymbol{\Pi}^t + \boldsymbol{b}_0 = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}_0 \qquad \text{and} \qquad \boldsymbol{\Pi}^t \cdot \boldsymbol{N} = \boldsymbol{0} \quad \text{on} \quad \partial\mathcal{B}_0^t. \tag{4}$$

Here the following definitions for the spatial motion Piola stress $\boldsymbol{\Pi}^t$ and the spatial volume force $\boldsymbol{b}_0$ (e.g. due to gravity) were made

$$\boldsymbol{\Pi}^t := \mathrm{D}_{\boldsymbol{F}} U_0 \quad \text{and} \quad \boldsymbol{b}_0 := -\partial_{\boldsymbol{\varphi}} U_0. \tag{5}$$

Next, considering carefully the spatial motion kinematics with the spatial motion Jacobian $J := \det \boldsymbol{F}(\boldsymbol{X})$, the variation $\mathrm{D}_\delta\{\bullet\}$ of the spatial volume element $\mathrm{d}v = J\,\mathrm{d}V$ at fixed material placement follows as

$$\mathrm{D}_\delta\,\mathrm{d}v = \mathrm{D}_\delta(J\,\mathrm{d}V) = [\,\mathrm{D}_\delta J]\,\mathrm{d}V = [J\boldsymbol{F}^{-t} : \mathrm{D}_\delta \boldsymbol{F}]\,\mathrm{d}V = [\boldsymbol{I} : [\,\mathrm{D}_\delta \boldsymbol{F} \cdot \boldsymbol{F}^{-1}]]\,\mathrm{d}v = [\boldsymbol{I} : \mathrm{grad}\,\mathrm{D}_\delta \boldsymbol{\varphi}]\,\mathrm{d}v. \tag{6}$$

Consequently, the variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional, now expressed by the potential energy density $U_t = J^{-1}U_0$ per unit volume in the spatial configuration $\mathcal{B}_t$ is alternatively computed as

$$\mathrm{D}_\delta I(\boldsymbol{\varphi}) = \int_{\mathcal{B}_t} [[U_t \boldsymbol{I} + \mathrm{D}_{\boldsymbol{F}} U_t \cdot \boldsymbol{F}^t] : \mathrm{grad}\,\mathrm{D}_\delta \boldsymbol{\varphi} + \partial_{\boldsymbol{\varphi}} U_t \cdot \mathrm{D}_\delta \boldsymbol{\varphi}]\,\mathrm{d}v \tag{7}$$

and renders the alternative option to express the quasi-static balance of spatial momentum and the corresponding Neumann boundary condition in terms of zero spatial traction

$$\mathrm{div}\boldsymbol{\sigma}^t + \boldsymbol{b}_t = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}_t \qquad \text{and} \qquad \boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial\mathcal{B}_t^t. \tag{8}$$

Thereby, with $\mathrm{D}_{\boldsymbol{F}} U_t \cdot \boldsymbol{F}^t = -\boldsymbol{F}^{-t} \cdot \mathrm{D}_{\boldsymbol{F}^{-1}} U_t$, the following definitions for the Cauchy stress $\boldsymbol{\sigma}^t$ in energy-momentum format and the spatial volume force $\boldsymbol{b}_t = J^{-1}\boldsymbol{b}_0$ were made

$$\boldsymbol{\sigma}^t := U_t \boldsymbol{I} - \boldsymbol{F}^{-t} \cdot \mathrm{D}_{\boldsymbol{F}^{-1}} U_t \quad \text{and} \quad \boldsymbol{b}_t := -\partial_{\boldsymbol{\varphi}} U_t. \tag{9}$$

It is essentially the above energy-momentum format of the Cauchy stress that gives rise to the question whether or not the zero level-set of the potential energy density has an effect, e.g. on the Cauchy stress and thus on the value of the spatial traction.

