Free Convection Boundary Layer over a Vertical Cone in a Non-Newtonian Fluid Saturated Porous Medium with Internal Heat Generation

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The steady free convection boundary layer over a vertical cone embedded in a porous medium filled with a non-Newtonian fluid with an exponential decaying internal heat generation is studied in this paper. We adopt for the non-Newtonian fluid the well-known model of a power-law fluid. Two cases are considered, variable wall temperature (VWT) and variable heat flux (VHF), respectively. As for the classical problem without internal heat generation, it has been found that similarity solutions exist for the case studied here, when internal heat generation is present in the porous medium. The similarity equations are solved numerically using a very efficient implicit finite-difference scheme. The obtained results are compared with those from the open literature and it is shown that they are in very good agreement.

1 Introduction

The study of heat transfer in fluid–saturated porous media has gained considerable attention in the last years, because of its numerous applications in geophysics and energy related problems, such as thermal insulation of buildings, enhanced recovery of petroleum resources, geophysical flows, packed bed reactors and sensible heat storage bed, water waste disposal and others. Recent monographs by Ingham and Pop (1998, 2002), Nield and Bejan (1999), Vafai (2000), Pop and Ingham (2001), and Bejan and Kraus (2003) give an excellent summary of the work on the subject.

A large number of physical phenomena involve free convection driven by internal heat generation. The most important applications are in the field of nuclear energy and also to fire and combustion modelling, the development of metal waste form from spent nuclear fuel and for storage of spent nuclear fuel (see Horvat et al., 2001). Studies in natural convection driven by internal heat generation have been done by Roberts (1967), Jahn and Reinke (1974), Hardee and Nilson (1977), Stewart and Dona (1988), etc. A couple of recent papers have been devoted to the subject of similarity solutions for free or mixed convection with internal heat generation in porous media for several geometric configurations, see Bagai (2003), Postelnicu and Pop (1999), Postelnicu et al. (2000, 2001).

On the other hand, many fluids involved in practical applications present a non-Newtonian behaviour. Such practical applications in porous media could be encountered in fields like ceramics production, filtration and oil recovery, certain separation processes, polymer engineering, petroleum production, see for instance Nakayama and Shenoy (1993), and Nakayama and Koyama (1991).

Chen and Chen (1988), Wang and Tu (1989), Mehta and Rao (1994) have studied the problem of free convection in non-Newtonian fluids for vertical surfaces using different methods. Also, Chen and Chen (1987) and Chamkha (1997) have solved the similar problems for horizontal surfaces. Free convection problems in porous media involving other surfaces have also been studied: cone (Yih, 1998), horizontal cylinders and spheres, as well as symmetric bodies (Nakayama and Koyama, 1991).

The objective of the present paper is to present similarity solutions for the problem of steady free convection over a heated vertical cone embedded in a porous medium saturated with a non-Newtonian power law fluid driven by internal heat generation.

2 Basic Equations

Under the Boussinesq and boundary layer approximations, the basic equations can be written as

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \tag{1}$$

$$u^n = \frac{g \cos\gamma \, K^*(n)\beta}{v^*}(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{q_m'''}{\rho C_p} \tag{3}$$

where $r = x\sin\gamma$ is the cone radius, x and y are the Cartesian co-ordinates along and normal to the generator , respectively, as shown in Figure 1, u and v are the velocity components along x and y axes, T is the fluid temperature, g is the acceleration due to gravity, β is the coefficient of thermal expansion, α_m is the effective thermal diffusivity, v^* is the modified kinematic viscosity, q_m''' is the internal heat generation source, C_p is the specific heat at constant pressure, n is the power law index and $K^*(n)$ is the modified permeability, which is given by
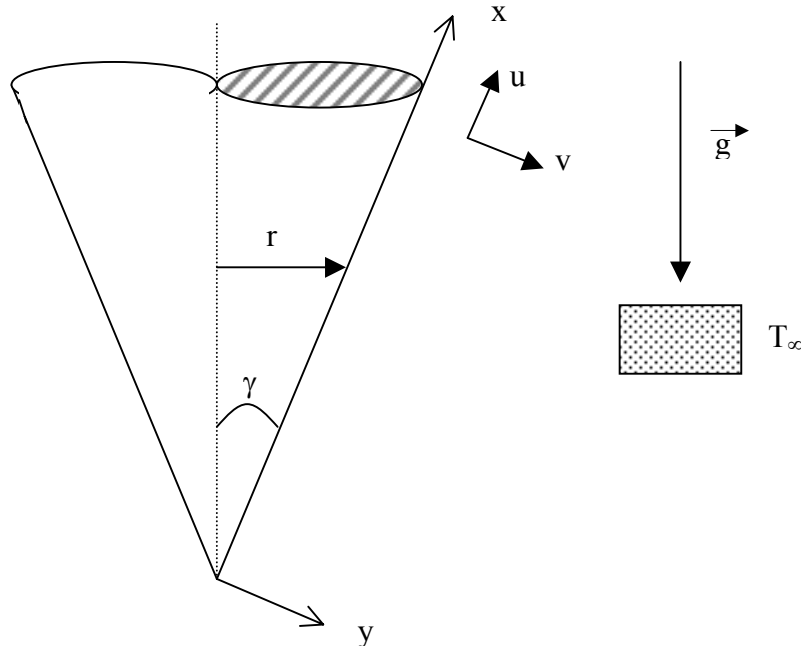

Figure 1. Physical Model and Co-ordinate System

$$K^*(n) = \frac{6}{25}\left(\frac{n\varphi}{3n+1}\right)^n \left(\frac{\varphi d}{3(1-\varphi)}\right)^{n+1} , \text{(Christopher and Middleman, 1965)} \tag{4a}$$

$$K^*(n) = \frac{2}{\varphi}\left(\frac{d\varphi^2}{8(1-\varphi)}\right)^{n+1} \frac{6n+1}{10n-3}\left(\frac{16}{75}\right)^{\frac{3(10n-3)}{10n+11}} , \text{(Darmadhikari and Kale, 1985)} \tag{4b}$$

where d is the particle diameter and φ is the porosity.

The boundary conditions for equations (1)-(3) are

$$y = 0 : \ v = 0 , \ T_w = T_\infty + Ax^\lambda \ \text{(VWT)} , \ \text{or} \ q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = Ax^\lambda \ \text{(VHF)} \tag{5a}$$

$$u \to 0 , \ T \to T_\infty \ \text{as} \ y \to \infty \tag{5b}$$

where VWT means variable wall temperature and VHF variable heat flux, respectively.

