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Formulation of Kirchhoff Rod Based on Quasi-coordinates

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The quasi-coordinates are applied to formulate Kirchhoff's rod. The potential energy of the rod expressed by the quasi-coordinates has a similar form as the kinetic energy and complementary kinetic energy in dynamics. The conjugate quasi-momentum is defined and the canonical equations due to the quasi-coordinates are given. Kirchhoff's equations can be derived directly from Boltzman-Hamel's equations or its canonical form with arc length s as independent variables. Lagrange's theorem is extended to determine the stability of equilibrium configuration of the elastic rod, and is proved using the Lyapunov's direct method. It is noticed that the condition of stability has a different physical explanation than in dynamics.

1 Introduction

The study of equilibrium and stability of a thin elastic rod was started by Daniel Bernoulli and Euler (1730) in the classic mechanics. The theoretical problem has a practical background in engineering, particularly, in the molecular biology as a macroscopic model of DNA. Kirchhoff (1859) found the analogy of equations of equilibrium of a thin rod with the equations of a heavy rigid body turning about a fixed point. The Kirchhoff equations of the elastic rod were derived analytically with the Euler angles as generalized coordinates. In this paper the quasi-coordinates are applied to formulate the Kirchhoff rod. The potential energy of the rod expressed by the quasi-coordinates has a similar form as the kinetic energy and complementary kinetic energy in dynamics (Rimrott, 1993). The conjugate quasi-momentum is defined and the canonical equations due to the quasi-coordinates are given. It is shown that Kirchhoff's equations can be derived directly from the Boltzman-Hamel equations or its canonical form with arc length s as an independent variable. Lagrange's theorem is extended to determine the stability of equilibrium configuration of the elastic rod, and is proved using Lyapunov's direct method. Although the condition of stability has a same form as in dynamics, it has a different physical explanation.

2 Generalized Coordinates and Quasi-coordinates

Consider a thin elastic rod with noncircular cross section and length L. The rod is assumed as homogeneous, inextensible, unshearable and without volume force and contact force. Then the configuration of the continuum rod is simplified to the attitude of its rigidified cross section with three degrees of freedom. We establish an arc-coordinate s along the centerline of the rod with the end point P_0 as originale. We assume that an external force $\boldsymbol{F}$ acts on P_0, while $-\boldsymbol{F}$ acts on another end point P_L. Let P be an arbitrary point on the centerline, and (P-xyz) be the principal coordinate frame of the cross section in P with z-axis along the tangent of centerline. When P moves along the centerline with unique velocity, the cross section rotates with angular velocity $\boldsymbol{\omega}$, which can be regarded as the turning rate of the cross section with respect to arc length s and is called twist vector (Nizzete and Goriely, 1999). Denote the components of vector $\boldsymbol{\omega}$ in (P-xyz) by ω_i $(i=1,2,3)$, the components of the resultant force $\boldsymbol{F}$ and torque $\boldsymbol{M}$ on the section P by F_i and M_i $(i=1,2,3)$. When the volume force and contact force are ignored, $\boldsymbol{F}$ is a constant vector, which can be selected as the direction of a fixed axis ζ. The linear constitutive relations between M_i and ω_i $(i=1,2,3)$ are assumed as

$$M_i = A_i\omega_i \ (i=1,2,3) \tag{1}$$

where A_i $(i=1,2,3)$ are the bending stiffnesses about x-, y-axes and the twisting stiffness about z-axis, respectively.

The attitude of the cross section of the rod relative to the inertial reference frame can be expressed by the Euler

angles $q_1 = \psi$, $q_2 = \vartheta$, $q_3 = \varphi$ as generalized coordinates, or by the quasi-coordinates π_i $(i = 1,2,3)$, defined as

$$\dot{\pi}_i = \omega_i \ (i = 1,2,3) \tag{2}$$

where the top dot denotes the differentiation with respect to the arc length *s*. The following relations between q_j and π_i $(i, j = 1,2,3)$ can be derived as

$$\dot{\pi}_i = \sum_{j=1}^{3} a_{ij}\dot{q}_j \quad , \quad \dot{q}_j = \sum_{j=1}^{3} b_{ji}\dot{\pi}_i \quad (i, j = 1,2,3) \tag{3}$$

where (a_{ij}) and (b_{ji}) are reversible matrices,

$$(a_{ij}) = \begin{pmatrix} \sin q_2 \sin q_3 & \cos q_3 & 0 \\ \sin q_2 \cos q_3 & -\sin q_3 & 0 \\ \cos q_2 & 0 & 1 \end{pmatrix}, \ (b_{ji}) = \begin{pmatrix} \csc q_2 \sin q_3 & \csc q_2 \cos q_3 & 0 \\ \cos q_3 & -\sin q_3 & 0 \\ -\cot q_2 \sin q_3 & \cot q_2 \cos q_3 & 1 \end{pmatrix} \tag{4}$$

