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Modelling Parametric Vibration of Gear-Pair Systems as a Tool for Aiding Gear Fault Diagnosis

Nguyen Van Khang, Thai Manh Cau, Nguyen Phong Dien

The main components in gear vibration spectra are the tooth-meshing frequency and its harmonics, together with sideband structures due to modulation effects. Sideband structures can be used as an important diagnostic symptom for gear fault detection. The main objective of the present paper is to unravel amplitude modulation effects which are responsible for generating such sidebands. The parametrically excited vibration of a gear-pair system in mesh was investigated. A comparison between the model result and actual experimental data from a test rig was also presented. Some analytical expressions are derived to provide a logical explanation for observed phenomena. The modelling result can be used to predict sideband amplitude in presence of the distributed gear faults such as non-uniform tooth wear, pittings. It may serve as a tool for aiding the gear fault diagnosis.

1 Introduction

Gearboxes are frequently used in machine systems for power transmission, speed variation and/or working direction. Dynamic modelling of gear vibration offers a better understanding of the vibration generation mechanisms as well as the dynamic behavior of the gear transmission in the presence of gear tooth damage. Because of their ubiquity and importance, gearboxes have received a considerable amount of attention in this respect. A significant number of paper have been published concerning the problem. In some studies, e.g. Bartelmus (2001), Howard et al. (2001), Vinayak and Singh (1998), Huang and Liu (2000), the researchers developed sophisticated models which take into consideration the most important dynamic factors in gearboxes such as periodic changes in tooth stiffness, the excitation from gear transmission errors, the coupling effect between the torsional and lateral vibrations of the gears and shafts. In addition, another approach focused on modelling the tooth mesh since the main source of vibration in a geared transmission system is usually the meshing action of the gears, e.g. Parker (2000), Theodossiades and Natsiavas (2000), Velex and Maatar (1996). The intent of these studies is to unravel some of the unknown aspects related to the interaction of non-linear effects such as effects of friction forces at the meshing interface and gear backlash with the time-varying mesh stiffness.

Vibration analysis has become a very important tool for detection of gear faults, and many signal processing procedures have been developed to extract information about incipient faults from the externally measured vibration signals (Forrester, 1996; Meltzer and Nguyen Phong Dien, 2003; Meltzer and Nguyen Phong Dien, 2004). It is well known that, the most important components in gear vibration spectra are the tooth-meshing frequency and its harmonics, together with sideband structures due to modulation effects. The increment in the number and amplitude of sidebands may indicate a gear fault condition, and the spacing of the sidebands is related to their source (Dalpiaz et.al., 2000). Consequently, sideband structures can be used as a important diagnostic feature for gear fault detection. The main objective of the present paper is to unravel amplitude modulation effects which are responsible for generating such sidebands. The paper uses a relative simple model of a pair of helical gears in mesh to produce typical vibration signals resulting from tooth deflection under load, variations in mesh stiffness and geometrical errors caused by machining errors and non-uniform wear. This model is characterized as a parametrically excited system. The harmonic balance method is employed for solving the differential equation of motion. A comparison between the model result and actual experimental data from a test rig is also presented.

2 Modelling of Gear-Pair System

The mechanical model of the gear-pair system in mesh investigated in this paper is shown in Fig. 1. The gear mesh is modelled as a pair of rigid disks connected by a spring-damper set along the line of contact. This kind of the model is also considered in ref. Parker (2000), Velex and Maatar (1996).

