

Symbolic Linearization of Differential/Algebraic Equations Based on Cartesian Coordinates

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A computer algebraic method for linearizing the equations of multibody system dynamics is discussed in this paper. Based on Cartesian coordinates, a symbolic linearization technique for differential/algebraic equations (DAE) of multibody system dynamics is obtained in a simple and effective way. The technique avoids some drawbacks of the numerical perturbation method. Two examples are performed to demonstrate the proposed method.

1 Introduction

Since the 1980's, two different modeling strategies have been presented in studying multibody system dynamics. The two strategies differ mainly by the description of the configuration of the rigid body. There are two representative modeling methods. One of them, the Roberson-Wittenburg's method (1977), uses relative coordinates to form the dynamical system, in which the displacement and rotational angles between the adjacent bodies are taken as generalized coordinates. The other method, represented by Haug (1989), uses absolute coordinates which contain the Cartesian coordinates of each body's mass center and the Euler parameters as generalized coordinates. The resulting dynamic equations usually form a system of differential/algebraic equations. The Cartesian coordinates, used by Garcia and Bayo (1986, 1991) to describe multibody systems, is another kind of absolute coordinates. The above dynamic equations, mostly for systems with large displacements, are highly non-linear and can be solved only by special numerical integration methods. For a vibration system with small displacements, however, these numerical integrations are not efficient. The successive linearization method, proposed by Wallrapp (1990), is effective for such problems, but it can not be used for those with large displacements. In the works of Liang (1986) and Sohoni (1986), a numerical perturbation method is used to set up a linear model for the system. The method requires computing of the Jacobian matrix, which causes iterative errors, requires an additional convergence condition and is inefficient. There are many other approaches which deal with the problem. For example, Lin and Yae (1994) set up another successive linearization method by using relative coordinates. Trom and Vanderploeg (1994) suggested an analytical/numerical method to overcome the disadvantages of successive linearization. Both of them require a large amount of manual work in deducing linearized constraints. Ni et al. (1997) set up a differential/algebraic equation system for the multibody system by using Lagrange equations of the first kind, and a Taylor expansion of the generalized mass matrix, the constraints and the generalized forces in the neighborhood of the equilibrium position to linearize the equations.

Based on multibody dynamic models described by Cartesian coordinates, a computer algebraic method designed to present symbolic linearization of the differential/algebraic equations of multibody system dynamics, is presented in this paper. In the linearization process, a successive linearization technique is introduced to obtain linearized equations by using a Taylor expansion of a the generalized mass matrix, the constraints and the generalized force vector in the neighborhood of the equilibrium position. The technique avoids some drawbacks of numerical perturbation methods and does not require setting up the linearized constraints library. The resulting linearized equations are explicit analytical expressions generated by symbolic computation, which are convenient in dealing with problems such as the computation of frequency responses, eigenvalue analysis and control design etc.

2 The Dynamic Model Described by Cartesian Coordinates

In a Cartesian coordinates system, a new dependent coordinate system is defined so that the position of a body is determined by the Cartesian coordinates of at least two of its points and the Cartesian components of at least one unit vector rigidly attached to the body. An interesting feature of the Cartesian coordinates is that the points and the unit vectors can be shared by two adjacent elements, which contributes to the definition of the position of both and therefore leads to a model of fewer total coordinates. Neither Euler's angles nor Euler's parameters are introduced to describe the configuration of the body, so the constraint equations in the Cartesian coordinates system are always quadratic. As a consequence, the elements of the Jacobian matrix are linear functions of the Cartesian coordinates, which is one of the reasons for the simplicity and efficiency of this formulation.

The constraint equations, usually obtained by using the constraints of rigid bodies or using constraints of the hinges connecting the adjoining bodies (Garcia de Jalón et al. 1993), form a set of nonlinear (quadratic) equations which can be described in vector form

$$\Phi(\mathbf{q}, t) = 0 \quad (1)$$

where Φ represents the constraint vector function, \mathbf{q} are the Cartesian coordinates and t is the time. Differentiating equation (1) with respect to t leads to

$$\Phi_{\mathbf{q}}(\mathbf{q}, t)\dot{\mathbf{q}} = -\dot{\Phi}_t = \mathbf{b} \quad (2)$$

$$\Phi_{\mathbf{q}}(\mathbf{q}, t)\ddot{\mathbf{q}} = -\ddot{\Phi}_t - \dot{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{c} \quad (3)$$

where $\Phi_{\mathbf{q}}$ is the Jacobian matrix of Φ . The right-hand sides of equation (2) and (3) are known functions of \mathbf{q} and t , $\dot{\mathbf{q}}$ and t , respectively.