(i) Variable wall temperature (VWT)

We introduce the following similarity variables

$$\psi = \alpha_m r \left(Ra_x^*\right)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \left(Ra_x^*\right)^{1/2}(y/x) \tag{6}$$

which are precisely those used by Cheng et al. (1985). Further, ψ is the stream function given by

$$ru = \frac{\partial \psi}{\partial y}, \quad rv = -\frac{\partial \psi}{\partial x} \tag{7}$$

and Ra_x^* is the generalised local Rayleigh number for a porous medium, which is defined as

$$Ra_x^* = \left(\frac{g\beta K^*(n) \cos\gamma \left(T_w - T_\infty\right)x^n}{\nu^* \alpha_m^n}\right)^{\frac{1}{n}} \tag{8}$$

In order that similarity solutions of equations (1)-(3) exist, we assume, following Postelnicu and Pop (1999), and Postelnicu *et al.* (2000, 2001) that the internal heat generation q_m''' is given by

$$q_m''' = \frac{k_m\left(T_w - T_\infty\right)}{x^2} Ra_x^* e^{-\eta} \tag{9}$$

where k_m is the effective thermal conductivity of porous medium.

On using (6) and (7), equations (2) and (3) become

$$(f')^n = \theta \tag{10}$$

$$\theta'' + \frac{1}{2}\left(3 + \frac{\lambda}{n}\right)f\theta' - \lambda f'\theta + e^{-\eta} = 0 \tag{11}$$

and the boundary conditions (5) reduce to

$$f(0) = 0 , \ \theta(0) = 1 \tag{12a}$$

$$\theta \to 0 \ \text{as} \ \eta \to \infty \tag{12b}$$

The main physical quantity of interest in this problem is the local Nusselt number, given by

$$Nu_x \big/ \left(Ra_x^*\right)^{1/2} = -\theta'(0) \tag{13}$$

(ii) Variable heat flux (VHF)

In this case, we use the following variables

$$\psi = \alpha_m r \left(Ra_x^*\right)^{1/3} f(\eta) \ , \ \theta(\eta) = \frac{T - T_\infty}{\left(q_w x / k\right)\left(Ra_x^*\right)^{-1/3}} \ , \ \eta = \left(Ra_x^*\right)^{1/3} \frac{y}{x} \tag{14}$$

see also Yih (1998), where the generalized modified Rayleigh number is now defined as

$$Ra_x^* = \left(\frac{x}{\alpha_m}\right)^{\frac{3n}{2n+1}} \left(\frac{g\beta K^*(n) q_w x \cos\gamma}{\nu^* k}\right)^{\frac{3}{2n+1}} \tag{15}$$

The internal heat generation term q_m''' is taken now as

$$q_m''' = \frac{\alpha_m q_w (T_w - T_\infty)}{kx} \left(Ra_x^*\right)^{-1/3} e^{-\eta} \tag{16}$$

With the transformation (14), the governing equations (2) and (3) become

$$(f')^n = \theta \tag{17}$$

$$\theta'' + \frac{\lambda + 3n + 2}{2n+1} f\theta' - \frac{(2\lambda + 1)n}{2n+1} f'\theta + e^{-\eta} = 0 \tag{18}$$

subject to the boundary conditions

$$f(0) = 0 \ , \ \theta'(0) = -1 \tag{19a}$$

$$\theta \to 0 \quad \text{as } \eta \to \infty \tag{19b}$$

In this case the following relation gives the local Nusselt number

$$Nu_x \Big/ \left(Ra_x^*\right)^{1/3} = \frac{1}{\theta(0)} \tag{20}$$

We point out to this end that equations (10)–(12) reduce to (11)–(14) found by Cheng *et al.* (1985) for $n = 1$ (Newtonian fluids) and the internal heat generation term set to zero. Another comparison may be done, when the internal heat generation is absent, see the above equations (10)–(12) and (17)–(19), and equations (8)–(10) and (14)–(17) in the paper by Yih (1998).

3 Results and Discussions

Equations (10) – (12) and (17) – (19) have been solved numerically for some values of the parameter *n* using a version of the shooting method as proposed by Chakraborty (1998). Table 1 gives some values of $[-\theta'(n,0)]$, with and without heat generation, in the case of VWT when $\lambda = 0$, 1/3 and 1/2. This table contains also values of $[-\theta'(n,0)]$, previously obtained by Cheng et al. (1985) and Yih (1998), when the internal heat generation is absent. It is clearly seen that our results are in excellent agreement with those known from the open literature. It is also seen that the presence of the internal heat generation within the porous medium leads to the decrease of the local heat transfer.

n	$\lambda = 0$				$\lambda = 1/3$			$\lambda = 1/2$		
	WIHG			IHG	WIHG		IHG	WIHG		IHG
	Cheng et al. (1985)	Yih (1998)	present results	present results	Cheng et al. (1985)	present results	present results	Cheng et al. (1985)	present results	present results
0.5	-	0.6522	0.6527	0.0939	-	0.8172	0.3009	-	0.8828	0.3881
0.8	-	0.7339	0.7340	0.1646	-	0.8884	0.3642	-	0.9574	0.4513
1.0	0.7685	0.7686	0.7686	0.1962	09210	0.9211	0.3936	0.9896	0.9897	0.4799
1.5	-	0.8233	0.8233	0.2477	-	0.9729	0.4413	-	1.0409	0.5271
2.0	-	0.8552	0.8552	0.2787	-	1.0033	0.4702	-	1.0710	0.5558
2.5	-	-	0.8762	0.2994	-	1.0233	0.4896	-	1.0908	0.5749

Table 1.Values of $[-\theta\,'(n, 0)]$ in the case of VWT: without internal heat generation (WIHG) and with internal heat generation (IHG).

n	$\lambda = 0$				$\lambda = 1/3$			$\lambda = 1/2$		
	WIHG			IHG	WIHG		IHG	WIHG		IHG
	Cheng et al. (1985)	Yih (1998)	present results	present results	Cheng et al. (1985)	present results	present results	Cheng et al. (1985)	present results	present results
0.5	-	1.1358	1.1357	1.3869	-	1.0857	1.3194	-	1.0644	1.2907
0.8	-	1.0839	1.0838	1.3930	-	1.0238	1.3063	-	0.9984	1.2698
1.0	1.0562	1.0564	1.0563	1.3890	0.9922	0.9923	1.2937	0.9653	0.9654	1.2539
1.5	-	0.9871	1.0074	1.3742	-	0.9377	1.2654	-	0.9086	1.2203
2.0	-	0.9760	0.9760	1.3608	-	0.9035	1.2440	-	0.8733	1.1959
2.5	-	-	0.9545	1.3500	-	0.8803	1.2282	-	0.8495	1.1781

Table 2. Values of $\theta(n, 0)$ in the case of VHF: without internal heat generation (WIHG) and with internal heat generation(IHG).