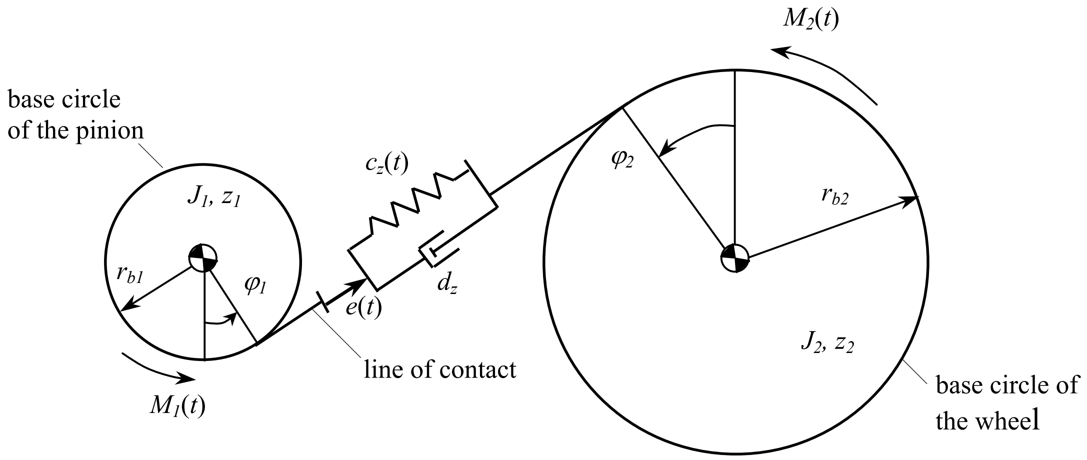

Figure 1. Mechanical model of the gear-pair system with faulty meshing

The model takes into account influences of the static transmission error which is simulated by a displacement excitation $e(t)$ at the mesh. This transmissions error arises from several sources, such as tooth deflection under load, non-uniform tooth spacing, tooth profile errors caused by machining errors as well as pitting, scuffing of teeth flanks. The mesh stiffness $c_z(t)$ is expressed as a time-varying function. The gear-pair is assumed to operate under high torque condition with zero backlash. Effects of friction forces at the meshing interface are neglected on the basis that in particular, the coefficient of friction is low (approx. 6%). The viscous damping coefficient of the gear mesh d_z is assumed to be constant.

The differential equations of motion for this system can be expressed in the form

$$J_1\ddot{\varphi}_1 + r_{b1}c_z(t)\left[r_{b1}\varphi_1 + r_{b2}\varphi_2 + e(t)\right] + r_{b1}d_z\left[r_{b1}\dot{\varphi}_1 + r_{b2}\dot{\varphi}_2 + \dot{e}(t)\right] = M_1(t) \tag{1}$$

$$J_2\ddot{\varphi}_2 + r_{b2}c_z(t)\left[r_{b1}\varphi_1 + r_{b2}\varphi_2 + e(t)\right] + r_{b2}d_z\left[r_{b1}\dot{\varphi}_1 + r_{b2}\dot{\varphi}_2 + \dot{e}(t)\right] = M_2(t) \tag{2}$$

where φ_i, $\dot{\varphi}_i$, $\ddot{\varphi}_i$ (i = 1,2) are rotation angle, angular velocity, angular acceleration of the input pinion and the output wheel respectively. J_1 and J_2 are the mass moments of inertia of the gears. $M_1(t)$ and $M_2(t)$ denote the external torques load applied on the system. r_{b1} and r_{b2} represent the base radii of the gears.

By introducing the composite coordinate

$$q = r_{b1}\varphi_1 + r_{b2}\varphi_2 \tag{3}$$

Eqs. (1), (2) yield a single differential equation in the following form

$$m_{red}\ddot{q} + c_z(t)q + d_z\dot{q} = F(t) - c_z(t)e(t) - d_z\dot{e}(t) \tag{4}$$

where $$m_{red} = \frac{J_1J_2}{J_1r_{b2}^2 + J_2r_{b1}^2}\ ;\quad F(t) = m_{red}\left(\frac{M_1(t)r_{b1}}{J_1} + \frac{M_2(t)r_{b2}}{J_2}\right) \tag{5}$$

Note that in Eq. (4) the rigid-body rotation from the original mathematical model is eliminated. The new coordinate $q(t)$ is expressed as the dynamic transmission error of the gear-pair system (Parker, 2000). For a specific gear-pair, the mesh stiffness $c_z(t)$ is obtained by means of LVR-Software (Boerner, 1999), based on the finite element analysis and the contact mechanics. In steady state motion of the gear system, the mesh stiffness can be approximately represented by a truncated Fourier series (Theodossiades and Natsiavas, 2000)

$$c_z(t) = c_0 + \sum_{k=1}^{K} c_k \cos(k\omega_z t + \gamma_k) \tag{6}$$

where ω_z is the gear meshing angular frequency which is equal to the number of gear teeth times the shaft angular frequency and K is the number of terms of the series.
In general, the error components are not identical for each gear tooth and will produce displacement excitation which is periodic with the gear rotation (i.e. repeated each time the tooth is in contact). Therefore, the excitation function $e(t)$ can be expressed in a Fourier series with the fundamental frequency corresponding to the rotation speed of the faulted gear. For instance, when the errors are situated at the teeth of the pinion, $e(t)$ may be taken in the form

$$e(t) = \sum_{i=1}^{I} e_i \cos(i\,\omega_1 t + \alpha_i) \tag{7}$$

Where ω_1 is the rotating frequency of the pinion shaft.