2.2 Material Motion

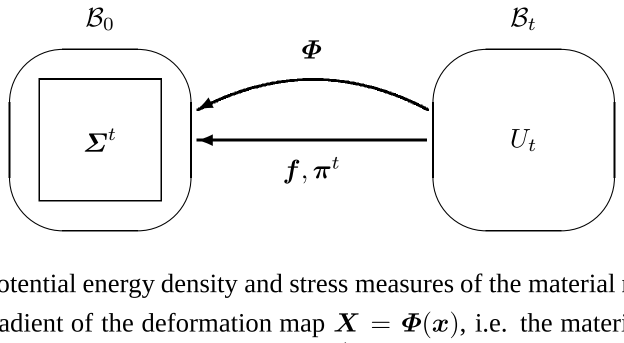

Figure 2. Kinematics, potential energy density and stress measures of the material motion problem: The spatial gradient of the deformation map $\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x})$, i.e. the material motion deformation gradient $\boldsymbol{f} = \mathrm{grad}\boldsymbol{\Phi}$, and the stress measure $\boldsymbol{\pi}^t$ are tensors in two-point description, the Eshelby stress $\boldsymbol{\Sigma}^t$ is a tensor in fully material description, the potential energy density U_t is measured per unit volume in the spatial configuration $\mathcal{B}_t$.

Next, for the material motion problem as illustrated in Fig. 2 the total potential energy $I = I(\boldsymbol{\Phi})$ is a functional of the material deformation map $\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{X}$, whereby its density U_t per unit volume in the spatial configuration $\mathcal{B}_t$ depends on $\boldsymbol{\Phi}(\boldsymbol{x})$ together with its spatial gradient $\boldsymbol{f} := \mathrm{grad}\boldsymbol{\Phi}(\boldsymbol{x})$ and is explicitly parametrised in $\boldsymbol{x}$, thus $I(\boldsymbol{\Phi})$ is expressed as

$$I(\boldsymbol{\Phi}(\boldsymbol{x})) = \int_{\mathcal{B}_t} U_t(\boldsymbol{\Phi}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}); \boldsymbol{x})\,\mathrm{d}v. \tag{10}$$

Here the relations to the kinematics of the spatial motion problem as described in the above are $\boldsymbol{\Phi}^{-1} = \varphi(\boldsymbol{X})$ and $\boldsymbol{f}^{-1} = \boldsymbol{F} \circ \boldsymbol{\Phi}(\boldsymbol{x})$. In the sequel we shall adopt the somewhat sloppy but simplified notation $\boldsymbol{f}^{-1} = \boldsymbol{F}$ and $\boldsymbol{F}^{-1} = \boldsymbol{f}$.

It is now important to recognise that taking a variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional at fixed spatial placement only leads to a stationary point $\mathrm{d}_\delta I = 0$ for the case of configurational equilibrium. Nevertheless, in more general cases configurational or rather material tractions $\boldsymbol{T}_t$ acting on the boundary $\partial\mathcal{B}_t$ and material forces, acting on all kinds of defects (vacancies, interfaces, dislocations, cracks and thelike), have to be considered. Note that material forces capture the energetic changes that go along with material motions of the defects relative to the ambient material. Here we shall for simplicity only consider a defect free continuum body, thus only material tractions $\boldsymbol{T}_t$ acting on the boundary $\partial\mathcal{B}_t$ appear that are power conjugate to variations $\mathrm{d}_\delta\boldsymbol{\Phi}$ of the boundary $\partial\mathcal{B}_t$

$$\mathrm{d}_\delta I(\boldsymbol{\Phi}) =: \int_{\partial\mathcal{B}_t} \boldsymbol{T}_t \cdot \mathrm{d}_\delta\boldsymbol{\Phi}\,\mathrm{d}a \leq 0. \tag{11}$$

The inequality is a reminder of the second law, since changes in configuration $\mathrm{d}_\delta\boldsymbol{\Phi}$ are only admissible if potential energy is released, compare the classical arguments related to the possible extension of cracks. Also observe that the material tractions $\boldsymbol{T}_t$ are introduced as a definition to capture the energetic changes associated with configurational variations.

The variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional $I(\boldsymbol{\Phi})$ is then computed in explicit format as

$$\mathrm{d}_\delta I(\boldsymbol{\Phi}) = \int_{\mathcal{B}_t} [\,\mathrm{d}_{\boldsymbol{f}} U_t : \mathrm{grad}\,\mathrm{d}_\delta\boldsymbol{\Phi} + \partial_{\boldsymbol{\Phi}} U_t \cdot \mathrm{d}_\delta\boldsymbol{\Phi}]\,\mathrm{d}v \tag{12}$$

and renders the primary option for the quasi-static balance of material momentum and the corresponding Neumann boundary condition, which come as a definition for the material traction

$$\mathrm{div}\boldsymbol{\pi}^t + \boldsymbol{B}_t = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}_t \qquad \text{and} \qquad \boldsymbol{\pi}^t \cdot \boldsymbol{n} =: \boldsymbol{T}_t \quad \text{on} \quad \partial\mathcal{B}_t. \tag{13}$$

Here the following definitions for the material motion Piola stress $\boldsymbol{\pi}^t$ and the material volume force $\boldsymbol{B}_t$ (capturing inhomogeneities) were made

$$\boldsymbol{\pi}^t := \mathrm{d}_{\boldsymbol{f}} U_t \quad \text{and} \quad \boldsymbol{B}_t := -\partial_{\boldsymbol{\Phi}} U_t. \tag{14}$$

Next, considering carefully the material motion kinematics with $j = \det \boldsymbol{f}(\boldsymbol{x})$ and $j^{-1} = J$, the variation $\mathrm{d}_\delta\{\bullet\}$ of the material volume element $\mathrm{d}V = j\,\mathrm{d}v$ at fixed spatial placement follows as

$$\mathrm{d}_\delta\,\mathrm{d}V = \mathrm{d}_\delta(j\,\mathrm{d}v) = [\,\mathrm{d}_\delta j]\,\mathrm{d}v = [j\boldsymbol{F}^t : \mathrm{d}_\delta\boldsymbol{f}]\,\mathrm{d}v = [\boldsymbol{I} : [\,\mathrm{d}_\delta\boldsymbol{f}\cdot\boldsymbol{F}]]\,\mathrm{d}V = [\boldsymbol{I} : \mathrm{Grad}\,\mathrm{d}_\delta\boldsymbol{\Phi}]\,\mathrm{d}V. \tag{15}$$

Consequently, the variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional, now expressed by the potential energy density $U_0 = JU_t$ per unit volume in the material configuration $\mathcal{B}_0$ is alternatively computed as

$$\mathrm{d}_\delta I(\boldsymbol{\Phi}) = \int_{\mathcal{B}_0} [[U_0\boldsymbol{I} + \mathrm{d}_{\boldsymbol{f}} U_0 \cdot \boldsymbol{f}^t] : \mathrm{Grad}\,\mathrm{d}_\delta\boldsymbol{\Phi} + \partial_{\boldsymbol{\Phi}} U_0 \cdot \mathrm{d}_\delta\boldsymbol{\Phi}]\,\mathrm{d}V \tag{16}$$

and renders the alternative option to express the quasi-static balance of material momentum and the corresponding Neumann boundary condition for the material traction $\boldsymbol{T}_0 = J\boldsymbol{T}_t$

$$\mathrm{Div}\boldsymbol{\Sigma}^t + \boldsymbol{B}_0 = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}_0 \qquad \text{and} \qquad \boldsymbol{\Sigma}^t \cdot \boldsymbol{N} =: \boldsymbol{T}_0 \quad \text{on} \quad \partial\mathcal{B}_0. \tag{17}$$

Thereby, with $\mathrm{d}_{\boldsymbol{f}} U_0 \cdot \boldsymbol{f}^t = -\boldsymbol{F}^t \cdot \mathrm{d}_{\boldsymbol{F}} U_0$, the following definitions for the Eshelby stress $\boldsymbol{\Sigma}^t$ in energy-momentum format and the corresponding material volume force $\boldsymbol{B}_0 = J\boldsymbol{B}_t$ were made

$$\boldsymbol{\Sigma}^t := U_0\boldsymbol{I} - \boldsymbol{F}^t \cdot \mathrm{d}_{\boldsymbol{F}} U_0 \quad \text{and} \quad \boldsymbol{B}_0 := -\partial_{\boldsymbol{\Phi}} U_0. \tag{18}$$

Again, it is essentially the above energy-momentum format of the Eshelby stress that gives rise to the question whether or not the zero level-set of the potential energy density has an effect, e.g. on the Eshelby stress and thus on the value of the material traction.