3 Potential Energy of the Rod

The total potential energy *E* of a thin elastic rod is composed of the elastic strain energy E_e and the potential energy E_p of external force.

$$E = E_\mathrm{e} + E_\mathrm{p} \tag{5}$$

The elastic strain energy E_e can be written as an integral along the centerline of the rod

$$E_\mathrm{e} = \int_0^L \Gamma_\mathrm{e} \, \mathrm{d}s \tag{6}$$

where Γ_e is the density of elastic strain energy,

$$\Gamma_\mathrm{e} = \frac{1}{2}\sum_{i=1}^{3} A_i \omega_i^2 \tag{7}$$

Let $\boldsymbol{r}_0$ and $\boldsymbol{r}_L$ be the vectors from a fixed point *O* to P_0 and P_L, and $\boldsymbol{R} = \boldsymbol{r}_0 - \boldsymbol{r}_L$. For a given virtual displacement $\delta\boldsymbol{R}$, the variation of energy E_p is equal to the negative virtual work of the external forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ as

$$\delta E_\mathrm{p} = -\boldsymbol{F} \cdot \delta\boldsymbol{R} = -\boldsymbol{F} \cdot \delta\int_0^L \boldsymbol{T}\mathrm{d}s = -\delta\int_0^L F\gamma \, \mathrm{d}s \tag{8}$$

where $\boldsymbol{T}$ denotes the unit vector along *z*- axis, $\gamma = \cos\vartheta$ is the cosine of the angle between ζ - and *z*- axes. Then the variation of the total potential energy is

$$\delta E = \delta E_e + \delta E_p = \delta\int_0^L \Gamma \, \mathrm{d}s \tag{9}$$

where Γ is the density of total potential energy

$$\Gamma = \frac{1}{2}\sum_{i=1}^{3} A_i \omega_i^2 - F\gamma \tag{10}$$

Defining the generalized momentum conjugated to the quasi-coordinate π_i $(i = 1,2,3)$ as

$$p_i = \frac{\partial}{\partial \omega_i}\Gamma(q,\omega) \quad (i = 1,2,3) \tag{11}$$

we obtain

$$p_i = A_i\omega_i = M_i \quad (i = 1,2,3) \tag{12}$$

which satisfy the equality of Poisson's brackets as (Marsden and Ratiu, 1994)

$$(p_1, p_2) = -p_3\ ,\ (p_2, p_3) = -p_1\ ,\ (p_3, p_1) = -p_2 \tag{13}$$

The quasi-velocity ω_i $(i = 1,2,3)$ can be obtained by partial differentiation of Γ as

$$\omega_i = \frac{\partial}{\partial p_i}\Gamma(q,p) \qquad (i = 1,2,3) \tag{14}$$

The function Γ can be expressed by p_i $(i = 1,2,3)$ as

$$\Gamma = \frac{1}{2}\sum_{i=1}^{3} \frac{p_i^2}{A_i} - F\gamma \tag{15}$$

According to the Kirchhoff kinetic analogy, both expressions of the density of potential energy (10) and (15) correspond to the kinetic energy and complementary kinetic energy in dynamics (Rimrott, 1993)

$$\Gamma = \sum_{i=1}^{3}\int \omega_i(q,p)\mathrm{d}p_i - F\gamma = \sum_{i=1}^{3}\int p_i(q,\omega)\mathrm{d}\omega_i - F\gamma \tag{16}$$

4 Boltzman-Hamel's Equations

According to the principle of minimal potential energy in the elasticity theory, the equilibrium conditions of the rod yields

$$\delta\int_0^L \Gamma\ \mathrm{d}s\ = 0 \tag{17}$$

Equation (17) has a same form as Hamilton's principle of least action when the arc length *s* is changed by the time variable *t*. Treating $\Gamma(q,\dot{q})$ as a function of the generalized coordinates q_j $(j = 1,2,3)$ and its derivatives, we obtain the Lagrange's equations with arc-coordinate *s* from the variation of integral (17), (Westcott et al., 1995)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Gamma}{\partial\dot{q}_j}\right) - \frac{\partial\Gamma}{\partial q_j} = 0 \quad (j = 1,2,3) \tag{18}$$