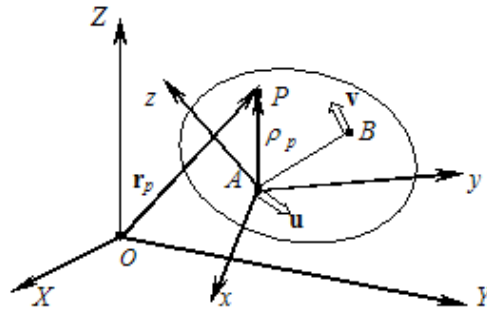


Figure 1. A rigid body described by Cartesian coordinates

In a Cartesian coordinate system, dynamic equations are established based on Lagrange's equations or on the principle of virtual power. Let $O-XYZ$ be an inertial reference frame and $A-xyz$ be a frame fixed on a rigid body (Figure 1). By choosing two points A and B as reference points and \mathbf{u} and \mathbf{v} as two base unit base vectors, the virtual power can be expressed in the form

$$W = \int_v (\delta \dot{\mathbf{r}}_p)^T \ddot{\mathbf{r}}_p dm \quad (4)$$

where $\delta \dot{\mathbf{r}}_p$ and $\ddot{\mathbf{r}}_p$ are the vectors of virtual velocity and actual acceleration of a point P on the rigid body, respectively. The position vector of point P in an inertial frame can be expressed as

$$\mathbf{r}_p = \mathbf{r}_A + \mathbf{A}\boldsymbol{\rho}_p \quad (5)$$

where \mathbf{A} is the coordinate transformation matrix and $\boldsymbol{\rho}_p$ is the position vector of point P in a moving frame fixed to the rigid-body with its origin at point A . In the inertial reference frame, the base vectors of the Cartesian coordinates form an order 3 matrix $\mathbf{S} = [\mathbf{r}_B - \mathbf{r}_A, \mathbf{u}, \mathbf{v}]$ which satisfies

$$\mathbf{S} = \mathbf{A}\mathbf{S}_0 \quad (6)$$

The matrix \mathbf{S}_0 in equation (6) is the value of matrix \mathbf{S} at the initial position of the rigid body. Differentiating equation (5) and equation (6) yields

$$\dot{\mathbf{r}}_p = \dot{\mathbf{r}}_A + \dot{\mathbf{A}}\boldsymbol{\rho}_p \quad \ddot{\mathbf{r}}_p = \ddot{\mathbf{r}}_A + \ddot{\mathbf{A}}\boldsymbol{\rho}_p \quad (7)$$

$$\dot{\mathbf{A}} = \dot{\mathbf{S}}\mathbf{S}_0^{-1} \quad \ddot{\mathbf{A}} = \ddot{\mathbf{S}}\mathbf{S}_0^{-1} \quad (8)$$

Substituting equations (7) and (8) into equation (4) leads to the following result (Garcia de Jalón et al. 1993)

$$\mathbf{W} = [\delta \dot{\mathbf{r}}_A^T \quad \delta \dot{\mathbf{r}}_B^T \quad \delta \dot{\mathbf{u}}^T \quad \delta \dot{\mathbf{v}}^T] \mathbf{M}^e [\ddot{\mathbf{r}}_A \quad \ddot{\mathbf{r}}_B \quad \ddot{\mathbf{u}} \quad \ddot{\mathbf{v}}]^T \quad (9)$$

where \mathbf{M}^e is the inertia matrix of the rigid body

$$\mathbf{M}^e = \begin{bmatrix} (m-2a_1+b_{11})\mathbf{I} & (a_1-b_{11})\mathbf{I} & (a_2-b_{12})\mathbf{I} & (a_3-b_{13})\mathbf{I} \\ & b_{11}\mathbf{I} & b_{12}\mathbf{I} & b_{13}\mathbf{I} \\ & & b_{22}\mathbf{I} & b_{23}\mathbf{I} \\ \text{symmetry} & & & b_{33}\mathbf{I} \end{bmatrix} \quad (10)$$

