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On the Mathematical Simulation of the Measuring of the Intraocular Pressure by Maklakov Method

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Maklakov's method for measurement of the intraocular pressure (IOP) is based on approach, in which an eyeball is modeled as a thin-walled spherical liquid-filled shell. Measuring the IOP one estimates the diameter of the circular contact area of the cornea and the tonometer. In the clinic special tables are used to estimate the IOP relating to the measured diameter. However nowadays the calculating of such tables is based on the empirical values of the IOP. In the present paper the mathematical simulation of the measuring of the intraocular pressure by Maklakov method is considered.

1 Maklakov's Method and Physical Background

To measure the intraocular pressure (IOP) A.N. Maklakov (Maklakov, 1984) proposed to apply the load P (usually $P = 5$ or 10 g) to the cornea, that eventually deforms under this load. The proposed method is realized in Maklakov tonometers, where a flat-bottomed load is applied to the cornea and then the diameter, d, of a circular contact area of cornea and tonometer is measured. Each tonometer is supplied with special tables which are lately used to estimate the IOP corresponding to the measured value of the diameter,d. The tables to determine the IOP are calculated in analysis of the changes in the internal pressure in the thin-walled shell under the loading. It is assumed that the elastic properties of the shell are homogeneous and one-parametric, i.e. they depend on the only constant that is the same for all patients. The value of this constant is determined in analysis of the values of the IOP calculated, estimates or measured by different methods.

In 1920-1930, when methods of the measurement of the IOP were developed, the analysis of stress-strain state in shells was a difficult problem. It explains why for estimation of the change in the IOP inside a loaded shell it was assumed that this value was equal to the change of the IOP in the unloaded shell with the additionally injected volume of liquid ΔV. Here ΔV is the volume of the spherical segment with a base equals to the contact area of the cornea and the tonometer. Presumably, this main assumption based on incompressibility of the vitreous body was firstly formulated by P. Romer. This assumption is used until nowadays to calculate the tables for calculation of the IOP and to develop different methods for the measuring of the IOP (Nestorov et al., 1974). With this assumption the pressure in the shell before loading (actual intraocular pressure) p_0 is given by relation (Bauer et al., 2000)

$$p_0 = p_t - \frac{Eh}{2\pi R^4(1-\nu)}\Delta V = p_t - K\Delta V, \quad p_t = \frac{4P}{\pi d^2}, \quad K \equiv \frac{Eh}{2\pi R^4(1-\nu)} \qquad (1)$$

where p_t is the tonometric pressure, P is the load, d is the diameter of the contact area, R is the radius of the shell, h is the shell thickness, E is Young's modulus, ν is Poisson's ratio and ΔV is the volume of liquid injected into the shell called in ophthalmology "the volume of corneal indentation". It follows from (1), that, if the main assumption is realistic, then with two tonometric tests with different weights one could find the IOP inside the unloaded shell p_0 and the constant K for the elasticity relations for the shell material. From clinical point of view the simulation of the eye shell with the elastic shell helps to estimate the individual IOP and the elastic properties of sclera. Such method of the measurement of the IOP was proposed by P. Romer as an empirical method, but it was not later elaborated apparently since the obtained values of the IOP agreed not so well with other empirical data. Anyway it is hard to say what was the real reason, but the development of methods of the measurement of the IOP went another way. In 1937 Friedenvald proposed the ratio that differed from (1) ratio, for which the pressure in the unloaded shell p_0, the tonometric pressure p_t and the volume of injected in the shell liquid ΔV are related as:

$$\ln p_0 = \ln p_t + A\Delta V, \quad A = const. \qquad (2)$$

The term A in equation (2) is called the ocular rigidity. Ocular rigidity is sometimes called scleral rigidity because when an internal pressure increase stretches the eye, it is largely the sclera that is stretched. Relation (2) agrees with (1). If the changes in the volume ΔV and the pressure $\Delta p = p_t - p_0$ are relatively small with respect to the initial values p_0 and $V = \frac{3}{4}\pi R^3$ then we denote

