TECHNISCHE MECHANIK, Band 24, Heft 1, (2004), 37-50
Manuskripteingang: 18. Dezember 2003

Auto-Balancing of Anisotropically Supported Rigid Rotors

B. Ryzhik, L. Sperling, H. Duckstein

This paper presents a generalisation of investigations into single-plane auto-balancing of statically and dynamically unbalanced rigid rotors considering the influence of support anisotropy. In the analytical part conditions for existence and stability of forward whirl are derived. The question of simultaneous backward whirl compensation is answered, and two special cases of supporting are considered. The results are confirmed by numerical simulations. Further simulation results for the two-plane device are included.

1 Introduction

In Sperling et al. (2004) the authors have investigated in detail single-plane auto-balancing of isotropically supported rigid rotors. However, in practice the isotropy of supporting is not always ensured, for example because of the properties of journal bearings or foundations.
Anisotropy of supports leads to an elliptical shape of the rotor vibration trajectory and to an increase in the number of critical speeds. Nevertheless, the averaged motions are similar to the case of isotropic supports, so the same approach to the analytical study is applicable.
The present paper is based on the aforementioned investigations. The authors waive the replication of identical parts and concentrate only on the specific details and results.
Again, conditions for the existence and stability of compensatory ball motions are derived and interpreted. Primarily, the existence conditions are valid for the forward whirl. However, in practically important cases the simultaneous backward whirl compensation is possible. Instead of a general analysis of the stability condition as in the case of isotropic supports, the investigation of stability is restricted on two specific simpler cases.
The results are confirmed and completed by numerical simulations, including also the case of the long rotor with a two-plane device. In compact form, some results of the present paper have already been published in Duckstein et al. (2003 a) and Duckstein et al. (2003 b).

2 Model and Equations of Motion

Assuming the rotor to be mounted on two orthotropic elastic damped supports with the same directions of the principal axes and the resultant stiffnesses $k_{11}, k_{12}, k_{22}, k_{33}, k_{34}, k_{44}$ and damping factors $c_{11}, c_{12}, c_{22}, c_{33}, c_{34}, c_{44}$ with respect to the vibrational co-ordinates $\mathbf{q_V}$, based on the model definition and the explanation of the symbols in Sperling et al. (2004) we obtain the following Lagrange's equations for the system under investigation, linearized in the vibrational co-ordinates $\mathbf{q_V}$:

$$M\ddot{r}_x + \sum_{k=1}^{n+m} m_k z_k \ddot{\psi}_y + c_{11}\dot{r}_x + c_{12}\dot{\psi}_y + k_{11} r_x + k_{12}\psi_y = \sum_{k=1}^{n+m} m_k \varepsilon_k \left(\ddot{\varphi}_k \sin\varphi_k + \dot{\varphi}_k^2 \cos\varphi_k\right), \tag{1}$$

$$\begin{aligned} &\sum_{k=1}^{n+m} m_k z_k \ddot{r}_x + J_a \ddot{\psi}_y + \tfrac{1}{2}\sum_{k=1}^{n+m} m_k \varepsilon_k^2 \left[-\ddot{\psi}_x \sin 2\varphi_k + \ddot{\psi}_y \left(1+\cos 2\varphi_k\right)\right] \\ &+ c_{12}\dot{r}_x + c_{22}\dot{\psi}_y - \left[\tilde{J}_z \dot{\varphi}_R + \sum_{k=1}^{n+m} \tilde{J}_k \dot{\varphi}_k\right]\dot{\psi}_x - \sum_{k=1}^{n+m} m_k \varepsilon_k^2 \dot{\varphi}_k \left(\dot{\psi}_x \cos 2\varphi_k + \dot{\psi}_y \sin 2\varphi_k\right) \\ &+ k_{12} r_x + k_{22}\psi_y - \tfrac{1}{2}\left[\tilde{J}_z \ddot{\varphi}_R + \sum_{k=1}^{n+m} \tilde{J}_k \ddot{\varphi}_k\right]\psi_x = \sum_{k=1}^{n+m} m_k \varepsilon_k z_k \left(\ddot{\varphi}_k \sin\varphi_k + \dot{\varphi}_k^2 \cos\varphi_k\right), \end{aligned} \tag{2}$$

$$M\ddot{r}_y - \sum_{k=1}^{n+m} m_k z_k \ddot{\psi}_x + c_{33}\dot{r}_y - c_{34}\dot{\psi}_x + k_{33}r_y - k_{34}\psi_x = -\sum_{k=1}^{n+m} m_k \varepsilon_k \left(\ddot{\varphi}_k \cos\varphi_k - \dot{\varphi}_k^2 \sin\varphi_k\right), \tag{3}$$

$$\begin{aligned}
&-\sum_{k=1}^{n+m} m_k z_k \ddot{r}_y + J_a \ddot{\psi}_x + \tfrac{1}{2}\sum_{k=1}^{n+m} m_k \varepsilon_k^2 \left[\ddot{\psi}_x \left(1-\cos 2\varphi_k\right) - \ddot{\psi}_y \sin 2\varphi_k\right] \\
&- c_{34}\dot{r}_y + c_{44}\dot{\psi}_x + \left[\tilde{J}_z \dot{\varphi}_R + \sum_{k=1}^{n+m} \tilde{J}_k \dot{\varphi}_k\right]\dot{\psi}_y + \sum_{k=1}^{n+m} m_k \varepsilon_k^2 \dot{\varphi}_k \left(\dot{\psi}_x \sin 2\varphi_k - \dot{\psi}_y \cos 2\varphi_k\right) \\
&- k_{34} r_y + k_{44}\psi_x + \tfrac{1}{2}\left[\tilde{J}_z \ddot{\varphi}_R + \sum_{k=1}^{n+m} \tilde{J}_k \ddot{\varphi}_k\right]\psi_y = \sum_{k=1}^{n+m} m_k \varepsilon_k z_k \left(\ddot{\varphi}_k \cos\varphi_k - \dot{\varphi}_k^2 \sin\varphi_k\right),
\end{aligned} \tag{4}$$

$$J_z\ \ddot{\varphi}_R - \sum_{k=1}^{n} J_{kR}\ddot{\varphi}_k + \beta_R \dot{\varphi}_R - \sum_{k=1}^{n} \beta_k \dot{\varphi}_k - \sum_{k=n+1}^{n+m} m_k \varepsilon_k \left[\ddot{r}_{kx} \sin\varphi_k - \ddot{r}_{ky}\ \cos\varphi_k\right] = L_R\left(\dot{\varphi}_R\right), \tag{5}$$

$$-J_{iR}\ddot{\varphi}_R + J_i \ddot{\varphi}_i - \beta_i \dot{\varphi}_R + \beta_i \dot{\varphi}_i - m_i \varepsilon_i \left[\ddot{r}_{ix}\ \sin\varphi_i - \ddot{r}_{iy}\ \cos\varphi_i\right]\ = 0, \qquad i = 1,\ldots,n\,, \tag{6}$$

where

$$r_{ix} = r_x + z_i \psi_y, \quad r_{iy} = r_y - z_i \psi_x\,, \qquad i = 1,\ldots,n+m \tag{7}$$

are the co-ordinates of the path centres in the planes of the inherent unbalances and the balls. These equations are the basis of numerical simulations.