\mathbf{I} is the (3×3) unit matrix, m is the mass of the rigid body. The parameters a_i , b_{ij} ($i, j = 1, 2, 3$) in (10) are defined as

$$\mathbf{a}_i = m \mathbf{r}_c^T \mathbf{S}_{0i}^{-1}, \quad \mathbf{b}_{ij} = (\mathbf{S}_{0i}^{-1})^T \mathbf{J} \mathbf{S}_{0j}^{-1}, \quad \mathbf{J} = \int_V \boldsymbol{\rho}_p \boldsymbol{\rho}_p^T dm$$

where \mathbf{r}_c contains the coordinates of the center of mass in the body fixed frame, and \mathbf{S}_{0i}^{-1} is the i^{th} row of \mathbf{S}_0^{-1} . There are 10 parameters in the matrix (10). They denote the mass of the rigid body, the coordinates of the mass center and the inertia tensor. The inertial force is described in the inertial frame, so the inertia matrix is constant. There are no velocity parts in the inertial forces. Therefore, neither Coriolis effects nor centrifugal ones appear in the matrix. If the rigid-body does not have the considered configuration of two points and two unitary vectors, it is possible to find equivalent expressions for the inertia matrix \mathbf{M}^e , or to modify the formulation so as to be able to form matrix (10).

After finding the expression for the inertia matrix, applying the principle of virtual power to the whole system leads to

$$\delta \dot{\mathbf{q}}^T (\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q} + \Phi_q^T \lambda) = 0 \quad (11)$$

where \mathbf{M} is the inertia matrix of the system assembled from the element inertia matrices \mathbf{M}^e by a process similar to the finite element method. The vector \mathbf{Q} denotes the generalized forces vector varying with position, velocity, and time. By applying vector \mathbf{Q} , it is possible to introduce the formulation of springs and dampers with very general characteristics. Since the virtual natural velocities are not independent, it is necessary to introduce the constraints (1) by means of a Lagrange multiplier λ . From equation (11), it is always possible to choose the independent components of \mathbf{q} and λ that satisfy

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_q \lambda = \mathbf{Q} \quad (12)$$

Combining equation (12) with equation (3) forms the following DAE of multibody system dynamics

$$\begin{bmatrix} \mathbf{M} & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{c} \end{bmatrix} \quad (13)$$

Many algorithms for the solution of equation (13) can be found in the literature. However, most of them require some numerical treatments which are not suitable for symbolic computations. Generalized coordinates separation method (Wehage and Haug 1982) requires only computing the inverse of the matrixes, which shows some advantages in symbolic operation.

3 Symbolic Linearization Method for Differential/Algebraic Equation of Dynamics

Consider a multibody system the configuration which is characterized by n Cartesian coordinates

$$\mathbf{q} = [q_1, q_2, \dots, q_n]^T \quad (14)$$

which is interrelated through the m holonomic constraint conditions (1). Let \mathbf{x} be the $n-m$ independent generalized coordinates of the system, and \mathbf{y} be the non-independent ones. With

$$\mathbf{q} = [\mathbf{y}^T, \mathbf{x}^T]^T \quad (15)$$

the constraint equation (2) is

$$\Phi_x \dot{\mathbf{x}} + \Phi_y \dot{\mathbf{y}} = \mathbf{b} \quad (16)$$

Since Φ_y is of full rank, one has

$$\dot{\mathbf{y}} = -\Phi_y^{-1} \Phi_x \dot{\mathbf{x}} + \Phi_y^{-1} \mathbf{b} \quad (17)$$

Similarly, the acceleration constraint equation (3) can be reduced to

$$\ddot{\mathbf{y}} = -\Phi_y^{-1} \Phi_x \ddot{\mathbf{x}} + \Phi_y^{-1} \mathbf{c} \quad (18)$$

We denote

$$\mathbf{H}_x = \begin{bmatrix} -\Phi_y^{-1} \Phi_x \\ \mathbf{I} \end{bmatrix}, \quad \mathbf{H}_t = \begin{bmatrix} \Phi_y^{-1} \mathbf{b} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{H}_{tt} = \begin{bmatrix} \Phi_y^{-1} \mathbf{c} \\ \mathbf{0} \end{bmatrix} \quad (19)$$

With respect to (15) follows

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} = \mathbf{H}_x \dot{\mathbf{x}} + \mathbf{H}_t \quad (20)$$

$$\ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{y}} \\ \ddot{\mathbf{x}} \end{bmatrix} = \mathbf{H}_x \ddot{\mathbf{x}} + \mathbf{H}_{tt} \quad (21)$$

By left multiplying equations (12) by \mathbf{H}_x^T , and regarding equations (20) and (21), one obtains

$$\mathbf{H}_x^T \mathbf{M} (\mathbf{H}_x \ddot{\mathbf{x}} + \mathbf{H}_{tt}) + \mathbf{H}_x^T \Phi_q^T \lambda = \mathbf{H}_x^T \mathbf{Q} \quad (22)$$