In Table 2 there are presented the values of $\theta(n, 0)$, with and without heat generation for the case of VHF when $\lambda = 0$, 1/2 and 1/3. Again, our results compare favourably with those reported by Cheng et al. (1985) and Yih (1998), in the case without internal heat generation. Further, Figures 2, 3 and 4 show the non-dimensional temperature profiles for some values of the power law parameter n, in the case of VWT for $\lambda = 0$, 1/3 and 1/2. We can see that the boundary layer thickness increases when the effect of internal heat generation is considered. The boundary layer thickness also increases with the increase of the power law index n. It happens in both cases when the effect of internal heat generation is present or absent.

The variation of the local Nusselt number with the power law index n in the VWT case is presented in Figure 5. It is noticed that in the both cases with and without internal heat generation, the heat transfer is more significant in the case of dilatant fluids ($n > 1$) than in the cases of Newtonian ($n = 1$) and pseudoplastic fluids ($n < 1$). On the other hand, one can easily remark that the heat transfer with internal heat generation is less intensive than when the internal heat generation is absent.

a)

$n = 0.5, 0.8, 1.0, 1.5, 2.0, 2.5$

$\lambda = 0$

b)

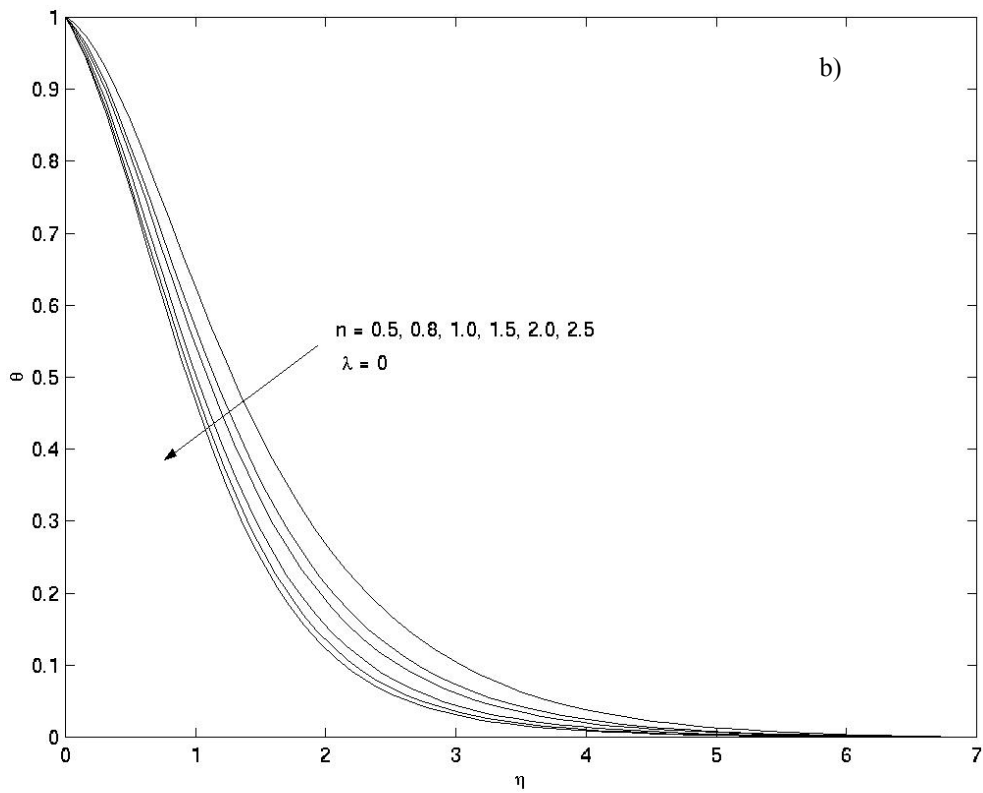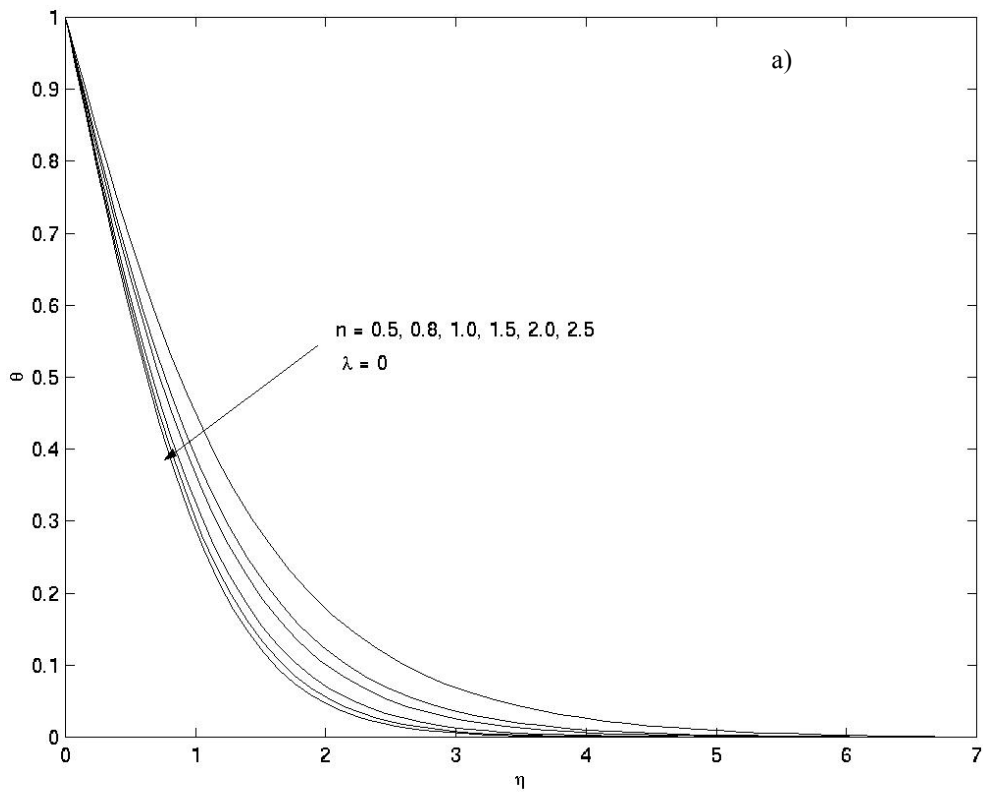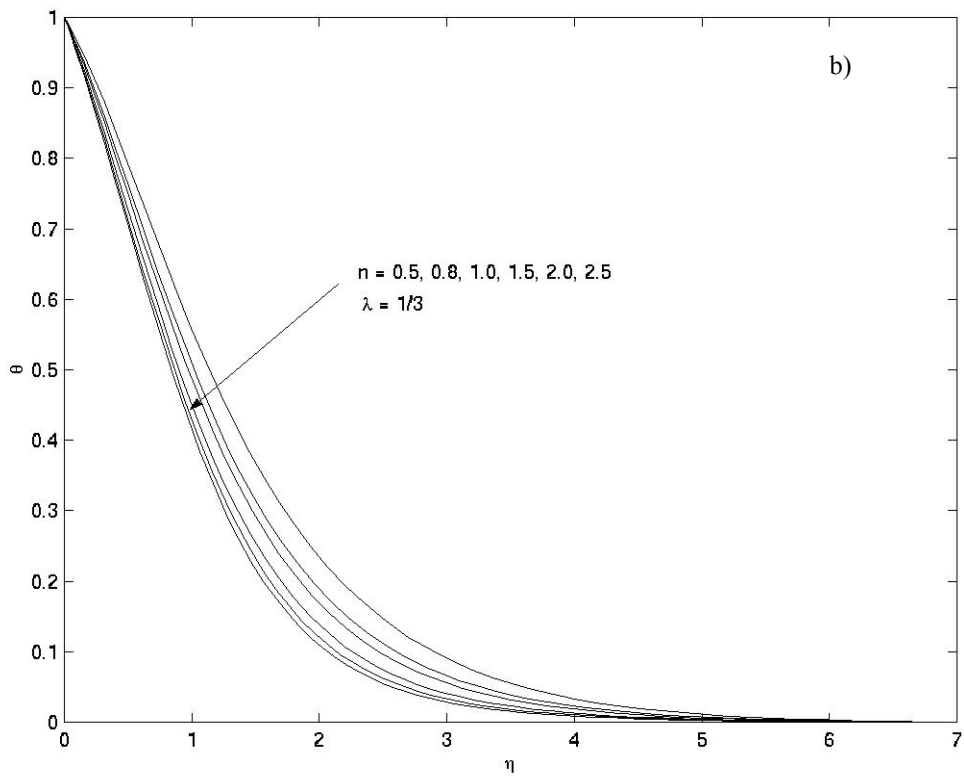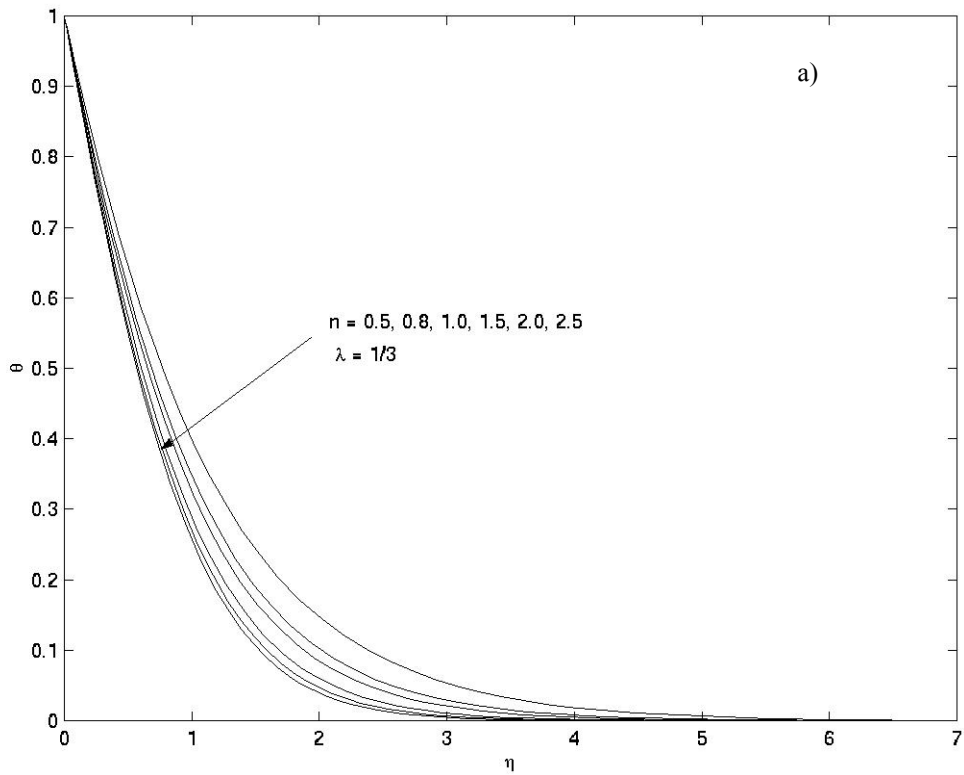$n = 0.5, 0.8, 1.0, 1.5, 2.0, 2.5$

$\lambda = 0$

Figure 2. Temperature profiles for VWT case with $\lambda = 0$:
a) without internal heat generation; b) with internal heat generation