3 General Conditions for Existence and Stability of Ball Motions Synchronous with the Rotor

Following the approach of Sperling et al. (2004) , section 4, we obtain the vibrational moments

$$V_i = \frac{1}{2\pi}\int_0^{2\pi} B_i d\,\Omega t\,, \qquad B_i = m_i \varepsilon_i \left[\ddot{r}_{ix} \sin\left(\Omega t + \alpha_i\right) - \ddot{r}_{iy} \cos\left(\Omega t + \alpha_i\right)\right]\ , \qquad i = 1,\ldots,n\,. \tag{8}$$

To reach an approximation for steady state vibrations, we simplify Eqs (1) – (4) as follows:

$$M\ddot{r}_x + k_{11}r_x + k_{12}\psi_y = \sum_{k=1}^{n+m} f_k \cos\left(\Omega t + \alpha_k\right), \tag{9}$$

$$J_a \ddot{\psi}_y - J_z \Omega \dot{\psi}_x + k_{12}r_x + k_{22}\psi_y = \sum_{k=1}^{n+m} z_k f_k \cos\left(\Omega t + \alpha_k\right), \tag{10}$$

$$M\ddot{r}_y + k_{33}r_y - k_{34}\psi_x = \sum_{k=1}^{n+m} f_k \sin\left(\Omega t + \alpha_k\right), \tag{11}$$

$$J_a \ddot{\psi}_x + J_z \Omega \dot{\psi}_y - k_{34}r_y + k_{44}\psi_x = -\sum_{k=1}^{n+m} z_k f_k \sin\left(\Omega t + \alpha_k\right) \tag{12}$$

with the centrifugal forces

$$f_i = m_i \varepsilon_i \Omega^2\ ,\ i = 1,\ldots,n+m\ . \tag{13}$$

Determining the steady-state solution of the Eqs. (9) – (12) for only the *k*-th member on each RHS in the form

$$\begin{aligned}
r_{xk} &= \hat{r}_{xk} \cos\left(\Omega t + \alpha_k\right), & \psi_{yk} &= \hat{\psi}_{yk} \cos\left(\Omega t + \alpha_k\right), \\
r_{yk} &= \hat{r}_{yk} \sin\left(\Omega t + \alpha_k\right), & \psi_{xk} &= \hat{\psi}_{xk} \sin\left(\Omega t + \alpha_k\right),
\end{aligned} \tag{14}$$

we obtain the amplitudes

$$\hat{r}_{xk} = \frac{\Delta_{rxk}^{an}}{\Delta^{an}} f_k\,, \qquad \hat{\psi}_{yk} = \frac{\Delta_{\psi yk}^{an}}{\Delta^{an}} f_k\,, \qquad \hat{r}_{yk} = \frac{\Delta_{ryk}^{an}}{\Delta^{an}} f_k\,, \qquad \hat{\psi}_{xk} = \frac{\Delta_{\psi xk}^{an}}{\Delta^{an}} f_k\,, \tag{15}$$

where

$$\begin{aligned}\Delta^{an} &= \left[\left(-M\Omega^2 + k_{11}\right)\left(-J_a\Omega^2 + k_{22}\right) - k_{12}{}^2\right]\left[\left(-M\Omega^2 + k_{33}\right)\left(-J_a\Omega^2 + k_{44}\right) - k_{34}{}^2\right] \\ &\quad - \left(-M\Omega^2 + k_{11}\right)\left(-M\Omega^2 + k_{33}\right)\left(J_z\Omega^2\right)^2\end{aligned} \tag{16}$$

and $\Delta_{rxk}^{an},\ldots$ are more complicated expressions.
Thus, the stationary orbital motion of the centres of the circular ball paths is described by the expressions

$$r_{ix} = \sum_{k=1}^{n+m} f_k A_{ik}^x \cos(\Omega t + \alpha_k)\,, \qquad r_{iy} = \sum_{k=1}^{n+m} f_k A_{ik}^y \sin(\Omega t + \alpha_k)\,, \qquad i = 1,\ldots,n\,, \tag{17}$$

where

$$A_{ik}^x = \frac{1}{\Delta^{an}}\left(\Delta_{rxk}^{an} + z_i \Delta_{\psi yk}^{an}\right), \quad A_{ik}^y = \frac{1}{\Delta^{an}}\left(\Delta_{ryk}^{an} - z_i \Delta_{\psi xk}^{an}\right) \tag{18}$$

are the corresponding harmonical influence coefficients.
In the isotropic case $k_{33}=k_{11}$, $k_{34}=k_{12}$, $k_{44}=k_{22}$, we find instead of the expressions (15) and (16) in accordance with the results in Sperling et al. (2004):

$$\Delta^{is} = \Delta\widetilde{\Delta}\,, \tag{19}$$

where

$$\begin{aligned}\Delta &= \left(-M\Omega^2 + k_{11}\right)\left(-\left(J_a - J_z\right)\Omega^2 + k_{22}\right) - k_{12}{}^2 \\ &= M\left(J_a - J_z\right)\Omega^4 - \left[Mk_{22} + \left(J_a - J_z\right)k_{11}\right]\Omega^2 + k_{11}k_{22} - k_{12}{}^2,\end{aligned} \tag{20}$$

$$\begin{aligned}\widetilde{\Delta} &= \left(-M\Omega^2 + k_{11}\right)\left(-\left(J_a + J_z\right)\Omega^2 + k_{22}\right) - k_{12}{}^2 \\ &= M\left(J_a - J_z\right)\Omega^4 - \left[Mk_{22} + \left(J_a + J_z\right)k_{11}\right]\Omega^2 + k_{11}k_{22} - k_{12}{}^2,\end{aligned} \tag{21}$$

and

$$\hat{r}_{xk} = \frac{\Delta_{rxk}}{\Delta} f_k\,, \qquad \hat{\psi}_{yk} = \frac{\Delta_{\psi yk}}{\Delta} f_k\,, \qquad \hat{r}_{yk} = \frac{\Delta_{ryk}}{\Delta} f_k\,, \qquad \hat{\psi}_{yk} = \frac{\Delta_{\psi xk}}{\Delta} f_k\,, \tag{22}$$

with

$$\Delta_{rxk} = \Delta_{ryk} = \left[-\left(J_a - J_z\right)\Omega^2 + k_{22} - k_{12} z_k\right]\,, \tag{23}$$

$$\Delta_{\psi yk} = -\Delta_{\psi xk} = \left[\left(-M\Omega^2 + k_{11}\right)z_k - k_{12}\right]\,. \tag{24}$$

In the anisotropic case, Eqs. (8) and (15) yield

$$B_i = -m_i\varepsilon_i\left[\ddot{r}_{ix}\sin(\Omega t + \alpha_i) - \ddot{r}_{iy}\cos(\Omega t + \alpha_i)\right]\,, \tag{25}$$

$$B_i = \frac{1}{2} f_i \sum_{k=1}^{n+m} f_k\left[\left(A_{ik}^x + A_{ik}^y\right)\sin(\alpha_i - \alpha_k) + \left(A_{ik}^x - A_{ik}^y\right)\sin(2\Omega t + \alpha_i + \alpha_k)\right], \quad i = 1,\ldots,n\,. \tag{26}$$

In contrast to the isotropic case, these moments B_i, acting between the rotor and the balls, are not constant and include 2Ω - frequency components.

Averaging yields the vibrational moments

$$V_i = \frac{1}{2} f_i \sum_{k=1}^{n+m} f_k \left(A_{ik}^x + A_{ik}^y\right)\sin\left(\alpha_i - \alpha_k\right), \quad i = 1,...,n . \tag{27}$$

Thus, the existence conditions for synchronous motions, i.e. motions with constant ball phases $\alpha_i = \alpha_i^*$, $i = 1,...,n$ in the average, are

$$V_i\left(\alpha_1 = \alpha_1^*,...,\alpha_n = \alpha_n^*\right) = \frac{1}{2} f_i \sum_{k=1}^{n+m} f_k \left(A_{ik}^x + A_{ik}^y\right)\sin\left(\alpha_i^* - \alpha_k^*\right) = 0 , \qquad i = 1,...,n , \tag{28}$$

where $\alpha_i^* \equiv \alpha_i, \; i = n+1,...,n+m$.

In the case of isotropic supporting $A_{ik}^x = A_{ik}^y = \frac{1}{2}\left(A_{ik}^x + A_{ik}^y\right) = A_{ik}$ and conditions (28) become identical with the existence conditions from Sperling et al. (2004).