It is assumed that when $\dot{\varphi}_1 = \omega_1 = const$, $\dot{\varphi}_2 = \omega_2 = const$, $d_z = 0$, $c_z(t) = c_0$, the dynamic transmission error of the gear-pair system q is equal to the static tooth deflection under constant load q_0 as

$$q = r_{b1}\varphi_1 + r_{b2}\varphi_2 = q_0 \tag{8}$$

Therefore, from the Eq. (4) one yields (Keppler, 1994)

$$F(t) \approx F_0(t) = c_0 q_0 + c_0 e(t) \tag{9}$$

3 Experimental Set-Up

The experiment was done at an ordinary back-to-back test rig (Fig. 2). The load torque was provided by a hydraulic rotary torque actuator which remains the external torque constant for any motor speed. The test gearbox operates at a nominal pinion speed of 1800 rpm (30 Hz), thus the meshing frequency f_z is 420 Hz. The input torque load applied to the system is 400 Nm. The test gears are dynamically isolated from the slave gears in the back-to-back configuration. The major parameter of the gear-pair are shown in Table 1. These parameters are also used in the model.

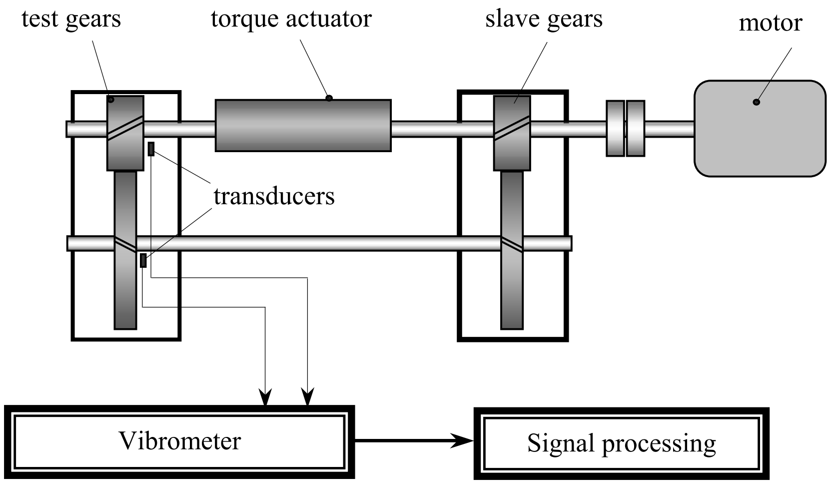

Figure 2. Test rig

A Laser Doppler Vibrometer was used for measuring oscillating parts of the angular speed of the shafts (i.e. oscillating part of $\dot{\varphi}_1$ and $\dot{\varphi}_2$) in order to determine experimentally the dynamic transmission error. The measurement was taken with two non-contacting transducers mounted in proximity to the shafts, positioned at the closest position to the test gears. The vibration signals were sampled at 10 kHz. The signal used in this study were recorded at the end of 12-hours total test time, at that time a surface fatigue failure occurred on some teeth of the pinion.