Notice that

$$\mathbf{H}_x^T \Phi_q^T = [\Phi_q \ \mathbf{H}_x]^T = \begin{bmatrix} \Phi_y & \Phi_x \end{bmatrix} \begin{bmatrix} -\Phi_y^{-1} \Phi_x \\ \mathbf{I} \end{bmatrix}^T = \mathbf{0} \quad (23)$$

One has

$$\mathbf{H}_x^T \mathbf{M} \mathbf{H}_x \ddot{\mathbf{x}} = \mathbf{H}_x^T \mathbf{Q} - \mathbf{H}_x^T \mathbf{M} \mathbf{H}_{tt} \quad (24)$$

Both sides of equation (24) are quadratic functions of the generalized coordinates, which means that a direct linearization of the equation is time consuming and the physical significance is not evident. Since the analytical solution of equation (24) can hardly be obtained, a successive linearization strategy is adopted in this paper. Consider a stationary system. Suppose $\tilde{\mathbf{q}}$ denotes an equilibrium position, and $\delta\mathbf{q}$ is a small disturbance. Then the disturbed motion is

$$\mathbf{q} = \tilde{\mathbf{q}} + \delta\mathbf{q} \quad (25)$$

The following equations are obtained from equation (20)

$$\delta\mathbf{q} = \mathbf{H}_x \delta\mathbf{x} \quad \delta\dot{\mathbf{q}} = \mathbf{H}_x \delta\dot{\mathbf{x}} \quad (26)$$

Using Taylor's expansion formula, the generalized mass matrix \mathbf{M} and the generalized force vector \mathbf{Q} are expanded as

$$\mathbf{M} = \tilde{\mathbf{M}} + O^1(\delta\mathbf{q}) \quad (27)$$

$$\mathbf{Q} = \tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}_q \delta\mathbf{q} + \tilde{\mathbf{Q}}_q \delta\dot{\mathbf{q}} + O^2(\delta\mathbf{q}, \delta\dot{\mathbf{q}}) \quad (28)$$

where $\tilde{\mathbf{M}}$, $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{Q}}_q$, $\tilde{\mathbf{Q}}_q$ are the values of \mathbf{M} , \mathbf{Q} , \mathbf{Q}_q , \mathbf{Q}_q in steady state movement at the equilibrium position, respectively, and $O^j(\delta\mathbf{q})$ denotes the order j or higher terms. Substituting equation (26) into (28) yields

$$\mathbf{Q} = \tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}_q \mathbf{H}_x \delta\mathbf{x} + \tilde{\mathbf{Q}}_q \mathbf{H}_x \delta\dot{\mathbf{x}} + O^2(\delta\mathbf{q}, \delta\dot{\mathbf{q}}) \quad (29)$$

\mathbf{H}_x can be expanded as

$$\mathbf{H}_x = \tilde{\mathbf{H}}_x + \mathbf{H}_x^1 + O^2(\delta\mathbf{q}) \quad (30)$$

where \mathbf{H}_x^1 is the first order term of the expansion. The following equations are obtained by (27), (29) and (30)

$$\mathbf{H}_x^T \mathbf{M} \mathbf{H}_x = \tilde{\mathbf{H}}_x^T \tilde{\mathbf{M}} \tilde{\mathbf{H}}_x + O^1(\delta\mathbf{q}) \quad (31)$$

$$\mathbf{H}_x^T \mathbf{Q} = \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}} + \mathbf{H}_x^{1T} \tilde{\mathbf{Q}} + \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x \delta\mathbf{x} + \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x \delta\dot{\mathbf{x}} + O^2(\delta\mathbf{q}, \delta\dot{\mathbf{q}}) \quad (32)$$

We denote

$$\mathbf{T} = \mathbf{H}_x^{1T} \tilde{\mathbf{Q}} \quad (33)$$

Substituting formula (25) and (26) into equation (33) and using a Taylor expansion yields

$$\mathbf{T} = \tilde{\mathbf{T}}_{\delta\mathbf{q}} \delta\mathbf{q} + O^2(\delta\mathbf{q}) = \tilde{\mathbf{T}}_{\delta\mathbf{q}} \tilde{\mathbf{H}}_x \delta\mathbf{x} + O^2(\delta\mathbf{q}) \quad (34)$$