The components of r_{ix} and r_{iy} (see Eq. (17)) in a frame ξ, η rotating with the rotor are

$$r_{i\xi} = r_{ix} \cos \Omega t + r_{iy} \sin \Omega t , \quad r_{i\eta} = -r_{ix} \sin \Omega t + r_{iy} \cos \Omega t . \tag{29}$$

They can be divided into the constant forward whirl parts

$$r_{i\xi}^f = \sum_{k=1}^{n+m} f_k A_{ik}^f \cos\alpha_k , \qquad r_{i\eta}^f = \sum_{k=1}^{n+m} f_k A_{ik}^f \sin\alpha_k \tag{30}$$

and the 2Ω-frequency backward whirl parts

$$r_{i\xi}^b = \sum_{k=1}^{n+m} f_k A_{ik}^b \cos\left(2\Omega t + \alpha_k\right), \qquad r_{i\eta}^b = \sum_{k=1}^{n+m} \left(-f_k A_{ik}^b\right)\sin\left(2\Omega t + \alpha_k\right), \tag{31}$$

where

$$A_{ik}^f = \frac{1}{2}\left(A_{ik}^x + A_{ik}^y\right), \qquad A_{ik}^b = \frac{1}{2}\left(A_{ik}^x - A_{ik}^y\right). \tag{32}$$

This yields the alternative form of the existence conditions

$$V_i = f_i \left[r_{i\xi}^f \left(\alpha_1^*,...,\alpha_n^*\right)\sin\alpha_i^* - r_{i\eta}^f \left(\alpha_1^*,...,\alpha_n^*\right)\cos\alpha_i^* \right] = 0 , \qquad i = 1,...,n . \tag{33}$$

Obviously, the „forward whirl compensation“ condition

$$r_{i\xi}^f \left(\alpha_1^*,...,\alpha_n^*\right) = 0 , \qquad r_{i\eta}^f \left(\alpha_1^*,...,\alpha_n^*\right) = 0 , \qquad i = \in \left(1,...,n\right) \tag{34}$$

constitutes a solution of the existence condition $V_i\left(\alpha_1 = \alpha_1^*,...,\alpha_n = \alpha_n^*\right) = 0$. „Forward whirl compensation“ means that the forward whirl of the *i*-th ball path centre is equal to zero.

If we define the rate of the forward whirl component due to the inherent unbalances

$$\hat{r}_{i\xi}^f = \hat{r}_i \cos\hat{\alpha}_i = \sum_{k=n+1}^{n+m} f_k A_{ik}^f \cos\alpha_k , \qquad \hat{r}_{i\eta}^f = \hat{r}_i \sin\hat{\alpha}_i = \sum_{k=n+1}^{n+m} f_k A_{ik}^f \sin\alpha_k , \tag{35}$$

we obtain as another form of the existence conditions

$$V_i = f_i \left[f_k A_{ik}^f \sin\left(\alpha_i^* - \alpha_k^*\right) + \hat{r}_i \sin\left(\alpha_i^* - \hat{\alpha}_i\right)\right] = 0 , \qquad i = 1,...,n . \tag{36}$$

If the forward whirl vanishes in at least two planes, the forward whirl is compensated for the whole rotor. This can be the case for a long rotor with a two plane device in the post-critical frequency range and for a rotor with a rigid bearing and a single-plane device in certain frequency ranges. In these cases according to Eq. (7) the forward whirl components of r_x, r_y, ψ_y, ψ_x vanish.

We demonstrate that consequently the backward whirl components will be equal to zero:
Eqs. (8) – (12) can be presented in the complex form. Introducing

$$\mathrm{r} = r_x + i r_y, \quad \psi = \psi_y - i\psi_x, \quad \mathrm{f}_k = f_k e^{i\alpha_k} \tag{37}$$

we obtain

$$M\ddot{\mathrm{r}} + \frac{1}{2}(k_{11}+k_{33})\,\mathrm{r} + \frac{1}{2}(k_{11}-k_{33})\,\bar{\mathrm{r}} + \frac{1}{2}(k_{12}+k_{34})\,\psi + \frac{1}{2}(k_{12}-k_{34})\,\bar{\psi} = \sum_{k=1}^{n+m} \mathrm{f}_k e^{i\Omega t}, \tag{38}$$

$$\begin{aligned} J_a\ddot{\psi} - J_z\Omega i\dot{\psi} + \frac{1}{2}(k_{12}+k_{34})\,\mathrm{r} + \frac{1}{2}(k_{12}-k_{34})\,\bar{\mathrm{r}} + \frac{1}{2}(k_{22}+k_{44})\,\psi + \frac{1}{2}(k_{22}-k_{44})\,\bar{\psi} = \\ = \sum_{k=1}^{n+m} z_k\,\mathrm{f}_k e^{i\Omega t}. \end{aligned} \tag{39}$$

The solution represents a combination of the forward and backward whirls

$$\mathrm{r} = \hat{\mathrm{r}}^f\, e^{i\Omega t} + \hat{\mathrm{r}}^b\, e^{-i\Omega t} \qquad \psi = \hat{\psi}^f\, e^{i\Omega t} + \hat{\psi}^b\, e^{-i\Omega t}. \tag{40}$$

Substitution of (40) into (38) and (39) gives

$$\begin{aligned} &\mathbf{L}^f\mathbf{R}^f + \mathbf{L}^{fb}\mathbf{R}^b = \mathbf{F}^f, \\ &\mathbf{L}^{fb}\mathbf{R}^f + \mathbf{L}^b\mathbf{R}^b = \mathbf{0} \end{aligned} \tag{41}$$

with the vectors

$$\mathbf{R}^f = \begin{pmatrix} \hat{\mathrm{r}}^f \\ \hat{\psi}^f \end{pmatrix}, \quad \mathbf{R}^b = \begin{pmatrix} \hat{\mathrm{r}}^b \\ \hat{\psi}^b \end{pmatrix}, \quad \mathbf{F}^f = \begin{pmatrix} \mathrm{f}_k \\ z_k\,\mathrm{f}_k \end{pmatrix} \tag{42}$$

and corresponding matrices $\mathbf{L}^f, \mathbf{L}^b, \mathbf{L}^{fb}$.
From the last equation (41) we obtain

$$\mathbf{R}^b = -\left(\mathbf{L}^b\right)^{-1}\mathbf{L}^{fb}\mathbf{R}^f, \tag{43}$$

which means, that if the forward whirl is equal to zero, the backward whirl also becomes equal to zero.
To evaluate the stability of the various solutions of the existence conditions, we have to analyze the equations in variations

$$J_i\ddot{\bar{\alpha}}_i + \beta_i\dot{\bar{\alpha}}_i + f_i\left[\sum_{k=1}^{n} f_k A_{ik}^f \cos\left(\alpha_i^* - \alpha_k^*\right)\left(\bar{\alpha}_i - \bar{\alpha}_k\right) + \bar{\alpha}_i\hat{r}_i\cos\left(\alpha_i^* - \hat{\alpha}_i\right)\right] = 0, \quad i = 1,\ldots,n., \tag{44}$$

If the "forward whirl compensation" conditions are fulfilled, they can be simplified

$$J_i\ddot{\bar{\alpha}}_i + \beta_i\dot{\bar{\alpha}}_i - f_i\sum_{k=1}^{n} f_k A_{ik}^f \cos\left(\alpha_i^* - \alpha_k^*\right)\bar{\alpha}_k = 0, \qquad i = 1,\ldots,n. \tag{45}$$

Assuming that $\beta_i > 0$, $i = 1,\ldots,n$, the positive definiteness of the „stiffness matrix" of Eqs. (44) or (45) is a necessary and sufficient condition for the asymptotic stability of any solution of the existence conditions.