Parameters	Pinion	Wheel
Gear type	helical, standard involute	
Material	steel	
Module (mm)	4,50	
Pressure angle (^{o})	20,00	
Helical angle (^{o})	14,56	
Number of teeth z	14	39
face width (mm)	67,00	45,00
base circle radius (mm)	30,46	84,86
Theoretical contact ratio	2,17	

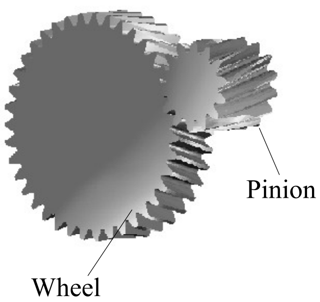

Table 1: Parameters of the test gear pair

4 Calculating Parametric Vibration and Experimental Comparisons

The vibration equation of gear-pair system is a differential equation with the periodic coefficients, which have the following form

$$m_{red}\ddot{q} + c_z(t)\,q + d_z\,\dot{q} - f(t) = 0 \tag{10}$$

where $$f(t) = c_0 q_0 - \left[c_z(t) - c_0\right]e(t) - d_z\dot{e}(t) \tag{11}$$

In this paper, four dominant coefficients c_0, c_1, c_2, c_3 in the Fourier series of the mesh stiffness expressed in Eq. (6) are taken into account

$$c_z(t) = c_0 + \sum_{k=1}^{3} c_k \cos(k\omega_z t + \gamma_k) = c_0 + \sum_{k=1}^{3}\left(\hat{c}_k \cos k\omega_z t + \hat{s}_k \sin k\omega_z t\right) \tag{12}$$

where $\omega_z = z_1\omega_1$ and the excitation function $e(t)$ is expressed by the two first terms of its Fourier series as follows

$$e(t) = \sum_{k=1}^{2} e_k \cos(k\omega_1 t + \alpha_k) \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11) one obtains

$$\begin{aligned} f(t) = {} & c_0 q_0 + d_z e_1\omega_1 \sin(\omega_1 t + \alpha_1) + 2d_z e_2\omega_1 \sin(2\omega_1 t + \alpha_2) \\ & - \frac{e_1}{2}\sum_{k=1}^{3}\Big\{\hat{c}_k \cos\left[(kz_1 - 1)\omega_1 t - \alpha_1\right] + \hat{s}_k \sin\left[(kz_1 - 1)\omega_1 t - \alpha_1\right] \\ & \qquad + \hat{c}_k \cos\left[(kz_1 + 1)\omega_1 t + \alpha_1\right] + \hat{s}_k \sin\left[(kz_1 + 1)\omega_1 t + \alpha_1\right]\Big\} \\ & - \frac{e_2}{2}\sum_{k=1}^{3}\Big\{\hat{c}_k \cos\left[(kz_1 - 2)\omega_1 t - \alpha_2\right] + \hat{s}_k \sin\left[(kz_1 - 2)\omega_1 t - \alpha_2\right] \\ & \qquad + \hat{c}_k \cos\left[(kz_1 + 2)\omega_1 t + \alpha_2\right] + \hat{s}_k \sin\left[(kz_1 + 2)\omega_1 t + \alpha_2\right]\Big\} \end{aligned} \tag{14}$$

The harmonic balance method is employed for solving the differential Eq. (10). Based on the analytic form of functions $c_z(t)$ and $f(t)$, it is now assumed that Eq. (10) has a solution which may be approximated by

$$\begin{aligned} q(t) = a_0 &+ \sum_{k=1}^{3}\left(a_k \cos k\omega_1 t + b_k \sin k\omega_1 t\right) + \sum_{k=1}^{3}\Big[a_{kz_1-2}\cos(kz_1-2)\omega_1 t \\ &+ b_{kz_1-2}\sin(kz_1-2)\omega_1 t + a_{kz_1-1}\cos(kz_1-1)\omega_1 t + b_{kz_1-1}\sin(kz_1-1)\omega_1 t \\ &+ a_{kz_1}\cos(kz_1)\omega_1 t + b_{kz_1}\sin(kz_1)\omega_1 t + a_{kz_1+1}\cos(kz_1+1)\omega_1 t \\ &+ b_{kz_1+1}\sin(kz_1+1)\omega_1 t + a_{kz_1+2}\cos(kz_1+2)\omega_1 t + b_{kz_1+2}\sin(kz_1+2)\omega_1 t\Big] \end{aligned} \tag{15}$$