Figure 3. Temperature profiles for VWT case with λ = 1/3:
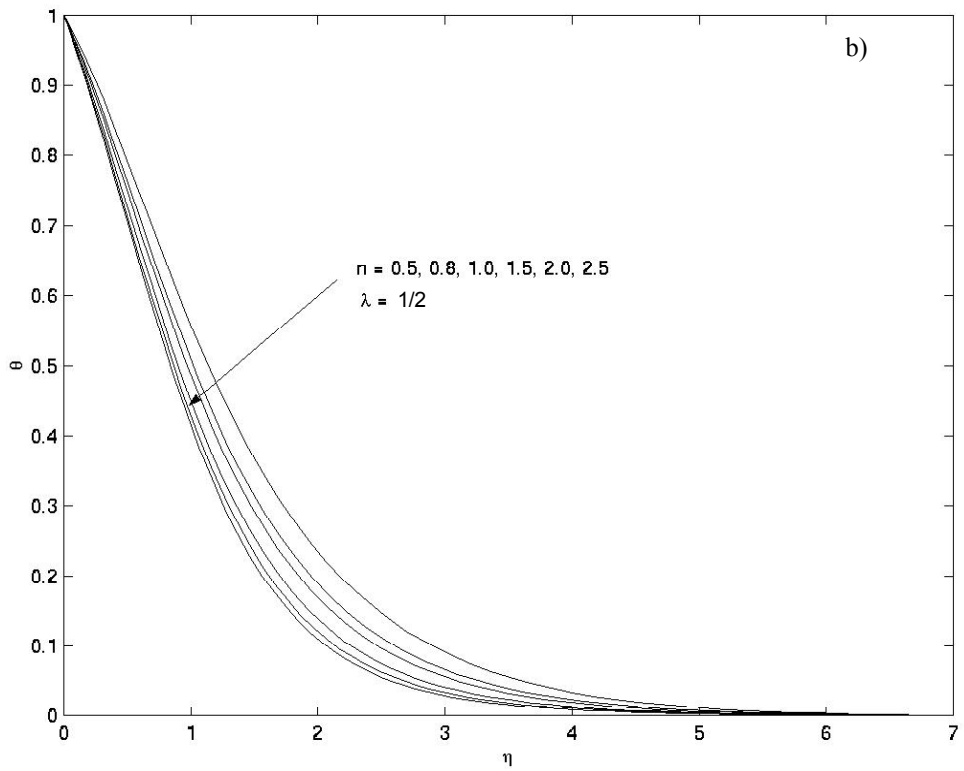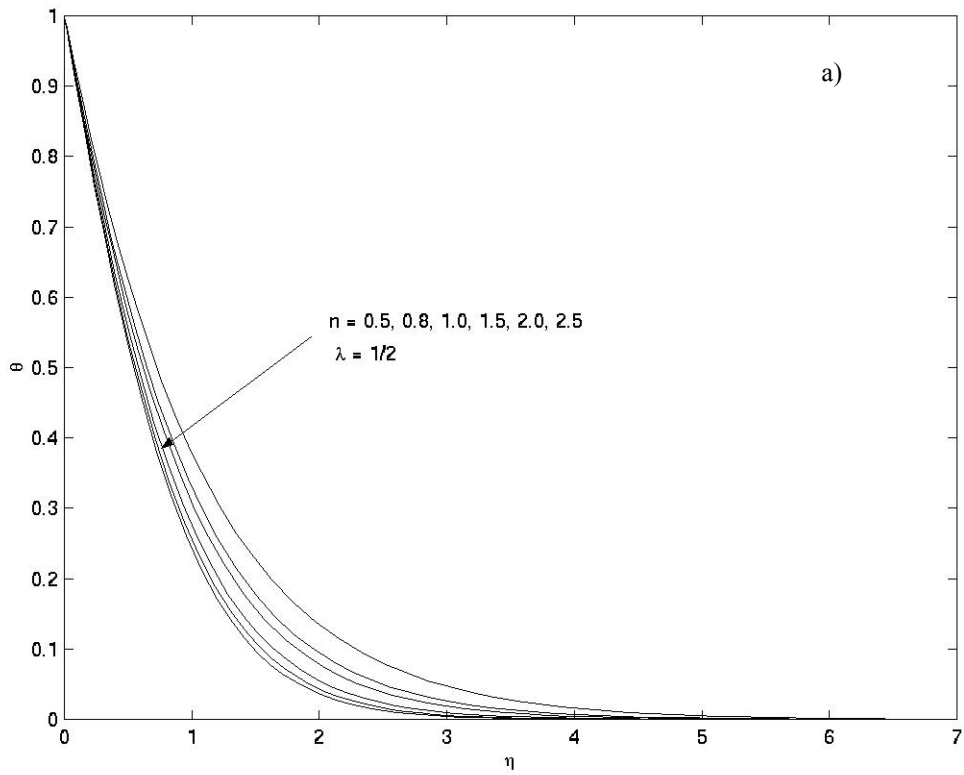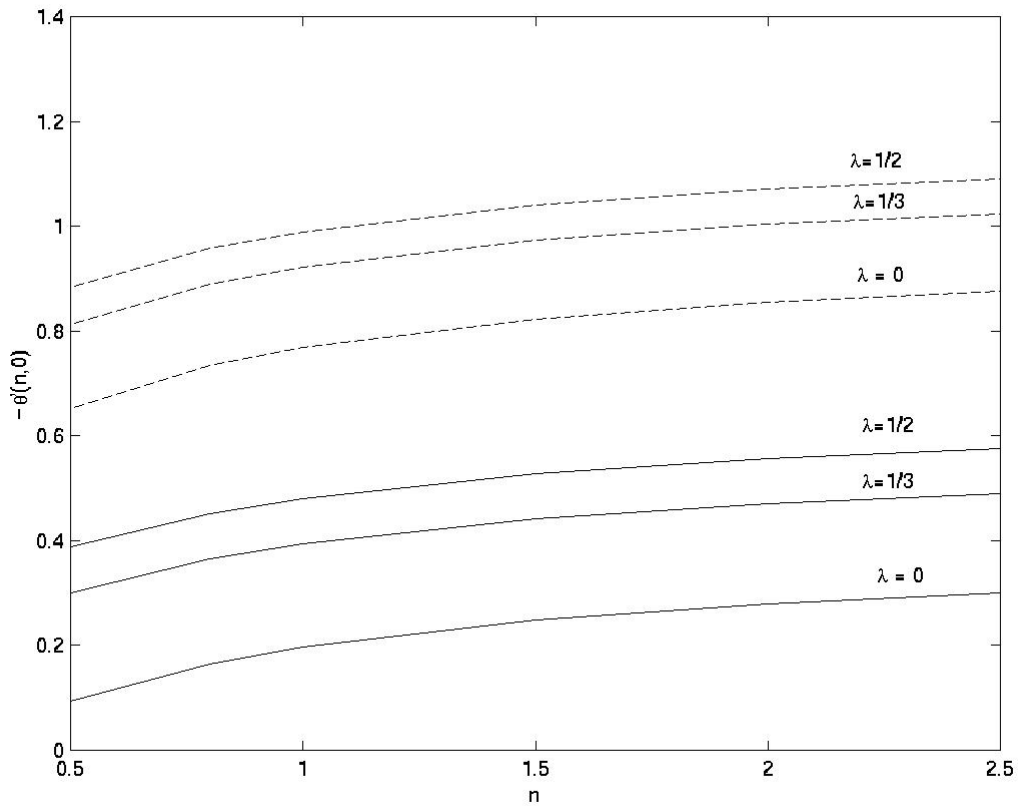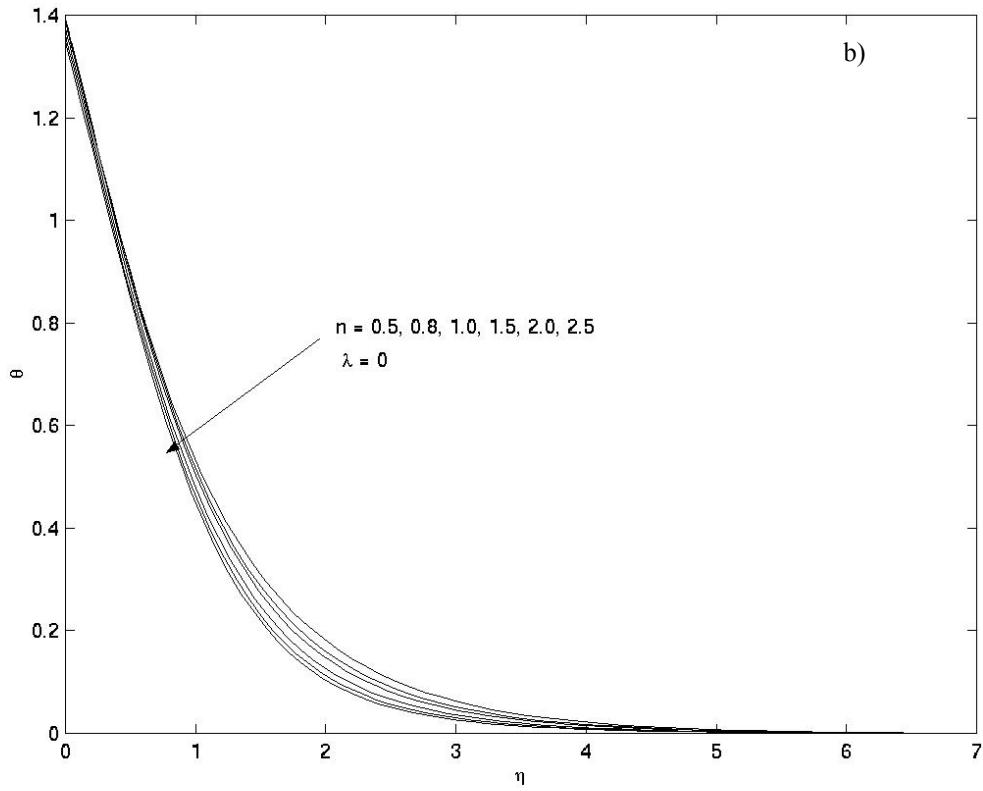a) without internal heat generation; b) with internal heat generation