4 Single-Plane Balancer with Two Identical Balls

In the case of a single-plan device ($z_2 = z_1, A_{2i}^f = A_{1i}^f, i = 1,\ldots,2+m, \hat{r}_2 = \hat{r}_1 = \hat{r}, \hat{\alpha}_2 = \hat{\alpha}_1 = \hat{\alpha}$) a solution of the type „forward whirl compensation" ensures only vanishing of the forward whirl in one plane, namely in the device plane, while as a rule tilting forward whirl motions of the rotor remain.
The backward whirl component in the device plane disappears simultaneously only for the exceptional condition

$$\frac{A_{11}^x}{A_{11}^y} = \frac{A_{12}^x}{A_{12}^y} = \ldots = \frac{A_{1,n+m}^x}{A_{1,n+m}^y}, \tag{46}$$

which means, when the proportions of the harmonical influence coefficients for x- and y-direction of all inherent and ball unbalances with respect to the path centre of the balls are identical.
In the case of two identical balls ($f_2 = f_1 = f$) the „forward whirl compensation" conditions are

$$\begin{aligned} &fA_{11}^f\left(\cos\alpha_1^* + \cos\alpha_2^*\right) + \hat{r}_1 \cos\hat{\alpha}_1 = 0, \\ &fA_{11}^f\left(\sin\alpha_1^* + \sin\alpha_2^*\right) + \hat{r}_1 \sin\hat{\alpha}_1 = 0. \end{aligned}, \tag{47}$$

The only solution

$$\alpha_1^* = \hat{\alpha}_1 + \gamma, \qquad \alpha_2^* = \hat{\alpha}_1 - \gamma, \qquad \gamma = \arccos\left(-\frac{\hat{r}_1}{2fA_{11}^f}\right) \tag{48}$$

exists on the condition that

$$|\kappa| \geq 1, \qquad \kappa = \kappa(\Omega) = \frac{2fA_{11}^f(\Omega)}{\hat{r}_1(\Omega)}. \tag{49}$$

From the equations in variations

$$J_i\ddot{\overline{\alpha}}_i + \beta_i\dot{\overline{\alpha}}_i - f^2 A_{11}^f \sum_{k=1}^{2} \cos\left(\alpha_i^* - \alpha_k^*\right)\overline{\alpha}_k = 0, \qquad i = 1, 2 \tag{50}$$

we obtain the necessary and sufficient stability condition

$$A_{11}^f < 0. \tag{51}$$

Thus, the combined condition for the existence and stability is

$$\kappa(\Omega) \leq -1. \tag{52}$$

All other solutions of the general existence conditions

$$\begin{aligned} &fA_{11}^f \sin\left(\alpha_1^* - \alpha_2^*\right) + \hat{r}_1 \sin\left(\alpha_1^* - \hat{\alpha}_1\right) = 0, \\ &fA_{11}^f \sin\left(\alpha_2^* - \alpha_1^*\right) + \hat{r}_1 \sin\left(\alpha_2^* - \hat{\alpha}_1\right) = 0 \end{aligned}, \tag{53}$$

are of the type $\sin\left(\alpha_1^* - \alpha_2^*\right) = 0$:

$$\begin{aligned} &\alpha_1^* = \hat{\alpha}_1, && \alpha_2^* = \hat{\alpha}_1, \\ &\alpha_1^* = \hat{\alpha}_1 + \pi, && \alpha_2^* = \hat{\alpha}_1 + \pi, \\ &\alpha_1^* = \hat{\alpha}_1, && \alpha_2^* = \hat{\alpha}_1 + \pi, \\ &\alpha_1^* = \hat{\alpha}_1 + \pi, && \alpha_2^* = \hat{\alpha}_1. \end{aligned} \tag{54}$$

These solutions correspond completely to the solutions, interpreted in Sperling et al. (2004), section 5.2 for the isotropic supporting. The only difference is that the interpretation here applies only on the forward whirl component of the motion.

5 Special Cases of Rotor Support

5.1 Purely Translational Vibrations

We consider a symmetrically supported rotor system with only static primary unbalance with the centrifugal force f_{pr} in the mid-plane, so the conditions

$$\psi_y \equiv 0, \qquad \psi_x \equiv 0 \tag{55}$$

are valid.

For this case, repeatedly dealt with in the literature, the harmonical influence coefficients are

$$A_{11}^{x} = \frac{1}{M\left(\omega_x^{\,2} - \Omega^2\right)}, \qquad A_{11}^{y} = \frac{1}{M\left(\omega_y^{\,2} - \Omega^2\right)}, \tag{56}$$

where

$$\omega_x^{\,2} = \frac{k_{11}}{M}, \qquad \omega_y^{\,2} = \frac{k_{33}}{M} \tag{57}$$

are the eigenfrequencies.
Because all centrifugal forces act in the same plane, condition (46) is fulfilled, so, when the forward whirl is compensated, no backward whirl appears. Hence the auto-balancing device in this case can completely compensate vibrations.
We rewrite parameter κ from Eq. (49) in the form

$$\kappa = \frac{2 f A_{11}^{f}}{f_{pr}\left|A_{11}^{f}\right|} \tag{58}$$

and obtain the existence condition for the compensation motion

$$|\kappa| = \frac{2f}{f_{pr}} \geq 1 . \tag{59}$$

Eqs. (32), (51), (56) yield the stability condition for the complete unbalance compensation

$$\frac{1}{\omega_x^{\,2} - \Omega^2} + \frac{1}{\omega_y^{\,2} - \Omega^2} < 0 . \tag{60}$$

If we assume, without lost of generality, $\omega_y > \omega_x$, the speed regions $\omega_x^{\,2} < \Omega^2 < \frac{1}{2}\left(\omega_x^{\,2} + \omega_y^{\,2}\right)$ and $\omega_y^{\,2} < \Omega^2$ are the regions of stable unbalance compensation.

5.2 Purely Tilting Vibrations

We consider a rotor system with a fixed-point, for instance a rigid bearing in the plane $z = 0$ so that the conditions

$$r_x = 0, \qquad r_y = 0 \tag{61}$$

are valid. Hence, the forward and backward whirls vanish for the whole rotor, if the existence conditions for the forward whirl compensation in the device plane are fulfilled.
The system of equations for the amplitudes

$$\begin{bmatrix} -J_a\Omega^2 + k_{22} & -J_z\Omega^2 \\ -J_z\Omega^2 & -J_a\Omega^2 + k_{44} \end{bmatrix} \begin{bmatrix} \hat{\psi}_{yk} \\ \hat{\psi}_{xk} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} z_k f_k \tag{62}$$

yields the equation for the critical speeds

$$\Delta = \left(J_a^{\,2} - J_z^{\,2}\right)\Omega^4 - J_a\left(k_{22} + k_{44}\right)\Omega^2 + k_{22}k_{44} = 0, \tag{63}$$

which has solutions

$$\Omega_{1,2}^{\;2} = \frac{J_a}{2\left(J_a^{\,2} - J_z^{\,2}\right)}\left[k_{22} + k_{44} \pm \sqrt{\left(k_{22} - k_{44}\right)^2 + 4\left(\frac{J_z}{J_a}\right)^2 k_{22}k_{44}}\,\right]. \tag{64}$$

Thus, for the “long” rotor ($J_z < J_a$) two critical speeds exist, whereas the “disc-shaft” rotor ($J_z > J_a$) exhibits only one critical speed.

From the expressions for the amplitudes and the harmonical influence coefficients

$$\hat{\psi}_{yk} = -\frac{1}{\Delta}\left[(J_a + J_z)\Omega^2 - k_{44}\right] z_k f_k = \Delta_{\psi yk} f_k \ ,$$
$$\hat{\psi}_{xk} = \frac{1}{\Delta}\left[(J_a + J_z)\Omega^2 - k_{22}\right] z_k f_k = \Delta_{\psi xk} f_k \ , \tag{65}$$

$$A_{ik}^{x} = z_i \Delta_{\psi yk} = -\frac{1}{\Delta}\left[(J_a + J_z)\Omega^2 - k_{44}\right] z_i z_k \ ,$$
$$A_{ik}^{y} = -z_i \Delta_{\psi xk} = -\frac{1}{\Delta}\left[(J_a + J_z)\Omega^2 - k_{22}\right] z_i z_k \tag{66}$$

we obtain the existence and stability parameter

$$\kappa = \frac{2f}{\hat{r}_1^f(\Omega)\Delta}\left[-(J_a - J_z)\Omega^2 + \frac{k_{22} + k_{44}}{2}\right] z_1^{\ 2} \ , \tag{67}$$

where

$$\hat{r}_1^f(\Omega) = +\sqrt{\left(\sum_{k=n+1}^{n+m} f_k A_{1k} \cos\alpha_k\right)^2 + \left(\sum_{k=n+1}^{n+m} f_k A_{1k} \sin\alpha_k\right)^2} \ . \tag{68}$$