Substituting equation (34) into equation (32) yields

$$\mathbf{H}_x^T \mathbf{Q} = \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}} + (\tilde{\mathbf{T}}_{\delta\mathbf{q}} \tilde{\mathbf{H}}_x + \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x) \delta\mathbf{x} + \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x \delta\dot{\mathbf{x}} + O^2(\delta\mathbf{q}, \delta\dot{\mathbf{q}}) \quad (35)$$

We set

$$\hat{\mathbf{x}} = \delta \mathbf{x} \quad (36)$$

Substituting equations (31), (35) and (36) into equation (24) and dropping quadratic and higher order terms, the linearized equation system of multibody system dynamics is obtained as follows

$$\hat{\mathbf{M}} \ddot{\hat{\mathbf{x}}} + \hat{\mathbf{C}} \dot{\hat{\mathbf{x}}} + \hat{\mathbf{K}} \hat{\mathbf{x}} = \hat{\mathbf{Q}} \quad (37)$$

where

$$\hat{\mathbf{M}} = (\mathbf{H}_x^T \mathbf{M} \mathbf{H}_x) \Big|_{x=\tilde{x}} = \tilde{\mathbf{H}}_x^T \tilde{\mathbf{M}} \tilde{\mathbf{H}}_x \quad (38)$$

$$\hat{\mathbf{K}} = -\frac{\partial(\mathbf{H}_x^T \mathbf{Q})}{\partial \mathbf{x}} \Big|_{x=\tilde{x}} = -(\mathbf{T}_{\delta \mathbf{q}} \tilde{\mathbf{H}}_x + \tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x) = -\left[\frac{d(\mathbf{H}_x^T)}{d\mathbf{X}^T} (\mathbf{I}_m \otimes \tilde{\mathbf{Q}}) + \tilde{\mathbf{H}}_x^T \frac{d\mathbf{Q}}{d\mathbf{X}^T} \right] \quad (39)$$

$$\hat{\mathbf{C}} = -(\mathbf{H}_x^T \mathbf{Q}_q \mathbf{H}_x) \Big|_{x=\tilde{x}} = -\tilde{\mathbf{H}}_x^T \tilde{\mathbf{Q}}_q \tilde{\mathbf{H}}_x \quad (40)$$

$$\hat{\mathbf{F}} = (\mathbf{H}_x^T \mathbf{Q}) \Big|_{x=\tilde{x}} = \mathbf{H}_x^T \tilde{\mathbf{Q}} \quad (41)$$

and \mathbf{I}_m in formula (39) denotes an $(m \times m)$ identity matrix. The vector \mathbf{T} in equation (34) is a function of \mathbf{H}_x . $\mathbf{T}_{\delta \mathbf{q}} \tilde{\mathbf{H}}_x$, a term appearing in $\hat{\mathbf{K}}$, is the coupling stiffness effect of constraints and generalized forces. Equation (19) shows that the inverse matrix of Φ_y and its Jacobian matrix have to be computed to obtain the term \mathbf{H}_x . In this paper, computer algebra operations and the symbolic linearization technique are used to obtain this term. Since Φ_y is usually a nonlinear matrix function, a symbolic inverse operation for it is very time consuming and the analytical solution is often too complicated to be of real use. To solve the problem, the following linear transformations of the generalized mass matrix are introduced

$$\frac{d(\mathbf{H}_x^T)}{d\mathbf{X}^T} = \left[\frac{d\mathbf{H}_x}{d\mathbf{X}} \right]^T \quad (42)$$

$$\frac{d(\Phi_y^{-1})}{d\mathbf{X}^T} = -(\mathbf{I}_m \otimes \tilde{\Phi}_y^{-1}) \frac{d(\Phi_y)}{d\mathbf{X}^T} \tilde{\Phi}_y^{-1} \quad (43)$$

$$\frac{d\mathbf{H}_x^T}{d\mathbf{X}^T} = \left[\begin{array}{c} (\mathbf{I}_m \otimes \tilde{\Phi}_y^{-1}) \frac{d\Phi_y}{d\mathbf{X}} \tilde{\Phi}_y^{-1} \tilde{\Phi}_x - (\mathbf{I}_m \otimes \tilde{\Phi}_y^{-1}) \frac{\partial \Phi_x}{\partial \mathbf{X}} \\ \mathbf{0} \end{array} \right]^T \quad (44)$$