Figure 4. Temperature profiles for VWT case with $\lambda = 1/2$:
a) without internal heat generation; b) with internal heat generation

Figure 5. Variation of the Nusselt number with the power law index *n* in the VWT case:
... without internal heat generation; ____ with internal heat generation

In Figures 6, 7 and 8 there are depicted the non-dimensional temperature profiles for some values of the power law parameter *n*, in the case of VHF with $\lambda = 0$, 1/3 and 1/2. The variation of the local Nusselt number with the power law index *n* in the case of VHF is presented in Figure 9. The same comments as in the case of VWT are again available.
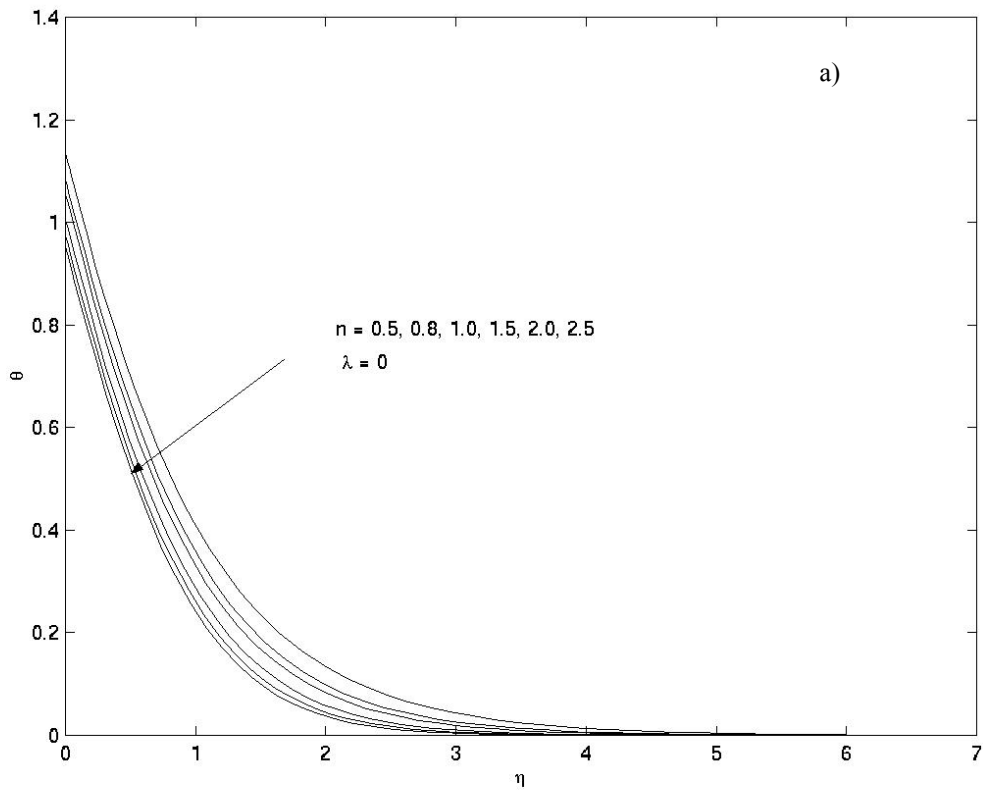
Figure 6. Temperature profiles for VHF case with $\lambda = 0$:
a) without internal heat generation; b) with internal heat generation
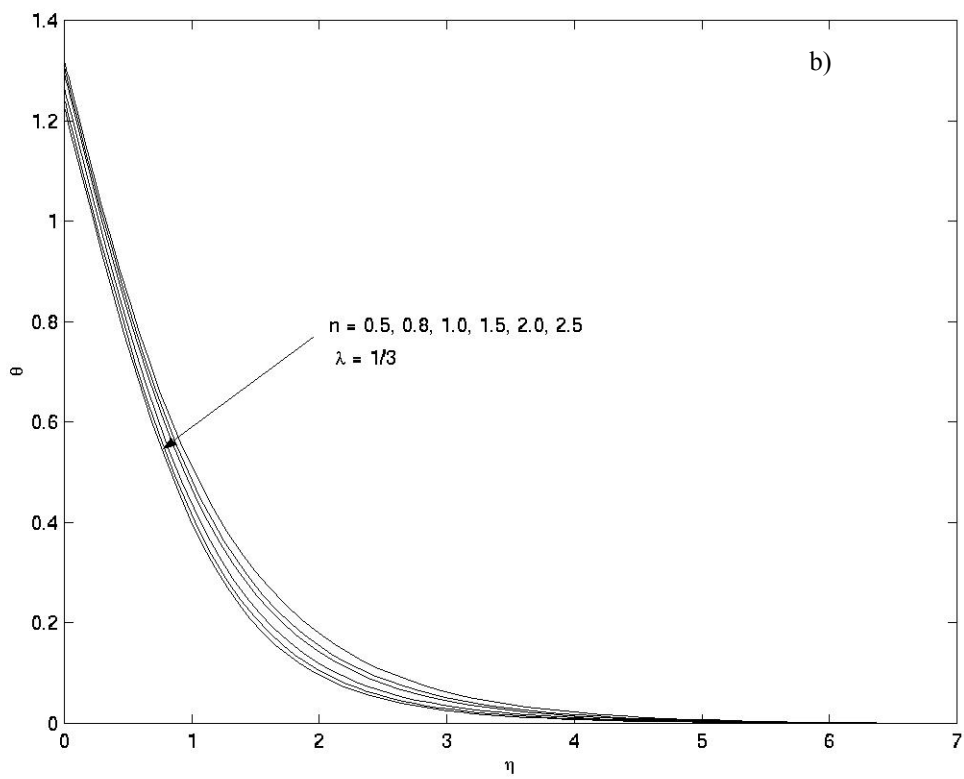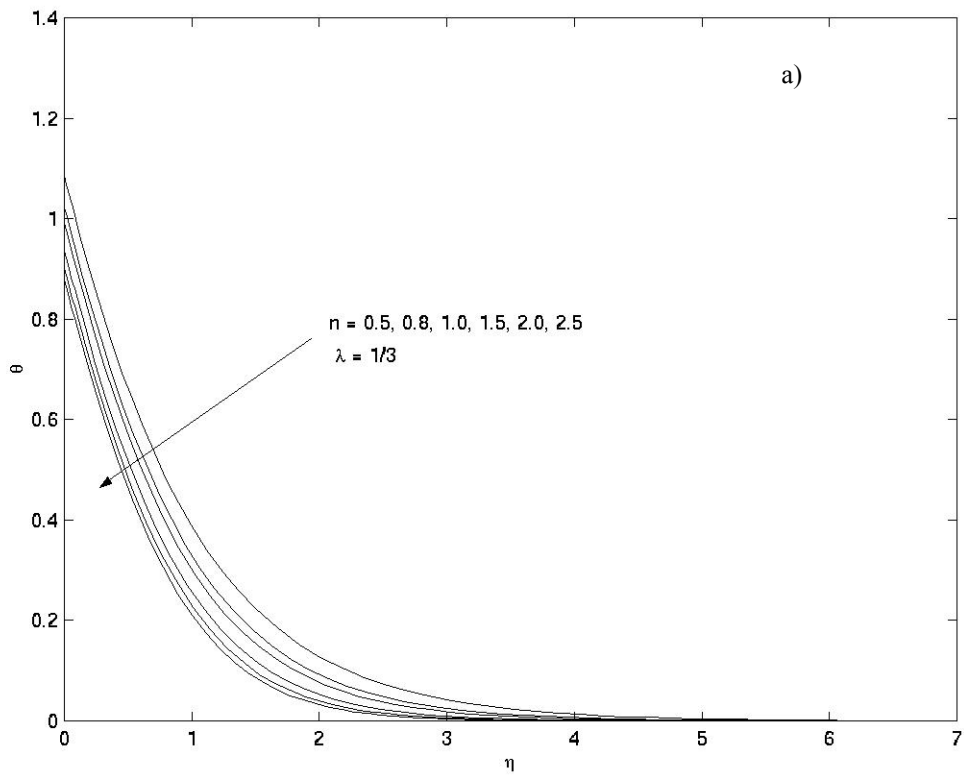
Figure 7. Temperature profiles for VHF case with $\lambda = 1/3$:
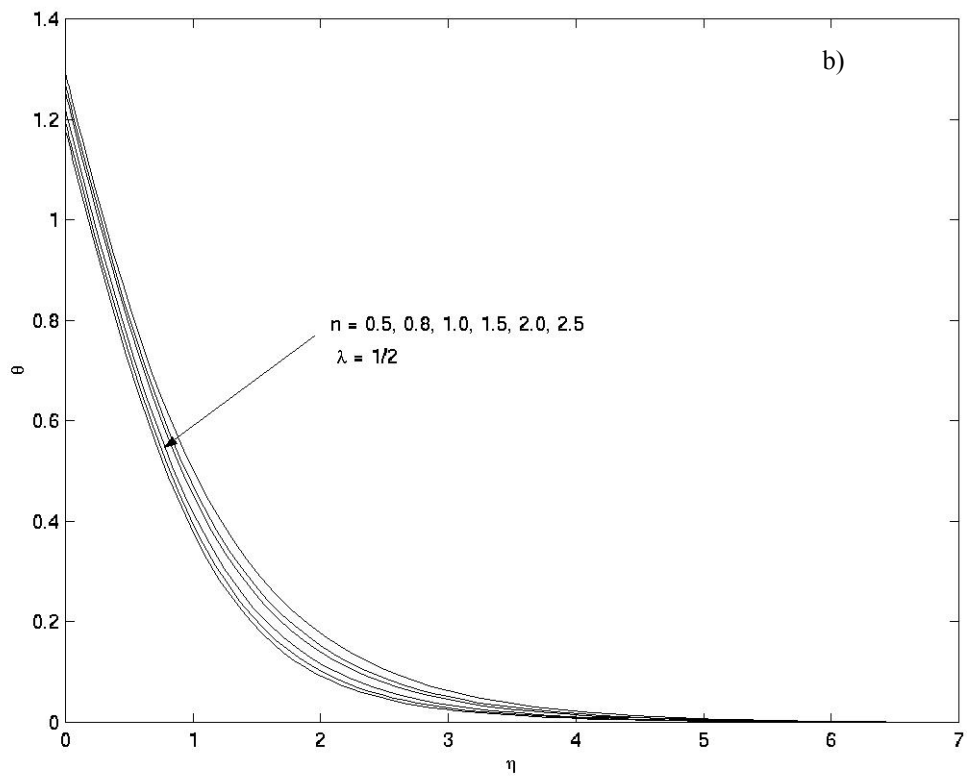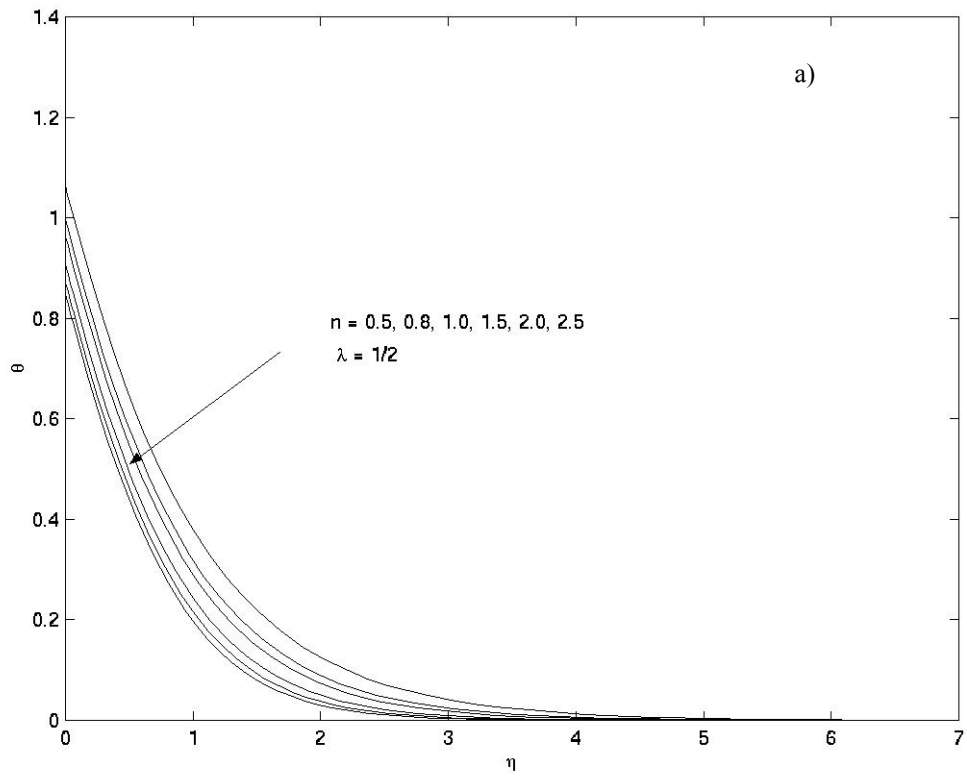a) without internal heat generation; b) with internal heat generation