Again, $|\kappa| \geq 1$ is the existence condition for the forward whirl compensation.
The stability condition for the forward whirl compensation $\kappa < 0$ has, following from Eq. (67), the form

$$\frac{-(J_a + J_z)\Omega^2 + \dfrac{k_{22} + k_{44}}{2}}{\Delta} = -\frac{\Omega^2 - \Omega_0^{\ 2}}{(J_a - J_z)\left(\Omega^2 - \Omega_1^{\ 2}\right)\left(\Omega^2 - \Omega_2^{\ 2}\right)} < 0 \tag{69}$$

where

$$\Omega_0^{\ 2} = \frac{k_{22} + k_{44}}{2(J_a + J_z)} \ . \tag{70}$$

In the case $J_z < J_a$, the relation

$$0 < \Omega_1^{\ 2} < \Omega_0^{\ 2} < \Omega_2^{\ 2} \tag{71}$$

is valid.
Hence, the speed regions $\Omega_1^{\ 2} < \Omega^2 < \Omega_0^{\ 2}$ and $\Omega_2^{\ 2} < \Omega^2$ are regions of stable forward whirl compensation.
In the contrary case of the "disk-shaft" rotor with $J_z > J_a$, we have

$$\Omega_2^{\ 2} < 0 \ , \qquad\qquad 0 < \Omega_1^{\ 2} < \Omega_0^{\ 2} \ . \tag{72}$$

In this case the area of stable unbalance compensation is

$$\Omega_1^{\ 2} < \Omega^2 < \Omega_0^{\ 2}. \tag{73}$$

This speed region is wide enough and has practical meaning only for the sufficiently anisotropic support, i.e. when the stiffness coefficients k_{22} and k_{44} are noticeably different.
In the more general case, when the rigid bearing is located in the plane $z = z_0$, so

$$r_x = -z_0 \psi_y \ , \qquad\qquad r_y = z_0 \psi_x \ , \tag{74}$$

the same results can be obtained by replacing J_a , z_k , k_{22} , k_{44} with the parameters

$$J_a^{\ *} = J_a + Mz_0^{\ 2} - 2z_0 \sum_{k=1}^{n+m} m_k z_k \ , \qquad\qquad z_k^{\ *} = z_k - z_0 \tag{75}$$

$$k_{22}^{\ *} = k_{22} - 2k_{12}z_0 + k_{11}z_0^{\ 2} \ , \qquad\qquad k_{44}^{\ *} = k_{44} - 2k_{34}z_0 + k_{33}z_0^{\ 2} \ . \tag{76}$$

6 Simulations

To illustrate the influence of support anisotropy on the vibration performance of rotor systems with auto-balancing devices, we present below simulation results obtained employing the Advanced Continuous Simulation Language (ACSL). We investigated transient processes of the rotor run-up to the "nominal" speed higher than critical speeds.

6.1 Purely Translational Vibrations

As an example of a symmetrical system with only static primary unbalance we consider a rotor with a mass of 1 kg mounted on two equal supports. We investigate two variants of supports, isotropic, with $k_{11} = k_{33} = 10000$ N/m and anisotropic, with $k_{11} = 10000$ N/m and $k_{33} = 20000$ N/m. In the first case the rotor exhibits only one critical speed, approximately 100 rad/s, in the second case there are two critical speeds, 100 rad/s and 140 rad/s.
Figs.1-6 demonstrate the results of simulations. Near critical speeds one can observe a noticeable Sommerfeld-effect; for the anisotropic supports the area of Sommerfeld-type motion is clearly wider, because the rotor passes two critical speeds.
In the case of anisotropic supports vibrations include a double-rotational-frequency component (Fig.6).
In the post-critical region the balls synchronize with the rotor and move to the compensation positions. As predicted by the theory, in this region the vibrations become equal to zero.

6.2 Purely Tilting Vibrations

Two rotors are considered: a "long" one ($J_a > J_z$) and a "disk-shaft" ($J_a < J_z$). Both rotors has a mass of 3.15 kg; for the first rotor the moments of inertia are $J_a = 0.075$ kg m^2 and $J_z = 0.0089$ kg m^2, for the second one they are $J_a = 0.075$ kg m^2 and $J_z = 0.3$ kg m^2. To obtain purely tilting vibrations one of the supports is supposed to be rigid, while the second one is elastic.
When the elastic support is isotropic, the selected "long" rotor exhibits one critical speed at 44 rad/s; when this support is anisotropic, there are two critical speeds, 44 rad/s and 88 rad/s. Figs.7-10 compare the vibration performance of a rotor with a single-plane auto-balancing device for isotropic and anisotropic supporting.
As predicted by the analytical study for the rotor with one rigid bearing, the single-plane auto-balancing device provides complete elimination of vibrations in the post-critical area, both, when elastic support is isotropic and anisotropic. In the anisotropic case an area of compensation is expected also directly beyond the first critical speed, but, due to the transient character of the motion in this region, this effect was not observed in presented simulations.
For the "disk-shaft" rotor the anisotropy of support radically changes the vibration "behavior" of the rotor system. The rotor on the isotropic support displays only forward whirl motion and exhibits no critical speeds due to the influence of the gyroscopic terms. When the elastic support is anisotropic, the back whirl appears and "brings in" the corresponding critical speed. The parameter κ is always positive in the isotropic case, which means that the auto-balancing device may only increase rotor vibrations, whereas in the anisotropic case vibrations can be eliminated in a certain limited area beyond the critical speed.
The influence of support anisotropy on the vibration performance of the "disk-shaft" rotor system with auto-balancing device is illustrated in Figs.11-14.

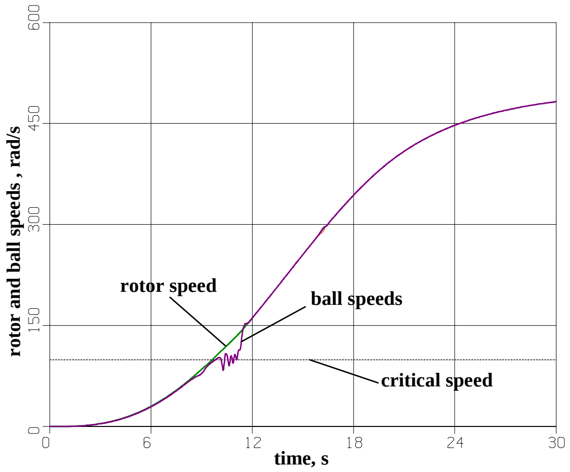

Figure 1. Purely translating vibrations.
Isotropic supports. Rotor and ball speeds.

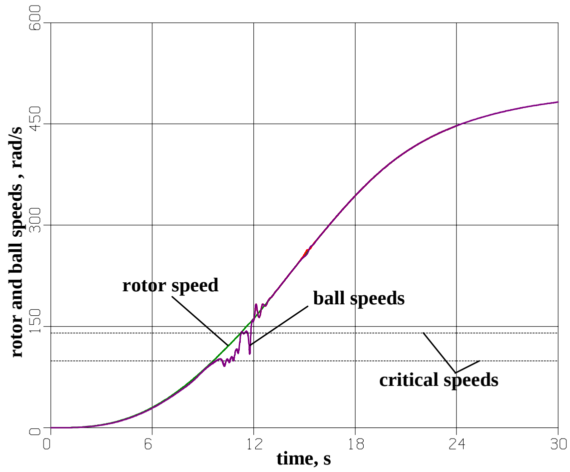

Figure 2. Purely translating vibrations.
Anisotropic supports. Rotor and ball speeds.

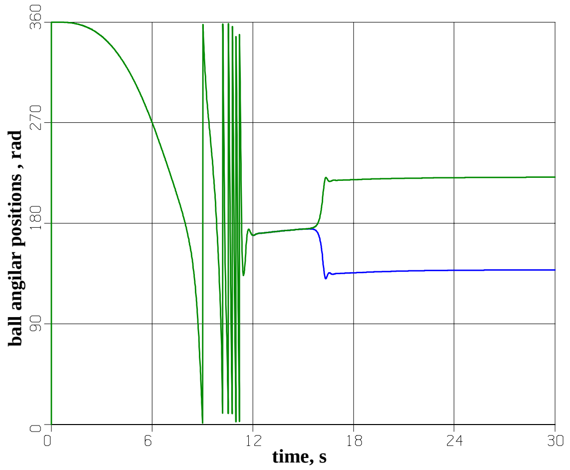

Figure 3. Purely translating vibrations.
Isotropic supports. Ball angular positions.