Substituting equations (19), (42), (43) and (44) into equation (39) yields

$$\begin{aligned} \hat{\mathbf{K}} &= -\frac{\partial(\mathbf{H}_x^T \mathbf{Q})}{\partial \mathbf{X}^T} \Big|_{x=\tilde{x}} \\ &= -\left\{ \left[\begin{array}{c} (\mathbf{I}_m \otimes \tilde{\Phi}_y^{-1}) \frac{d\Phi_y}{d\mathbf{X}} \tilde{\Phi}_y^{-1} \tilde{\Phi}_x - (\mathbf{I}_m \otimes \tilde{\Phi}_y^{-1}) \frac{\partial \Phi_x}{\partial \mathbf{X}} \\ \mathbf{0} \end{array} \right]^T (\mathbf{I}_m \otimes \tilde{\mathbf{Q}}) + \tilde{\mathbf{H}}_x^T \frac{d\mathbf{Q}}{d\mathbf{X}^T} \right\} \end{aligned} \quad (45)$$

Equation (45) means that the symbolic operation of Φ_y^{-1} can be changed to that of $\tilde{\Phi}_y^{-1}$, and the latter is much easier handle because it is independent of \mathbf{q} .

4 Simulation Examples

In this section, two examples are given to test the symbolic simulation procedure discussed above. All the calculations were performed by using the software for symbolic dynamic modeling and analysis of multibody systems, so the symbolic linearized equations can be translated automatically.

Example 1: Double Pendulum

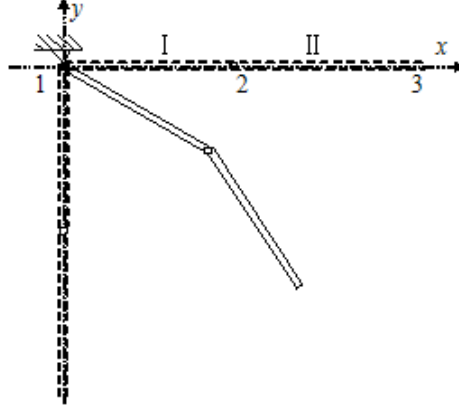


Figure 2. Double pendulum

The model of the double pendulum is shown in Figure 2. The lengths and masses of link rod I and II are L_1, L_2 and m_1, m_2 , respectively. The double pendulum sways slightly around Z-axis in the plane XOY. We take point 1 as the origin of XOY frame, and choose point 1 (x_1, y_1), point 2 (x_2, y_2) and point 3 (x_3, y_3) as the base points of the system, and $\mathbf{q} = [x_1 \ y_1 \ x_2 \ y_2 \ x_3 \ y_3]^T$ as the vector of generalized coordinates. The system has two degrees of freedom. We choose $\mathbf{x} = [x_2 \ x_3]^T$ as the independent generalized coordinates and $\mathbf{y} = [x_1 \ y_1 \ y_2 \ y_3]^T$ as the dependent ones. The constraints of the system are

$$\Phi(\mathbf{q}) \triangleq \begin{bmatrix} x_1 \\ y_1 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - L_1^2 \\ (y_3 - y_2)^2 + (x_3 - x_2)^2 - L_2^2 \end{bmatrix} = \mathbf{0} \quad (46)$$

The generalized mass matrix \mathbf{M} , the generalized force vector \mathbf{Q} and the Jacobian matrix of the constraint function vector are

$$\mathbf{M} = \begin{bmatrix} \frac{m_1}{3} & 0 & \frac{m_1}{6} & 0 & 0 & 0 \\ 0 & \frac{m_1}{3} & 0 & \frac{m_1}{6} & 0 & 0 \\ \frac{m_1}{6} & 0 & \frac{m_1 + m_2}{3} & 0 & \frac{m_2}{6} & 0 \\ 0 & \frac{m_1}{6} & 0 & \frac{m_1 + m_2}{3} & 0 & \frac{m_2}{6} \\ 0 & 0 & \frac{m_2}{6} & 0 & \frac{m_2}{3} & 0 \\ 0 & 0 & 0 & \frac{m_2}{6} & 0 & \frac{m_2}{3} \end{bmatrix} \quad (47)$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & -\frac{(m_1 + m_2)g}{2} & 0 & -\frac{m_2 g}{2} \end{bmatrix}^T \quad (48)$$

$$\Phi_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2(x_2-x_1) & -2(y_2-y_1) & 2(x_2-x_1) & 2(y_2-y_1) & 0 & 0 \\ 0 & 0 & -2(x_3-x_2) & -2(y_3-y_2) & 2(x_3-x_2) & 2(y_3-y_2) \end{bmatrix} \quad (49)$$