3 Shifts in Potential Energy Density

Next we shall carefully investigate the effect of adding arbitrary constants to the potential energy density, i.e. we shall clarify whether or not the stress measures of the spatial and the material motion problem remain invariant or change in a meaningful way.

3.1 Spatial Motion

Suppose first a shift of the spatial motion potential energy density U_0, i.e. the potential energy density per unit volume in the material configuration $\mathcal{B}_0$, by a constant c_0 to render the shifted potential energy density $\bar{U}_0$, i.e.

$$U_0 \to \bar{U}_0 = U_0 + c_0. \tag{19}$$

Accordingly, the potential energy density U_t per unit volume in the spatial configuration $\mathcal{B}_t$ shifts to $\bar{U}_t$ in a deformation dependend fashion, i.e.

$$U_t \to \bar{U}_t = U_t + jc_0. \tag{20}$$

As a consequence the shift of the total potential energy functional I of the hyperelastic body to its transformed counterpart $\bar{I}$ follows as

$$I \to \bar{I} = I + \int_{\mathcal{B}_0} c_0 \,\mathrm{d}V = I + \int_{\mathcal{B}_t} jc_0 \,\mathrm{d}v. \tag{21}$$

Thus it is obvious, that a variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional I at fixed material placement $\boldsymbol{X}$ is invariant with respect to the above shift, i.e.

$$\mathrm{D}_\delta \bar{I} = \mathrm{D}_\delta I, \tag{22}$$

whereas a variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional I at fixed spatial placement $\boldsymbol{x}$ depends directly on the above shift, i.e.

$$\mathrm{d}_\delta \bar{I} \neq \mathrm{d}_\delta I. \tag{23}$$

As an immediate result the spatial motion Piola stress $\boldsymbol{\Pi}^t$ is clearly invariant under a shift of the spatial motion potential energy density, i.e

$$\bar{\boldsymbol{\Pi}}^t = \mathrm{D}_{\boldsymbol{F}} \bar{U}_0 \equiv \mathrm{D}_{\boldsymbol{F}} U_0 = \boldsymbol{\Pi}^t, \tag{24}$$

which, due to $\mathrm{D}_{\boldsymbol{f}} j = j\boldsymbol{F}^t$ and thus $\mathrm{D}_{\boldsymbol{f}} \bar{U}_t = \mathrm{D}_{\boldsymbol{f}} U_t + jc_0 \boldsymbol{F}^t$, is also reflected in the energy-momentum format for the spatial motion Cauchy stress $\boldsymbol{\sigma}^t$, i.e.

$$\bar{\boldsymbol{\sigma}}^t = \bar{U}_t \boldsymbol{I} - \boldsymbol{f}^t \cdot \mathrm{D}_{\boldsymbol{f}} \bar{U}_t \equiv U_t \boldsymbol{I} - \boldsymbol{f}^t \cdot \mathrm{D}_{\boldsymbol{f}} U_t = \boldsymbol{\sigma}^t. \tag{25}$$

Clearly, this invariance is to be expected for the stress measures of the spatial motion problem, since they follow from a variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional at fixed material placement

$$\mathrm{D}_\delta \bar{I}(\boldsymbol{\varphi}) = \mathrm{D}_\delta \int_{\mathcal{B}_0} [U_0(\boldsymbol{\varphi}, \boldsymbol{F}; \boldsymbol{X}) + c_0] \,\mathrm{d}V = \int_{\mathcal{B}_0} \mathrm{D}_\delta U_0(\boldsymbol{\varphi}, \boldsymbol{F}; \boldsymbol{X}) \,\mathrm{d}V = \mathrm{D}_\delta I(\boldsymbol{\varphi}). \tag{26}$$

In other words an arbitrary constant c_0 added to the spatial motion potential energy density U_0 does not affect the variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional I which leaves the material volume $\mathrm{vol}(\mathcal{B}_0)$ unchanged, thus variations in total potential energy exclusively stem from variations of the arguments of U_0, i.e. from variations in the spatial motion deformation map $\boldsymbol{\varphi}$ and the corresponding deformation gradient $\boldsymbol{F}$.