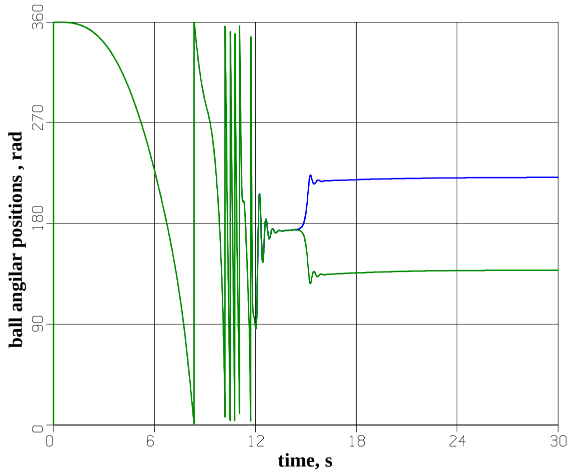

Figure 4. Purely translating vibrations.
Anisotropic supports. Ball angular positions.

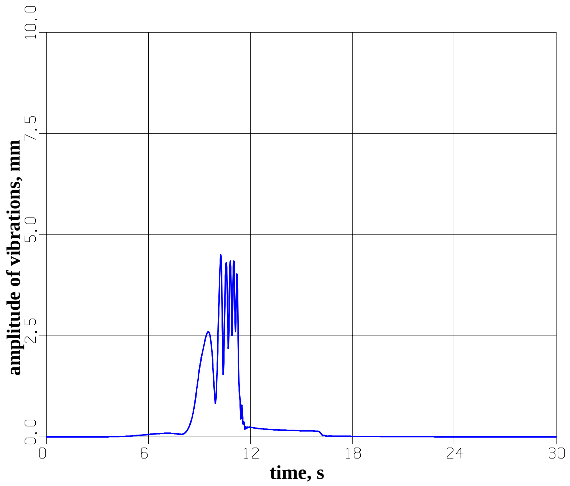

Figure 5. Purely translating vibrations.
Isotropic supports. Amplitude of vibrations.

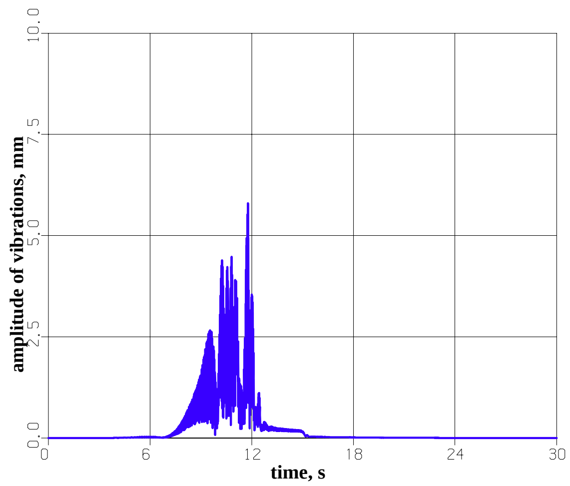

Figure 6. Purely translating vibrations.
Anisotropic supports. Amplitude of vibrations.

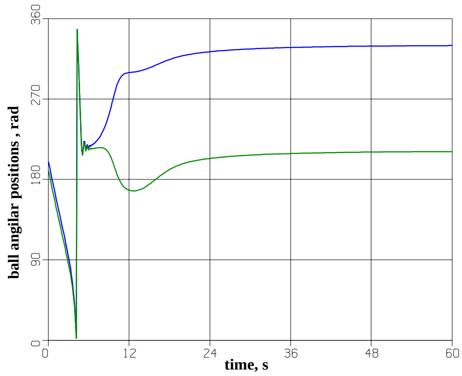

Figure 7. Purely tilting vibrations. Long rotor. Isotropic supports. Ball angular positions.

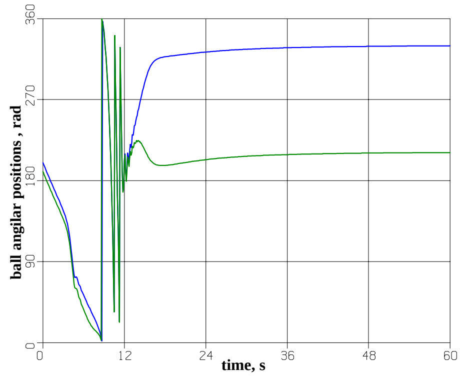

Figure 8. Purely tilting vibrations. Long rotor. Anisotropic supports. Ball angular positions.

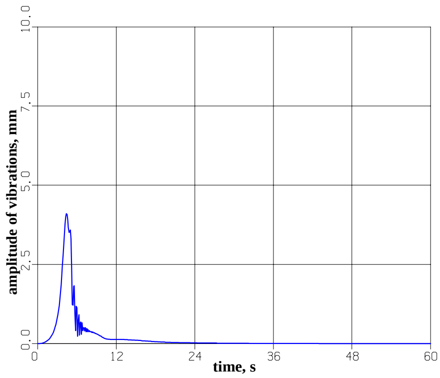

Fig.9. Purely tilting vibrations. Long rotor. Isotropic supports. Amplitude of vibrations.

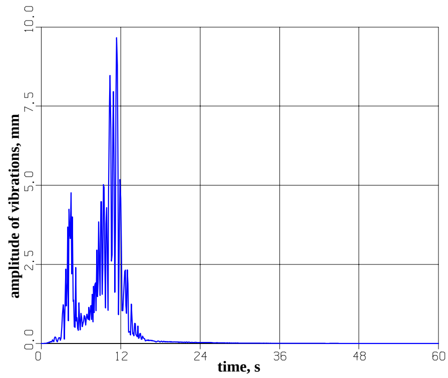

Figure 10. Purely tilting vibrations. Long rotor. Anisotropic supports. Amplitude of vibrations.

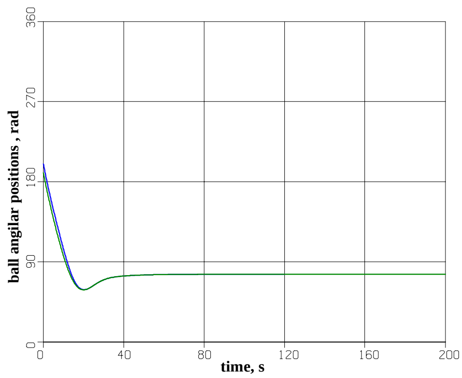

Figure 11. Purely tilting vibrations. Disk-shaft rotor. Isotropic supports. Ball angular positions.

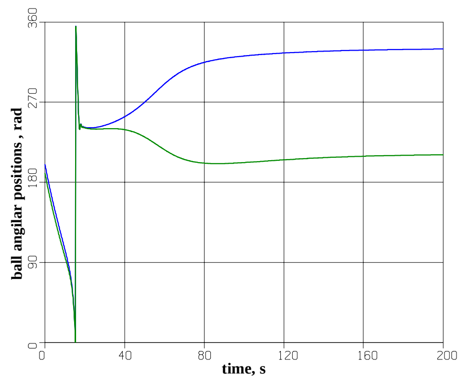

Fig.12. Purely tilting vibrations. Disk-shaft rotor. Anisotropic supports. Ball angular positions.

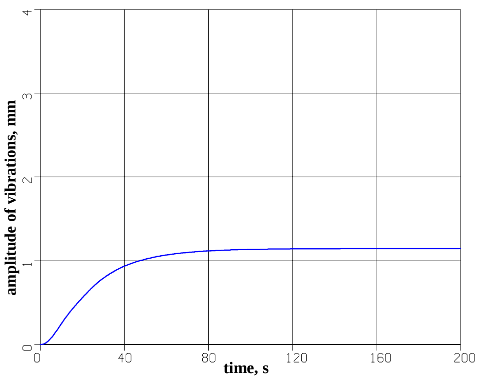

Figure 13. Purely tilting vibrations. Disk-shaft rotor. Isotropic supports. Amplitude of vibrations.

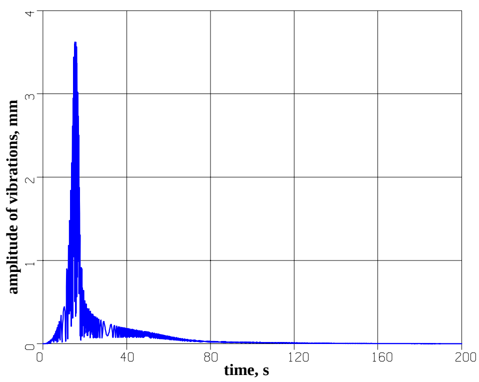

Figure 14. Purely tilting vibrations. Disk-shaft rotor. Anisotropic supports. Amplitude of vibrations.