This estimate now is introduced in Eq. (10) and its right-hand side is written as a sum of trigonometric functions. This differential equation then is approximately satisfied by setting the coefficients in this sum: cos0, $\cos\omega_1 t$, $\sin\omega_1 t$, $\cos 2\omega_1 t$, $\sin 2\omega_1 t$, $\cos(kz_1\text{-}2)\omega_1 t$, $\sin(kz_1\text{-}2)\omega_1 t$, $\cos(kz_1\text{-}1)\omega_1 t$, $\sin(kz_1\text{-}1)\omega_1 t$, $\cos(kz_1)\omega_1 t$, $\sin(kz_1)\omega_1 t$, $\cos(kz_1+1)\omega_1 t$, $\sin(kz_1+1)\omega_1 t$, $\cos(kz_1+2)\omega_1 t$, $\sin(kz_1+2)\omega_1 t$ (k = 1,2,3) equal to zero. This results in 35 algebraic equations for the unknowns $a_0, a_1, b_1, a_2, b_2, ..., a_{kz_1-2}, b_{kz_1-2}, ..., a_{kz_1+2}, b_{kz_1+2}$.

The following parameters of the model are used for numerical simulation: J_1= 9,3·10^{-2} (kgm^2); J_2 = 0,272 (kgm^2); m_{red} = 7,92 (kg). Another parameters of the gears are shown in Table 1. According to the experiment, a nominal pinion speed of 1800 rpm ($\omega_1 = 60\pi$ and f_1 = 30 Hz) is chosen.

By using the LVR-Software, the mesh stiffness of the test gear pair at particular meshing position was obtained as shown in Fig. 3. It gives the values of coefficients: c_0 = 8,04·10^8; c_1 = 0,304·10^8; c_2 = 0,185·10^8; c_3 = 0,050·10^8 (N/m) with corresponding phase angular γ_1 =1,02; γ_2 = -0,72; γ_3 = -0,93 (radian) and static tooth deflection q_0 = 1,2·10^{-5} (m), thus the static tooth force c_0q_0 is 9650,4 (N). The numerical simulation is realized with the program MAPLE®. The calculated dynamic transmission error is shown in Fig. 4. Figs. 5 and 6 present results obtained with different excitation functions *e(t)* .