$$A = \frac{K}{p} \simeq \frac{K}{p_0}. \tag{3}$$

It follows from (3) that in the general case the ocular rigidity A and therefore the IOP obtained due to (2) in two tonometric tests with different weights may depend on both the initial IOP and the weight. It is important that both methods for the estimation of IOP based on either (1) or (2) use same model of an eye and the main assumption on ΔV, but for this model (1) is an exact relation and relation (2) is approximate. For real conditions $\Delta V/V \sim 10^{-3}$, $\Delta p/p_0 \sim 10^{-1}$ and therefore, the difference in the values of IOP and constant K is about 10 $\%$.

Nevertheless the tonometry developed in the way characterized by relation (2). However in both approaches the difficulties are similar: the calculated values of IOP do not agree well enough with those obtained in the other tests. That is why there were attempts to refine relation (2). For example, in calculation of ΔV it was assumed in Nestorov and Vurgaft (1972), that the contact area included the narrow layer of a tear. Therefore the current tables for calculation of the IOP though based on the eye shell model described above are in fact the collections of empirical data based on the hundred years experience of the eye study. Nevertheless the model itself and the main assumption on the value ΔV are still the ground of such tables. It should be noted that in making the tables the geometrical and elastic properties of the model are supposed to be fixed. However, the geometrical parameters of an eye are approximately constant, but the elastic properties essentially vary for different people and change with age, different pathologies of vision or after surgery. For example, in Iomdina (2000), it was reported that Young's modulus for sclera may vary in the range $E \sim 1 - 45$ MPa. Therefore the IOP obtained due to the tables may differ for a patient from the real IOP. It is interesting to estimate the possible error in the measurement of the IOP. Below the solution of mechanical model problem is discussed. The eye is modeled as a thin-walled shell and the effect of the elastic properties of shells on the diameter of the contact area is considered.

2 The Equation of the Axisymmetric Deformation

The eye is modeled with two spherical segments (see Fig. 1).

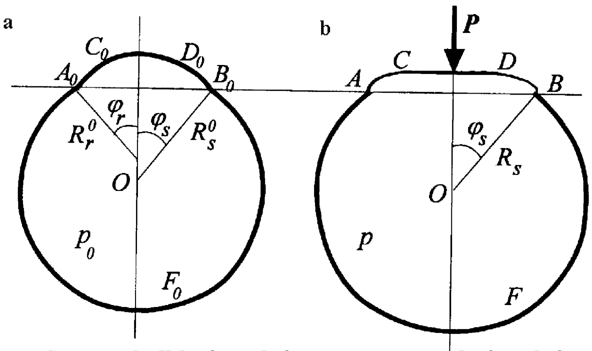

Figure 1: The eye shell before deformation (a) and after deformation (b)

These two segments shell is filled with almost uncompressible liquid under pressure p_0. The spherical segment of radius R models sclera (segment $A_0F_0B_0$ in Fig. 1a), and all variables associated with this part are denoted with the index s. The small segment (segment $A_0C_0D_0B_0$ in Fig. 1a) is a model of the cornea, and the variables corresponding to this part of the shell are marked with r. It is assumed that under the load the sclera deforms in such a way that the central angle for the contact line, φ_s is constant

$$A_0B_0 = 2R_s^0 \sin\varphi_s = 2R_r^0 \sin\varphi_r. \tag{4}$$

and the spherical segment modeling sclera remains a spherical segment with the new radius, R_s. It is also supposed that Young's modulus for cornea is significantly less than Young's modulus for sclera ($E_s >> E_r$) and the cornea is modeled as a soft shell. Later means that it does not resist the bending deformations. Under these assumptions the deformations in cornea are large and the cornea takes the form represented in Fig. 1b.

The pressure p is connected with load P by relation

$$P = pS, \qquad S = \pi r_d^2, \quad r_d = \frac{CD}{2},$$

where S is the contact area of the cornea and the tonometer.