Substituting equations (46)~(49) into (13), a differential/algebraic equation system is obtained to describe the dynamic model of the system. Through simple deduction and computation, the Jacobian matrices Φ_x , Φ_y and their values at the equilibrium position $\tilde{\Phi}_x$, $\tilde{\Phi}_y$, the generalized force vector \tilde{Q} and the matrix functions $\tilde{\Phi}_y^{-1}$ and \tilde{H}_x are obtained as follows

$$\Phi_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2(x_2-x_1) & 0 \\ -2(x_3-x_2) & 2(x_3-x_2) \end{bmatrix} \quad (50)$$

$$\Phi_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2(x_2-x_1) & -2(y_2-y_1) & 2(y_2-y_1) & 0 \\ 0 & 0 & -2(y_3-y_2) & 2(y_3-y_2) \end{bmatrix} \quad (51)$$

$$\Phi_x^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{\Phi}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2L_1 & -2L_1 & 0 \\ 0 & 0 & 2L_2 & -2L_2 \end{bmatrix} \quad (52)$$

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & 0 & -\frac{(m_1+m_2)g}{2} & 0 & -\frac{m_2g}{2} \end{bmatrix}^T \quad (53)$$

$$\tilde{\Phi}_y^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2L_1} & 0 \\ 0 & 1 & -\frac{1}{2L_1} & -\frac{1}{2L_2} \end{bmatrix} \quad \tilde{H}_x = \begin{bmatrix} -\tilde{\Phi}_y^{-1} \Phi_x \\ \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (54)$$

where \mathbf{I}_m in \tilde{H}_x is an (2×2) identity matrix. Its position in the generalized coordinate system is in correspondence with the position of the independent generalized coordinate system. We took the vertical position of the system as the equilibrium position and denote \hat{x} as $\hat{x} = x - \tilde{x}$. The following linearized matrices \hat{M} , \hat{K} and \hat{Q} are derived by equations (38), (39) and (41)

$$\hat{M} = \begin{bmatrix} \frac{(m_1+m_2)}{3} & \frac{m_2}{6} \\ \frac{m_2}{6} & \frac{m_2}{3} \end{bmatrix} \quad (55)$$

$$\hat{K} = \begin{bmatrix} -\frac{m_2g(2L_2+L_1)+m_1gL_2}{2L_1L_2} & \frac{m_2g}{2L_2} \\ \frac{m_2g}{2L_2} & -\frac{m_2g}{2L_2} \end{bmatrix} \quad (56)$$

$$\hat{\mathbf{Q}} = [0 \ 0]^T \quad (57)$$

Thus the following linearized dynamical equations of the system in the vicinity of the stabilized movement is obtained by equation (37)

$$\begin{bmatrix} \frac{(m_1 + m_2)}{3} & \frac{m_2}{6} \\ \frac{m_2}{6} & \frac{m_2}{3} \end{bmatrix} \begin{bmatrix} \ddot{\hat{x}}_2 \\ \ddot{\hat{x}}_3 \end{bmatrix} + \begin{bmatrix} -\frac{m_2 g(2L_2 + L_1) + m_1 g L_2}{2L_1 L_2} & \frac{m_2 g}{2L_2} \\ \frac{m_2 g}{2L_2} & -\frac{m_2 g}{2L_2} \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (58)$$