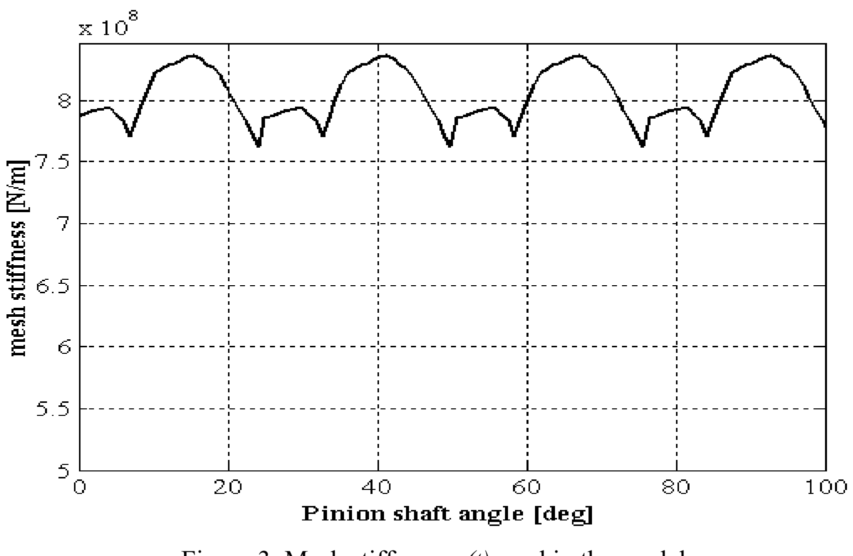

Figure 3. Mesh stiffness $c_z(t)$ used in the model

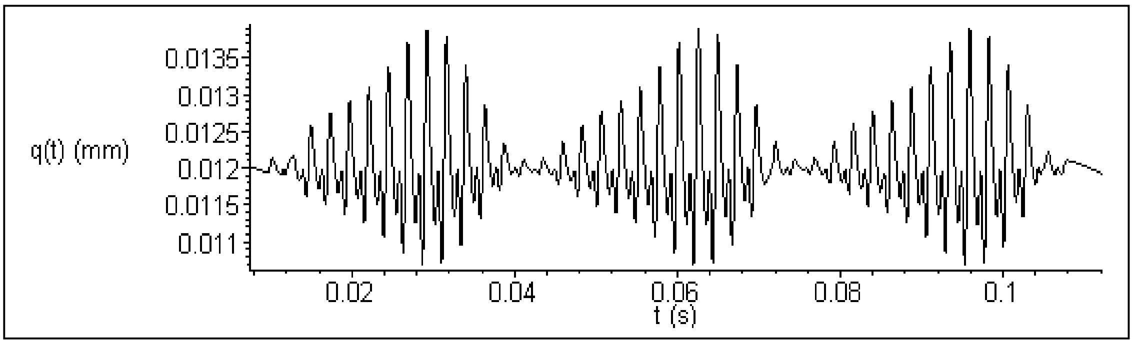

Figure 4. Modelling result: Time history of oscillating part of the dynamic transmission error *q(t)*

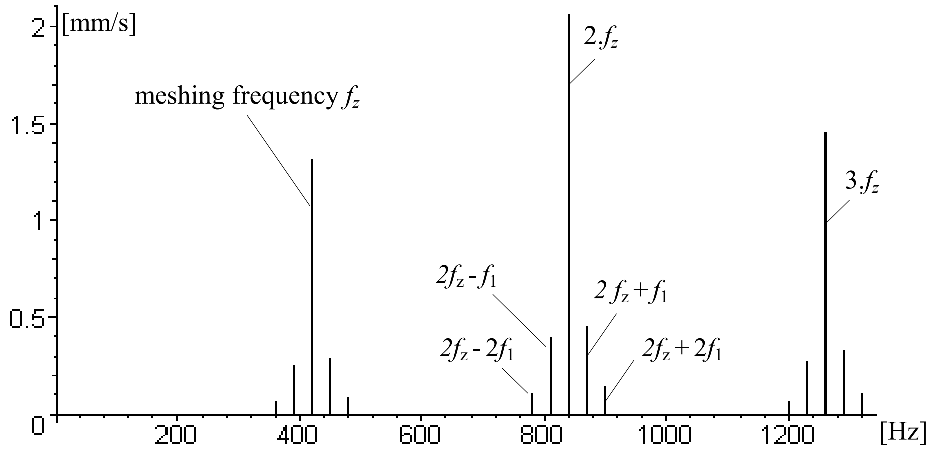

Figure 5. *Modelling result*: frequency spectrum of $\dot{q}(t)$
(Excitation function *e(t)* with coefficients: $e_1 = 0{,}005$ (mm), $e_2 = 0{,}0015$ (mm)
and phase angular $\alpha_1=\pi/3$, $\alpha_2=\pi/4$)

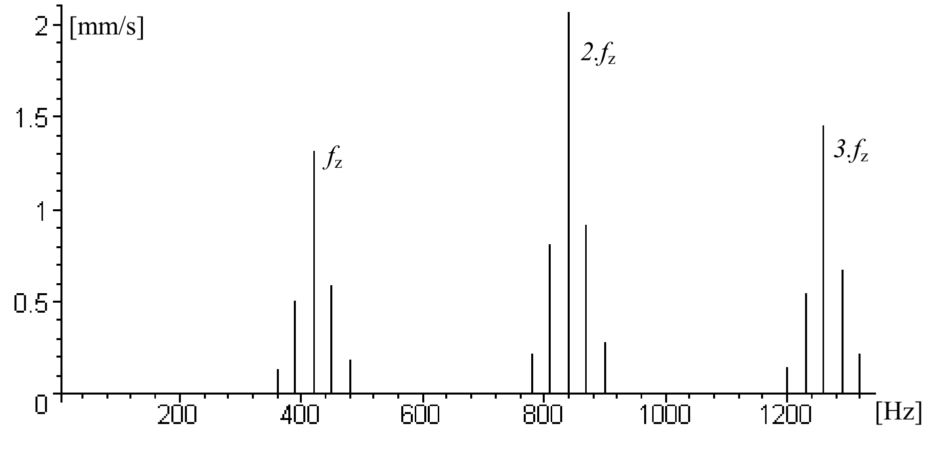

Figure 6. *Modelling result*: frequency spectrum of $\dot{q}(t)$
(Excitation function *e(t)* with larger coefficients: $e_1 = 0{,}01$ (mm), $e_2 = 0{,}003$ (mm)

and phase angular $\alpha_1=\pi/3$, $\alpha_2=\pi/4$)