On the other hand, now due to $\mathrm{d}_{\boldsymbol{f}} j = j \boldsymbol{F}^t$, the material motion Piola stress is equipped with an additional contribution under a shift of the spatial motion potential energy density, i.e

$$\bar{\boldsymbol{\pi}}^t = \mathrm{d}_{\boldsymbol{f}} \bar{U}_t = \mathrm{d}_{\boldsymbol{f}} U_t + j c_0 \boldsymbol{F}^t = \boldsymbol{\pi}^t + j c_0 \boldsymbol{F}^t, \tag{27}$$

which is clearly recognised as a (configurational) pressure-like term $c_0 \boldsymbol{I}$ when evaluating the corresponding material motion Cauchy or rather the Eshelby stress

$$\bar{\boldsymbol{\Sigma}}^t = J\, \mathrm{d}_{\boldsymbol{f}} \bar{U}_t \cdot \boldsymbol{f}^t = J\, \mathrm{d}_{\boldsymbol{f}} U_t \cdot \boldsymbol{f}^t + c_0 \boldsymbol{I} = \boldsymbol{\Sigma}^t + c_0 \boldsymbol{I}. \tag{28}$$

Since the Eshelby stress follows from a variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional I at fixed spatial placement it captures variations of the material configuration, i.e. configurational changes that go along with a change of the material volume $\mathrm{vol}(\mathcal{B}_0)$

$$\mathrm{d}_\delta \bar{I}(\boldsymbol{\Phi}) = \mathrm{d}_\delta \int_{\mathcal{B}_0} [U_0(\boldsymbol{\Phi}, \boldsymbol{f}; \boldsymbol{x}) + c_0]\, \mathrm{d}V = \int_{\mathcal{B}_t} \mathrm{d}_\delta U_t(\boldsymbol{\Phi}, \boldsymbol{f}; \boldsymbol{x})\, \mathrm{d}v + \int_{\mathcal{B}_0} c_0\, \mathrm{d}_\delta\, \mathrm{d}V. \tag{29}$$

Thus the Eshelby stress is directly affected by an arbitrary constant c_0 added to the spatial motion potential energy density U_0 since variations in total potential energy are then not only due to variations of the arguments of U_t, i.e. from variations in the material motion deformation map $\boldsymbol{\Phi}$ and the corresponding deformation gradient $\boldsymbol{f}$, but also due to changes in material volume $\mathrm{vol}(\mathcal{B}_0)$ via $\mathrm{d}_\delta\, \mathrm{d}V = \mathrm{Div}\, \mathrm{d}_\delta \boldsymbol{\Phi}\ \mathrm{d}V$. These changes in material volume are reflected by a pressure-like contribution to the Eshelby stress which constitutes the energy density part $\bar{U}_0 \boldsymbol{I}$ of the energy-momentum format.

3.2 Material Motion

Next, suppose a shift of the material motion potential energy density U_t, i.e. the potential energy density per unit volume in the spatial configuration $\mathcal{B}_t$, by a constant c_t to render the shifted potential energy density $\bar{U}_t$, i.e.

$$U_t \to \bar{U}_t = U_t + c_t. \tag{30}$$

Accordingly, now the potential energy density U_0 per unit volume in the material configuration $\mathcal{B}_0$ shifts to $\bar{U}_0$ in a configuration dependend fashion, i.e.