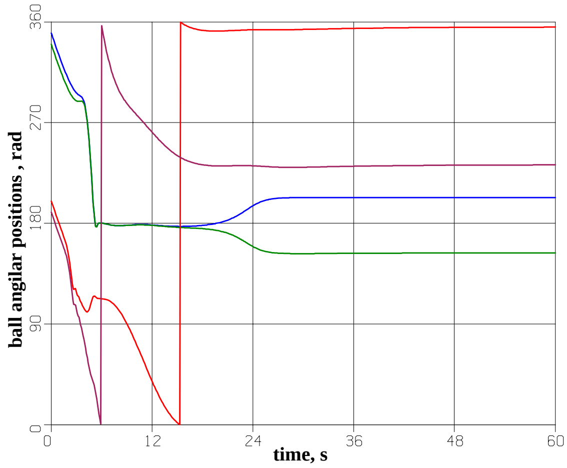

Figure 15. General case. Long rotor. Isotropic supports. Ball angular positions.

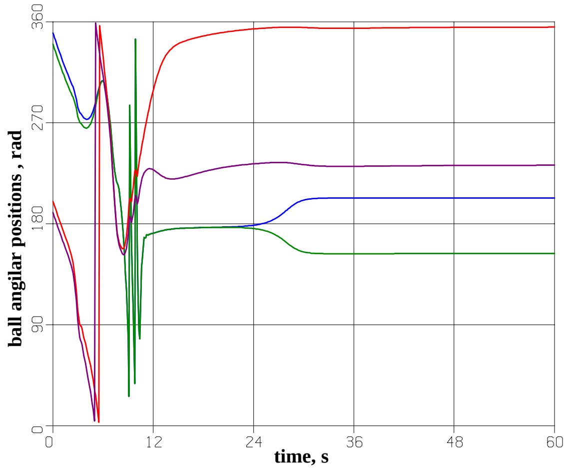

Figure 16. General case. Long rotor. Anisotropic supports. Ball angular positions.

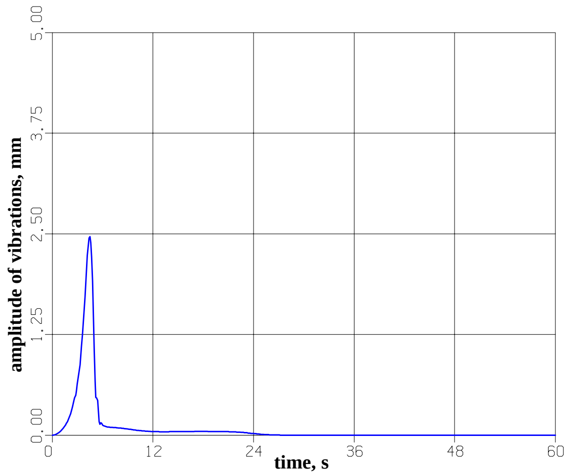

Figure 17. General case. Long rotor. Isotropic supports. Amplitude of vibrations in the plane I.

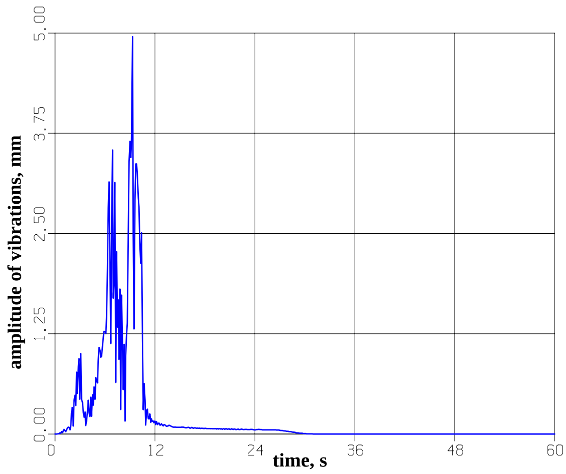

Figure 18. General case. Long rotor. Anisotropic supports. Amplitude of vibrations in the plane I.

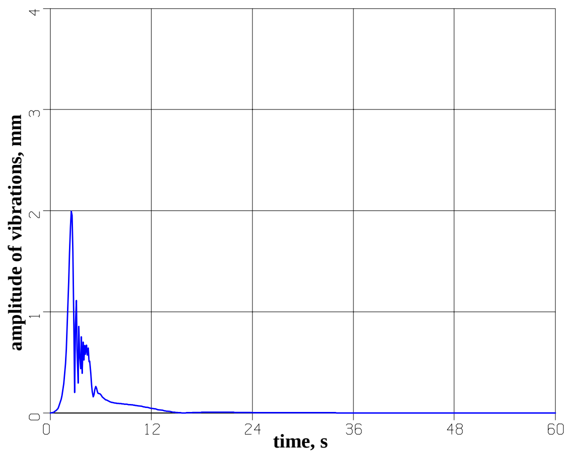

Figure 19. General case. Long rotor. Isotropic supports. Amplitude of vibrations in the plane II.

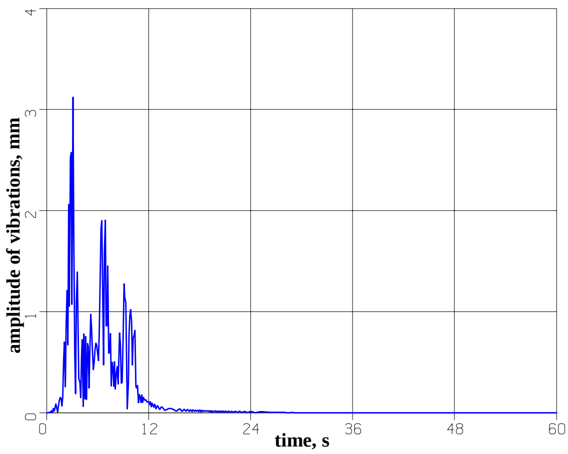

Figure 20. General case. Long rotor. Anisotropic supports. Amplitude of vibrations in the plane II.

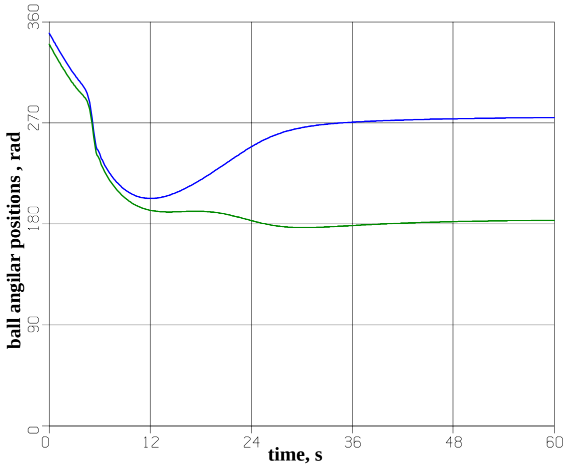

Figure 21. General case. Disk-shaft rotor. Isotropic supports. Ball angular positions.

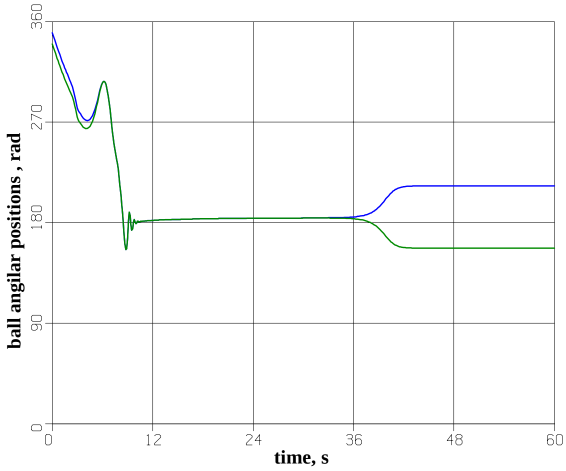

Figure 22. General case. Disk-shaft rotor. Anisotropic supports. Ball angular positions.