Let s_0 is the length of generatrix, r_0 is the distance between a point on the middle surface and the axis of rotation before deformation and $0 \le s_0 \le s_0^b = R_r^0 \varphi_r$. Let point s_0^d corresponds to the point D, the boundary of the contact area.

The equations of equilibrium of the shell and the geometrical relations on the part DB, i.e. for $s_d \le s_0 \le s_b$ have a form (Chernykh et al., 2002)

$$\begin{array}{lll} (r_0 T_1)' - T_2 \cos\varphi = 0, & \varphi_0 = \frac{s_0}{R_r^0}, & r_0 = R_r^0 \sin\varphi_0, \\ -T_2 \sin\varphi - r_0 \varphi' T_1 + l_1 r p = 0, & \varepsilon_2 = \frac{r}{r_0} - 1, & T_2 = \nu_r T_1 + E_r h_r \varepsilon_2 + T_0(1-\nu_r), \\ r' = l_1 \cos\varphi, & l_1 = 1 + \varepsilon_1, & \varepsilon_1 = \frac{(T_1 - T_0)(1-\nu_r^2)}{E_r h_r} - \nu_r \varepsilon_2, \\ & {}' = \dfrac{d}{ds_0} & \end{array} \tag{5}$$

where h_r is the cornea thickness, T_1 and T_2 are the tangential stress-resultants after the loading, T_0 is the stress-resultant in the initial state, φ is the angle between the normal to the shell and the axis of the rotation after deformation, ε_1, ε_2 are additional deformations of the shell, related to the stress-resultants as

$$T_1 - T_0 = \frac{E_r h_r (\varepsilon_1 + \nu_r \varepsilon_2)}{1 - \nu_r^2}, \qquad T_2 - T_0 = \frac{E_r h_r (\varepsilon_2 + \nu_r \varepsilon_1)}{1 - \nu_r^2}, \quad T_0 = \frac{p_0 R_r^0}{2}.$$

The main unknown values in system (5) are functions T_1, r, φ. The rest values could be express through the main ones due to (5).

On the section under tonometer ($0 \le s_0 \le s_d$) $\varphi = 0$ holds and the system (5) has the following form

$$(r_0 T_1)' - T_2 = 0, \qquad r' = l_1. \tag{6}$$

For arbitrary radius of the contact area r_d we should seek a solution of systems (5) and (6), which satisfies the following conditions: (i) the solution of systems (5) and (6) should be limited at the point $s_0 = 0$, (ii) functions T_1, r and φ should be smooth at the point D, i.e.

$$r(s_d) = r_d,$$

and at the point B

$$r(s_b) = r_b, \qquad r_b = R_s^0 \sin\varphi_s \left(1 + \frac{(1-\nu_s) R_s^0 (p - p_0)}{2 E_s h_s}\right), \tag{7}$$

where r_b is obtained from the condition of the sclera deformation and h_s is the thickness of sclera.

To evaluate the value r_b first we should estimate the decreasing of the volume ΔV under the section $ACDB$ after the loading by tonometer

$$\Delta V = \pi \int_0^{s_b} \left(r_0^2 \sin\varphi_0 - r^2 l_1 \sin\varphi\right) ds_0.$$

This decreasing of the volume should be equilibrated by extension of sclera and compressions of the vitreous body under the increasing the pressure

$$\Delta V_1 = \Lambda(p - p_0), \qquad \Lambda = \frac{3(1-\nu_s) R_s^0}{2 E_s h_s} V_s + \frac{1}{K}(V_s + V_r).$$

Here Λ is pliability of the eye under the increasing pressure, K is modulus of the volume stiffness of the vitreous body (for uncompressible vitreous body $1/K = 0$), V_s and V_r are the volumes of the segments $A_0C_0D_0B_0$ and $A_0F_0B_0$ correspondingly

$$V_s = \frac{\pi {R_s^0}^3}{3}(2 + 3\cos\varphi_s - \cos^3\varphi_s), \quad V_r = \frac{\pi {R_r^0}^3}{3}(2 - 3\cos\varphi_r + \cos^3\varphi_r).$$

In systems (5),(6) the values of P, r_d, p are given (see (4)), and p_0, s_d, ΔV could be found.