$$U_0 \to \bar{U}_0 = U_0 + J c_t. \tag{31}$$

As a consequence the shift of the total potential energy functional I of the hyperelastic body to its transformed counterpart $\bar{I}$ follows as

$$I \to \bar{I} = I + \int_{\mathcal{B}_t} c_t\, \mathrm{d}v = I + \int_{\mathcal{B}_0} J c_t\, \mathrm{d}V. \tag{32}$$

Thus it directly follows that a variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional at fixed spatial placement $\boldsymbol{x}$ is invariant with respect to the above shift, i.e.

$$\mathrm{d}_\delta \bar{I} = \mathrm{d}_\delta I, \tag{33}$$

whereas a variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional at fixed material placement $\boldsymbol{X}$ depends directly on the above shift, i.e.

$$\mathrm{D}_\delta \bar{I} \neq \mathrm{D}_\delta I. \tag{34}$$

As an immediate result here the material motion Piola stress $\boldsymbol{\pi}^t$ is obviously invariant under a shift of the material motion potential energy density, i.e

$$\bar{\boldsymbol{\pi}}^t = \mathrm{d}_{\boldsymbol{f}} \bar{U}_t \equiv \mathrm{d}_{\boldsymbol{f}} U_t = \boldsymbol{\pi}^t, \tag{35}$$

which, due to $\mathrm{d}_{\boldsymbol{F}} J = J\boldsymbol{f}^t$ and thus $\mathrm{d}_{\boldsymbol{F}} \bar{U}_0 = \mathrm{d}_{\boldsymbol{F}} U_0 + J c_t \boldsymbol{f}^t$, is also reflected in the energy-momentum format for the material motion Cauchy or rather the Eshelby stress $\boldsymbol{\Sigma}^t$, i.e.

$$\bar{\boldsymbol{\Sigma}}^t = \bar{U}_0 \boldsymbol{I} - \boldsymbol{F}^t \cdot \mathrm{d}_{\boldsymbol{F}} \bar{U}_0 \equiv U_0 \boldsymbol{I} - \boldsymbol{F}^t \cdot \mathrm{d}_{\boldsymbol{F}} U_0 = \boldsymbol{\Sigma}^t. \tag{36}$$

Again, this invariance is in accordance with intuition for the stress measures of the material motion problem if we recognise, that they follow from a variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional at fixed spatial placement

$$\mathrm{d}_\delta \bar{I}(\boldsymbol{\Phi}) = \mathrm{d}_\delta \int_{\mathcal{B}_t} [U_t(\boldsymbol{\Phi}, \boldsymbol{f}; \boldsymbol{x}) + c_t] \,\mathrm{d}v = \int_{\mathcal{B}_t} \mathrm{d}_\delta U_t(\boldsymbol{\Phi}, \boldsymbol{f}; \boldsymbol{x}) \,\mathrm{d}v = \mathrm{d}_\delta I(\boldsymbol{\Phi}). \tag{37}$$

In other words an arbitrary constant c_t added to the material motion potential energy density U_t does not affect the variation $\mathrm{d}_\delta\{\bullet\}$ of the total potential energy functional I which leaves the spatial volume vol$(\mathcal{B}_t)$ unchanged, thus variations in total potential energy exclusively stem from variations of the arguments of U_t, i.e. from variations in the material motion deformation map $\boldsymbol{\Phi}$ and the corresponding deformation gradient $\boldsymbol{f}$.

On the other hand, again due to $\mathrm{D}_{\boldsymbol{F}} J = J\boldsymbol{f}^t$, the spatial motion Piola stress is equipped with an additional contribution under a shift of the material motion potential energy density, i.e

$$\bar{\boldsymbol{\Pi}}^t = \mathrm{D}_{\boldsymbol{F}} \bar{U}_0 = \mathrm{D}_{\boldsymbol{F}} U_0 + J c_t \boldsymbol{f}^t = \boldsymbol{\Pi}^t + J c_t \boldsymbol{f}^t, \tag{38}$$

which is easily recognised as a (deformational) pressure-like term $c_t \boldsymbol{I}$ when evaluating the corresponding spatial motion Cauchy stress

$$\bar{\boldsymbol{\sigma}}^t = j\, \mathrm{D}_{\boldsymbol{F}} \bar{U}_0 \cdot \boldsymbol{F}^t = j\, \mathrm{D}_{\boldsymbol{F}} U_0 \cdot \boldsymbol{F}^t + c_t \boldsymbol{I} = \boldsymbol{\sigma}^t + c_t \boldsymbol{I}. \tag{39}$$