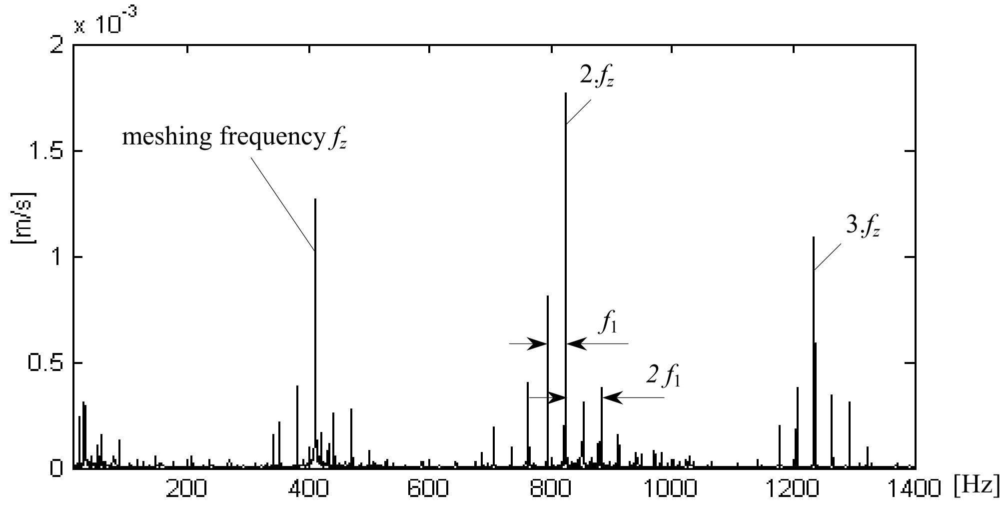

Figure 7. *Experimental result*: frequency spectrum of $\dot{q}(t)$

The spectra in Figures 5 and 6 show clearly the meshing frequency and its harmonics with sideband structures. As expected, the sidebands are spaced by the rotational frequency f_1 of the pinion. By comparing amplitude of these sidebands in both spectra, it can be concluded that the excitation function $e(t)$ caused by tooth errors is responsible for generating sidebands.

Figure 7 shows a frequency spectrum of the first derivative of the dynamic transmission error $\dot{q}(t)$ determined from the experimental data. The spectrum presents sidebands at the meshing frequency and its harmonics. In particular, the dominant sidebands are spaced by the rotational frequency of the pinion and characterised by high amplitude, which gives a clear indication of the presence of the faults on the pinion. Comparing the spectra displayed in Figure 6 and Figure 7, it can be observed that results of computer simulation agree closely with results of measurements on test rig.

5 Conclusions

In the above sections, the parametrically excited vibration of the gear-pair system in mesh was investigated. A comparison between the model result and actual experimental data was also presented. The target of the study is to provide the fundamental understanding of the physical mechanism related to the gear faults, which generate modulation sidebands in vibration spectra. Some analytical expressions are derived to provide a logical explanation for observed phenomena. The modelling result can be used to predict sideband amplitude in presence of the "distributed" gear faults such as non-uniform tooth wear and pittings. Consequently, it may serve as a tool for aiding the gear fault diagnosis.

Here it must be noted that although the mechanical model is a relative simple, but it can be able to reveal essential dynamic properties of the gear-pair in mesh. Rather, this study is intended only to explain the appearance of the sideband phenomenon generated by errors and distributed faults on gears. No attempt is made here to present a generous study on the effects of external load variation, variations of the mesh stiffness caused by local tooth faults (cracks), parametric resonances etc., and a mathematical treatment of this problem is let for future investigation. However, the obtained results seem promising and extension to more complicated geared systems and other types of fault.

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Address: Prof. Dr.-Ing. habil. Nguyen Van Khang, Dipl.-Ing. Thai Manh Cau, Dr.-Ing. Nguyen Phong Dien, Department of Applied Mechanics, Hanoi University of Technology, 1. Dai Co Viet, Hanoi, Vietnam.

e-mail: nvankhang@mail.hut.edu.vn