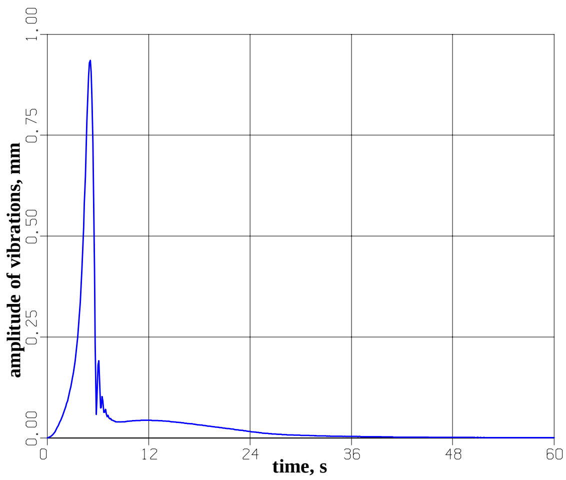

Figure 23. General case. Disk-shaft rotor. Isotropic supports. Amplitude of vibrations in the plane of device.

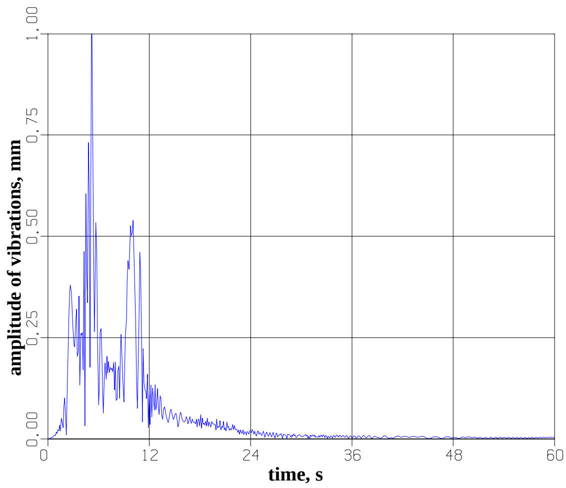

Fig.24. General case. Disk-shaft rotor. Anisotropic supports. Amplitude of vibrations in the plane of device.

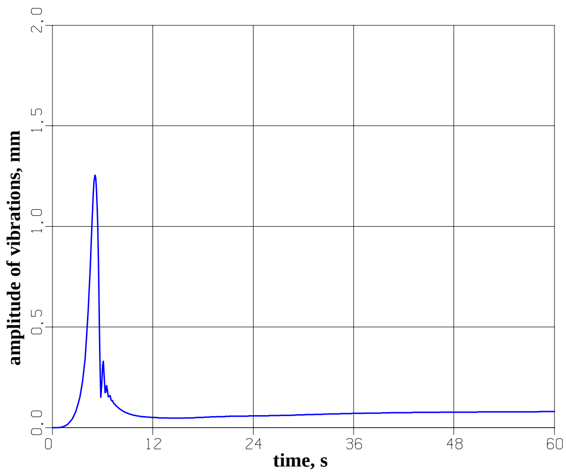

Figure 25. General case. Disk-shaft rotor. Isotropic supports. Amplitude of vibrations of the rotor edge.

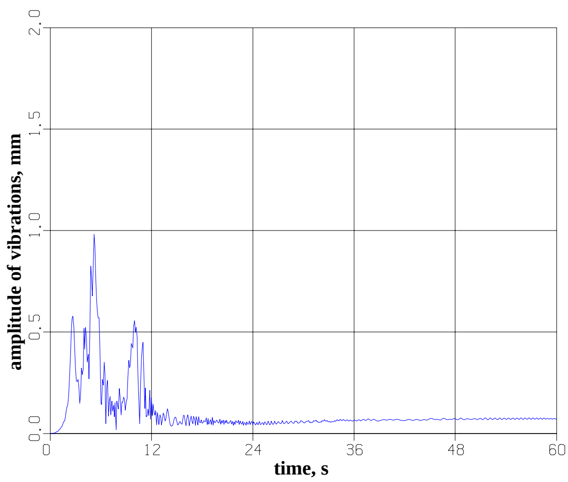

Figure 26. General case. Disk-shaft rotor. Anisotropic supports. Amplitude of vibrations of the rotor edge.

6.3 General Rigid Rotor

Figs.15-26 illustrate the performance of auto-balancing device for statically and dynamically unbalanced rigid rotors on elastic supports. We consider two rotors with the same parameters, as in part 6.2, but this time both supports are elastic. As above, we compare the vibrations in the case of anisotropic and isotropic supporting.
The “long rotor”, which exhibits two critical speeds in the case of isotropic supports (40 rad/s and 70 rad/s) and four critical speeds for anisotropic supporting (40 rad/s, 70 rad/s, 78 rad/s and 135 rad/s) was equipped with a two-plane auto-balancing device. Analyzing the simulation results (Figs.15-20), one can notice that the area of critical speeds in the case of anisotropic supports is wider and vibrations there contain noticeable double-frequency component, but in the post-critical range the vibration performance for both types of supports is similar and advantageous: auto-balancing device provides complete compensation of vibrations.
An analogous picture was observed for the disk-shaft rotor. This rotor exhibits one critical speed, 53 rad/s, when supports are isotropic, and three critical speeds, 34 rad/s, 60 rad/s and 101 rad/s, in the case of anisotropic supports. We select a single-plane auto-balancing device, because for rotors of the disk-shaft type a single-plane device often looks preferable. In the post-critical area such a device provides a partial compensation of vibrations, making vibrations in the plane of the device equal to zero.
The results of simulations are presented in Figs. 21-26. As predicted by the analytical study, in the case of anisotropic supporting the auto-balancing device “compensates” only forward whirl, leaving some residual backward whirl vibrations. Nevertheless, the positive effect from the auto-balancing device is noticeable and similar both in the case of isotropic and anisotropic supports.

7 Conclusions

The presented study demonstrates that an auto-balancing device can efficiently perform both in the case of isotropic and anisotropic supports. The anisotropy of supporting leads to an increase in the number of critical speeds. Therefore, the rotor system may have a wider area of elevated vibrations when passing critical speeds region than in the case of isotropic supports. But in the post-critical ranges, where the corresponding influence coefficients are negative, the auto-balancing balls seek compensation positions providing a decrease of vibrations, not depending on the type of supports.
In case of anisotropic supporting the rotor vibrations include forward and backward whirl. When the auto-balancing balls synchronize with the rotor, they compensate only forward whirl. In certain cases, which are considered in the presented paper, the auto-balancing device eliminates the forward whirl in the whole rotor. In such cases the backward whirl also disappears, so we obtain the complete compensation of vibrations. In other cases the forward whirl becomes equal to zero only in the plane of device (partial compensation of unbalance). That leaves some residual vibrations, both in the plane of device, where backward whirl remains, and in the whole rotor. But, when the device plane is selected correctly, these residual vibrations can be made satisfactorily low.

Acknowledgement
The authors gratefully acknowledge the financial support of the Deutsche Forschungsgemeinschaft within the project *Selbsttätiges Auswuchten starrer Rotoren* (SP 462/7-3).

References

Sperling, L.; Ryzhik, B.; Duckstein, H.: Single-plane auto-balancing of rigid rotors. *Technische Mechanik*, 24, 1 (2004), 1 – 24

Duckstein, H.; Ryzhik, B., Sperling, L.: Self-balancing of an anisotropically supported rigid rotor. Analytical part. *Proceedings of the 3rd Polyakhov Readings*, St. Petersburg (2003a), 111-116

Duckstein, H.; Ryzhik, B.; Sperling, L.: Self-balancing of an anisotropically supported rigid rotor. Simulation Results. *Proceedings of the 3rd Polyakhov Readings*, St. Petersburg (2003b) , 116-120

Address: Dr.-Ing. Boris Ryzhik, Prof. Dr.-Ing. habil. Lutz Sperling and Dr.-Ing. Henner Duckstein, Institut für Mechanik, Otto-von-GuerickeUniversität Magdeburg, Universitätsplatz 2, 39106 Magdeburg.
email: Boris.Ryzhik@mb.uni-magdeburg.de, Lutz.Sperling@mb.uni-magdeburg.de, Henner.Duckstein@mb.uni-magdeburg.de