Since the Cauchy stress follows from a variation $\mathrm{D}_\delta\{\bullet\}$ of the total potential energy functional at fixed material placement it captures variations of the spatial configuration, i.e. deformational changes that go along with a change of the spatial volume vol$(\mathcal{B}_t)$

$$\mathrm{D}_\delta \bar{I}(\boldsymbol{\varphi}) = \mathrm{D}_\delta \int_{\mathcal{B}_t} [U_t(\boldsymbol{\varphi}, \boldsymbol{F}; \boldsymbol{X}) + c_t] \,\mathrm{d}v = \int_{\mathcal{B}_0} \mathrm{D}_\delta U_0(\boldsymbol{\varphi}, \boldsymbol{F}; \boldsymbol{X}) \,\mathrm{d}V + \int_{\mathcal{B}_t} c_t \,\mathrm{D}_\delta \,\mathrm{d}v. \tag{40}$$

Thus the Cauchy stress is directly affected by an arbitrary constant c_t added to the material motion potential energy density U_t since variations in total potential energy are then not only due to variations of the arguments of U_0, i.e. from variations in the spatial motion deformation map $\boldsymbol{\varphi}$ and the corresponding deformation gradient $\boldsymbol{F}$, but also due to changes in spatial volume vol$(\mathcal{B}_t)$ via $\mathrm{D}_\delta \,\mathrm{d}v = \mathrm{div}\, \mathrm{D}_\delta \boldsymbol{\varphi} \;\mathrm{d}v$. These changes in spatial volume are reflected by a pressure-like contribution to the Cauchy stress which constitutes the energy density part $\bar{U}_t \boldsymbol{I}$ of the energy-momentum format.

4 Conclusion

The material motion problem of configurational mechanics opens the important possibility to investigate the tendency of all kind of defects to move relative to the ambient material. This being an important viewpoint in itself, the consideration of the spatial and the material motion problem reveals, on top of this, in general an intriguing duality in terms of the format of e.g. the balance equations, the kinematic quantities and the stress measures. It is thus interesting to note that the dualities between the spatial and the material motion problem also hold for the problem under question in a meaningful way.

	$\bar{U}_0 = U_0 + c_0$	$\bar{U}_t = U_t + c_t$
$\bar{\boldsymbol{\sigma}}^t$	$\boldsymbol{\sigma}^t$	$\boldsymbol{\sigma}^t + c_t \boldsymbol{I}$
$\bar{\boldsymbol{\Sigma}}^t$	$\boldsymbol{\Sigma}^t + c_0 \boldsymbol{I}$	$\boldsymbol{\Sigma}^t$

Table 1. Effect of potential energy density shifts on spatial and material stress measures: Depending on the motion problem in question the stress measures remain either invariant or are equipped with an additional pressure-like contribution.

The results of the present investigation are displayed in Tab. 1, whereby, summarising, the main thrust of this contribution is the following: naïvely one would expect that the zero-level set of the potential energy is completely irrelevant for the evaluation of the stress measures, however in the present context of the spatial and the material motion problem of hyperelasticity this only holds for the one set of stress measures considered, whereas the other set transforms in a meaningful way by being equipped with an additional pressure-like term. Clearly, this additional pressure-like term will not change the balance of momentum statements since the divergence of a constant tensor (or likewise the divergence of the cofactor of the corresponding deformation gradient) disappears. Nevertheless it will of course contribute to the Neumann boundary conditions. As an example consider the first column in Tab. 1 and recall that the material traction, which comes as a definition, is power conjugate to variations of the boundary to the material configuration. Thereby the additional contribution $c_0\boldsymbol{N}$ to the material traction obviously captures the changes of the material volume and thus the accompanying changes in potential energy due to its shift by c_0.

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