Example 2: Spring pendulum system

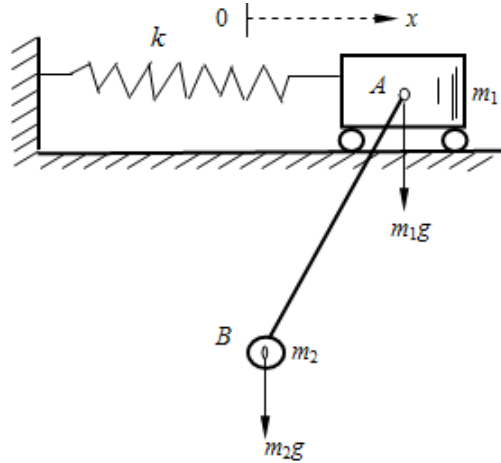


Figure 3 Spring pendulum system

The model of a spring pendulum is shown in Figure 3. The spring, with stiffness coefficient k , connects to the wall with its left end. At its right end it attaches a vehicle of mass m_1 . A pendulum with mass m_2 is attached to the mass center of the vehicle by a string of length L . Neglecting the masses of the spring and of the string, with some other simplifications, the system can be treated as a spring pendulum model. We took the mass center of the vehicle $A(x_1, y_1)$ and the mass center of the weight $B(x_2, y_2)$ as the base points of the system. We chose $\mathbf{q} = [x_1 \ y_1 \ x_2 \ y_2]^T$ as the generalized coordinates, $\mathbf{x} = [x_1 \ x_2]^T$ as the independent generalized coordinates and $\mathbf{y} = [y_1 \ y_2]^T$ as the non-independent ones. With two degrees of freedom, the system contains two constraint equations. After deriving the differential/algebraic equations of the dynamical system is a symbolic linearization is made to obtain $\hat{\mathbf{M}}$, $\hat{\mathbf{K}}$ and $\hat{\mathbf{Q}}$. The symbolic linearization of the dynamical equation in the vicinity of the equilibrium position is then derived as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\hat{x}}_1 \\ \ddot{\hat{x}}_2 \end{bmatrix} + \begin{bmatrix} \frac{m_2 g}{L} - \frac{k}{2} & -\frac{m_2 g}{L} \\ -\frac{m_2 g}{L} & \frac{m_2 g}{L} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{kx_1}{2} \\ 0 \end{bmatrix} \quad (59)$$

To verify the correctness of the symbolic linearization method and the efficiency of the software developed from the method, the Lagrangian equations are used to deduce the linearized dynamic equations. The kinetic energy T and the potential energy V of the system are

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left\{ \dot{x}_1^2 + \frac{L^2 (\dot{x}_2 - \dot{x}_1)^2}{L^2 - (x_2 - x_1)^2} + 2x_1 \frac{L(\dot{x}_2 - \dot{x}_1)}{\sqrt{L^2 - (x_2 - x_1)^2}} \cos[\arcsin \frac{x_2 - x_1}{L}] \right\} \quad (60)$$

$$V = -\frac{1}{2}kx_1^2 - m_2g L \cos[\arcsin \frac{x_2 - x_1}{L}] \quad (61)$$

The corresponding Lagrange approach leads to the equation of dynamics

$$m_1\ddot{x}_1 - \frac{m_2L^2(\ddot{x}_2 - \ddot{x}_1)}{L^2 - (x_2 - x_1)^2} + m_2(\ddot{x}_2 - \ddot{x}_1) + \frac{m_2(x_2 - x_1)\sqrt{L^2 - (x_2 - x_1)^2}(\dot{x}_2 - \dot{x}_1)^2}{[L^2 - (x_2 - x_1)^2]^{3/2}} - \frac{m_2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)^2}{L^2 - (x_2 - x_1)^2} - \frac{m_2L^2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)^2}{[L^2 - (x_2 - x_1)^2]^2} - \frac{m_2g(x_2 - x_1)}{\sqrt{L^2 - (x_2 - x_1)^2}} - kx_1 = 0 \quad (62)$$

$$\frac{m_2L^2(\ddot{x}_2 - \ddot{x}_1)}{L^2 - (x_2 - x_1)^2} + m_2\ddot{x}_1 + \frac{m_2L^2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)^2}{[L^2 - (x_2 - x_1)^2]^2} + \frac{m_2g(x_2 - x_1)}{\sqrt{L^2 - (x_2 - x_1)^2}} = 0 \quad (63)$$

We denote x_1 and x_2 as $x_1 = x_{10} + \Delta x_1$, $x_2 = x_{20} + \Delta x_2$, where x_{01} , x_{02} are the values of x_1, x_2 at equilibrium position and Δx_1 , Δx_2 denote small disturbances. Substituting them into (62) and (63), expanding the resulting functions and neglecting the order 2 and higher terms, the linearized dynamical equation system is obtained as follows

$$\begin{cases} m_1 \Delta \ddot{x}_1 + \frac{m_2g}{L}(\Delta x_1 - \Delta x_2) = k \Delta x_1 \\ m_2 \Delta \ddot{x}_2 + \frac{m_2g}{L}(\Delta x_2 - \Delta x_1) = 0 \end{cases} \quad (64)$$

By making a comparison between equation (59) and (64), it is clear that the linearized dynamical equations derived by the Lagrange approach is identical to that derived by the proposed symbolic linearization method.

5 Conclusion

Based on complete Cartesian coordinates, a symbolic method for deducing the linearized equation system of multibody system dynamics is presented in this paper, which overcomes some deficiency in traditional numerical modeling methods. The method can be taken as a linearization module of nonlinear systems which does not rely on a large special library of linearized constraints. It is an algorithm of general-purpose, of clear mechanical meaning and the facility of obtaining symbolic explicit expression. The examples show that the method can be applied to symbolic linearization analysis and optimum design of complicated mechanical systems.

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