Figure 8. Temperature profiles for VHF case with $\lambda = 1/2$:
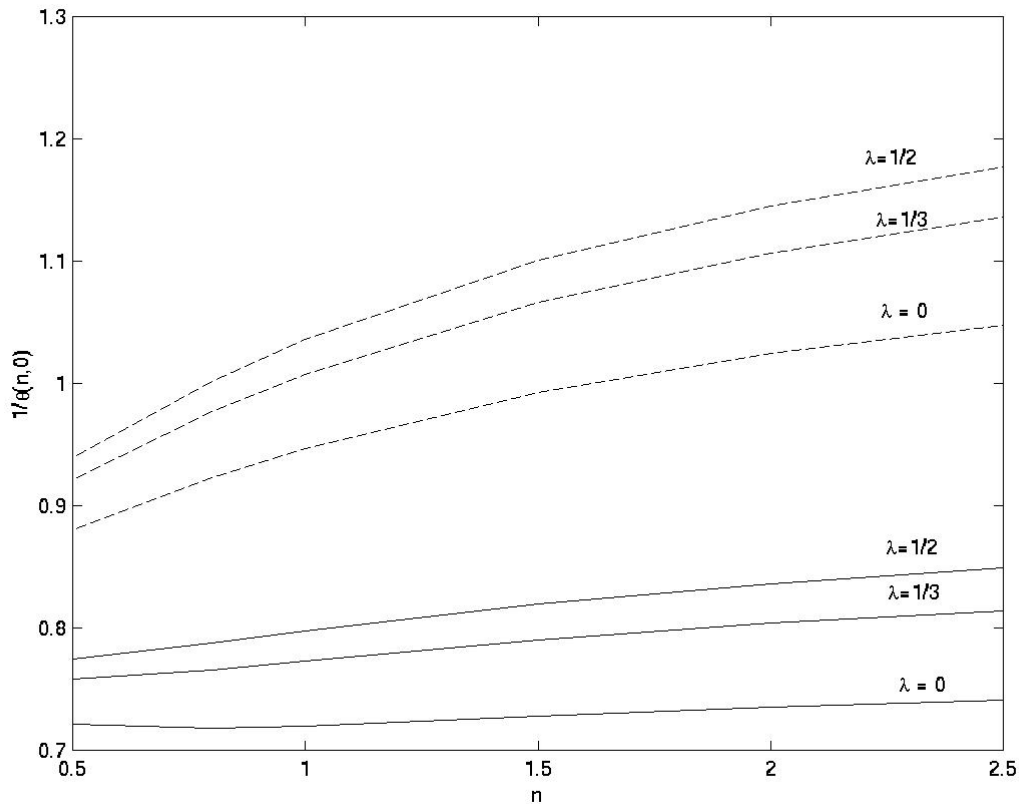a) without internal heat generation; b) with internal heat generation

Figure 9. Variation of the Nusselt number with the power law index *n* in the VHF case: ... without internal heat generation; ____ with internal heat generation

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