where the density of the potential energy $\Gamma(q,\dot{q})$ plays the role of a Lagrangian. If $\Gamma(\pi,\dot{\pi})$ is regarded as a function of quasi-coordinates π_i $(i = 1,2,3)$ and its derivatives, then the Boltzman-Hamel equations of the holonomic system can be derived from eq. (17) as (Hamel, 1904)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Gamma}{\partial \omega_k}\right)-\frac{\partial \Gamma}{\partial \pi_k}+\sum_{i=1}^{3}\sum_{m=1}^{3}A_i\omega_i\omega_m\gamma_{km}^{i}=0 \qquad (k=1,2,3) \tag{19}$$

where γ_{km}^{i} is the three-index Boltzman symbol defined as

$$\gamma_{km}^{i}=\sum_{j=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial a_{ij}}{\partial q_l}-\frac{\partial a_{il}}{\partial q_j}\right)b_{jk}b_{lm} \quad (i,k,m=1,2,3) \tag{20}$$

Similar to the definition in analytical mechanics, we define the Hamiltonian function of the rod as

$$H=\sum_{i=1}^{3}\omega_i\frac{\partial \Gamma}{\partial \omega_i}-\Gamma \tag{21}$$

Then we obtain

$$H=\frac{1}{2}\sum_{i=1}^{3}A_i\omega_i^2+F\gamma=\frac{1}{2}\sum_{i=1}^{3}\frac{p_i^2}{A_i}+F\gamma \tag{22}$$

which is similar to the function Γ, where $-F\gamma$ is changed by $+F\gamma$. Boltzman-Hamel's equations can be transferred to the Hamilton canonical form as

$$\dot{\pi}_k=\frac{\partial H}{\partial p_k}\ , \qquad \dot{p}_k=-\frac{\partial H}{\partial \pi_k}+\sum_{i=1}^{3}\sum_{m=1}^{3}A_i\omega_i\omega_m\gamma_{km}^{i} \qquad (k=1,2,3) \tag{23}$$

5 Kirchhoff's Equations and Jacobi's Integral

Kirchhoff's equations can be derived from Boltzman-Hamel's equations directly without tedious calculation with Euler angles (Westcott et al., 1995). Substituting eq. (4) into eqs. (19) or (23) , we obtain

$$\begin{aligned}
&A\frac{\mathrm{d}\omega_1}{\mathrm{d}s}+(C-B)\omega_2\omega_3-F_2=0\\
&B\frac{\mathrm{d}\omega_2}{\mathrm{d}s}+(A-C)\omega_3\omega_1+F_1=0\\
&C\frac{\mathrm{d}\omega_3}{\mathrm{d}s}+(B-A)\omega_1\omega_2=0
\end{aligned} \tag{24}$$

The constant force $\boldsymbol{F}$ yields the Poisson's equations as

$$\begin{aligned}
&\frac{\mathrm{d}F_1}{\mathrm{d}s}+\omega_2F_3-\omega_3F_2=0\\
&\frac{\mathrm{d}F_2}{\mathrm{d}s}+\omega_3F_1-\omega_1F_3=0\\
&\frac{\mathrm{d}F_3}{\mathrm{d}s}+\omega_1F_2-\omega_2F_1=0
\end{aligned} \tag{25}$$

Eqs. (24),(25) have the same form as the dynamical equations of a heavy rigid body about fixed point. There exists a Jacobi integral from eqs. (24), (25) as

$$H = \frac{1}{2}\sum_{i=1}^{3} A_i \omega_i^2 + F\gamma = \frac{1}{2}\sum_{i=1}^{3} \frac{p_i^2}{A_i} + F\gamma = \text{const} \tag{26}$$

It should be noted that the Jacobi's integral of elastic rod has a different physical significance than in dynamics. It means the conservation of the sum of the strain energy and the external work, but not the total potential energy of the rod.

6 Extended Lagrange's Theorem of Stability

A theorem of stability due to Lagrange in dynamics is stated as follows. In a conservative system an isolated equilibrium position corresponding to a minimum value of the potential energy is stable. In statics of elastic rod there is a similar theorem which can be called extended Lagrange's theorem: If an isolated equilibrium configuration of elastic rod corresponds to a minimum value of the sum of strain energy and external work, the equilibrium is stable.

To prove the theorem, we take the Hamiltonian function H as a Lyapunov function, which is positive definite when it has a minimum value. Since the function H is conserved, its total derivative with respect to arclength s is zero along the perturbed trajectory. According to the Lyapunov theorem, the equilibrium is stable.

Above-mentioned theory can be applied in stability problem of equilibrium of elastic rod, but its physical means cannot be explained as the minimum of total energy of the rod, except that the external work vanishes.

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