For numerical solution of systems (5) and (6) it's convenient to introduce function p_*

$$p_* = \begin{cases} 0 & \text{for} \quad r < r_d, \\ p & \text{for} \quad r \geq r_d, \end{cases}$$

then system (6) is a part of system (5).

System (6) has a pecular point $s_0 = 0$.

The asymptotic analysis of the system shows that in neighbourhood of that point

$$T_1(s_0) = T_{10} + O(s_0^2), \qquad \varphi = 0, \qquad r = r_1 s_0 + O(s_0^3), \quad r_1 = 1 + \frac{(1-\nu_r)(T_{10} - T_0)}{E_r h_r},$$

where T_{10} is constant.

Taking into account this expansions and arbitrary values p_0 and T_{10} we can solve numerically system (6), then changing the value T_{10}, satisfying condition (7), and changing the value p_0 we obtain the equality $\Delta V = \Delta V_1$.

3 Results

In Table the results of calculations due to this model are represented for following parameters: load $P = 5$ g, central angle for the contact line $\varphi_r = 38^o$, modulus of the volume stiffness of the vitreous body $K = 100$ MP, Young's modulus E_r=1.2 MPa and Poison's ratio $\nu_r = 0.5$ for cornea, the radius of the unloaded cornea R_r = 8 mm. For sclera h_s=1 mm, R_s=12 mm, $\nu_s = 0.45$. Young's modulus for sclera ($E_s = 6$ MPa, $E_s = 12$ MPa) and thickness of cornea $h_r = 0.5$ mm, $h_r = 0.3$ mm are shown in the table. Values p and p_0 are given in mm Hg. $d = 2r_d$ is the diameter of the contact area. The corresponding data of calibrating tables for elastotonometer are represented in the last column.

d	p	$h_r = 0.5$ $E_s = 6$	$h_r = 0.3$ $E_s = 6$	$h_r = 0.5$ $E_s = 12$	$h_r = 0.3$ $E_s = 12$	Nesterov and Vurgaft (1972)
3.5	38.2	36.7	35.6	35.6	34.9	50.7
4.0	29.3	27.5	26.8	26.3	25.5	36.9
4.5	23.1	21.1	20.4	19.7	18.9	27.1
5.0	18.7	16.4	15.7	14.7	13.9	20.0
5.5	15.5	12.7	12.1	10.7	10.0	15.2
6.0	13.0	9.7	8.9	7.3	6.6	11.2

One can see that the calculated intraocular pressure depends on parameters of cornea and sclera, and the change of Young's modulus for shells effects more essential than the change of the cornea thickness.

For load $P = 10g$ the results of calculations give better agreement with the data in calibrating tables. In Fig. 2 the results of calculations are represented for h_s=1 mm, h_r=0.5 mm, R_s=12 mm, R_r = 8 mm, E_r=1.2 MPa. The labels 1 and 2 correspond to different Young's modulus of sclera.

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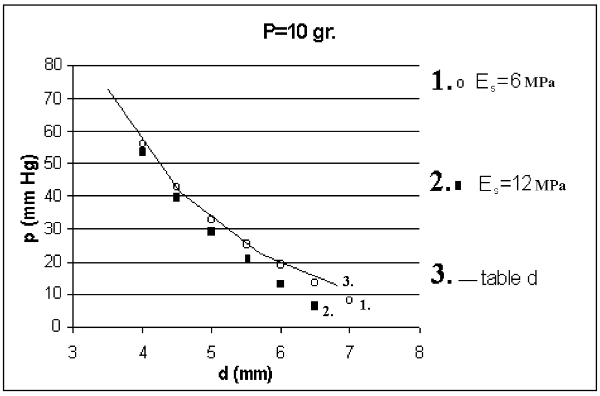

Figure 2: The numerical results vs. data in the calibration tables

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