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On the Localized Vibration Modes of Thin Elastic Shells *

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A short elaboration of the localized modes of free vibrations of thin elastic shells is presented. A two-dimensional system of equations based on the Kirchhoff–Love hypotheses is used. Asymptotic expansions for the natural frequencies and for the corresponding vibration modes in power series of the relative shell thickness are constructed.

1 Introduction

The investigation of the spectrum of small free vibrations of thin elastic shells and of the corresponding vibration modes is the basis for solutions of a large number of dynamical shell problems. Among them there are the forced vibrations, the parametric vibrations, the nonlinear vibrations, and so on. The general solution of this problem is given in the book of Goldenweizer et al. (1979) (see also Aslanian and Lidsky (1974), Oniashvili (1957), Skudrzyk (1968)). The shell spectrum is non-negative and discrete with the point of condensation at infinity. The low part of the shell spectrum is very complex. The distance between adjacent points of the spectrum is asymptotically small. The important characteristic of the spectrum is its density. In contrary to plates, for which the density is asymptotically constant, for shells the density may have points of maxima (see the already referred books and also the papers of Bolotin (1965), Goldenweizer (1970), Tovstik (1972)). Also in applications it is interesting to know the minimal shell frequency (see Tovstik (1975)).

In the simplest case of a circular cylindrical shell with simply supported edges, the vibration modes occupy the whole shell surface. Sometimes the same holds for high-frequency free vibrations of arbitrary shells. In contrary this paper discusses the various cases of vibration modes localization. The localized modes may appear due to the variation of the neutral shell surface curvatures or/and to the weak support of the shell edge. To construct localized modes, asymptotic expansions based on a geometric small parameter equal to the relative shell thickness is used. In cases of a varying shell curvature the so-called weakest points or lines appear. The deflection of the localized mode exponentially decreases with increasing distance from these points or lines. The asymptotical description of these modes contains turning points. The modes localized near the edge are also discussed. As a rule these modes appear near the free edge or near the weakly supported edge. The types of the weak support for shells of positive, zero and negative Gaussian curvature are indicated.

The vibration modes localization appears in the various problems of elastic bodies. For constructing these modes asymptotic methods are used. For a membrane the mode types of ”whispering gallery” and ”jumping ball” are found (see Babich and Buldyrev (1972)). The former are localized near the membrane edge, and the latter localized near some line (for example, near the short diameter of the elliptic membrane). The eigen-function localized near the free edge of a rectangular plate with two opposite edges simply supported is constructed by Ishlinski (1954). In the three-dimensional range the edge excitation of shells of revolution is studied by Kaplunov (1991). Also in the three-dimensional range the localized vibration modes of plates with variable thickness (see Tovstik (1994)) and of bodies of revolution with variable thickness (see Kaplunov et al. (2001)) are analysed.

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2 The Two-dimensional Shell Equations and the Boundary Conditions

We study a thin shell with the constant thickness h made of linearly elastic isotropic homogeneous material. The material parameters are Young's modulus E and Poisson's ratio ν. In the neutral shell surface we introduce the orthogonal curvilinear coordinates $\mathbf{x} = \{x, y\} \in \Omega$ where Ω is the area occupied by this surface and Γ denotes its boundary.

To deliver the two-dimensional equations, we use the Kirchhoff-Love hypothesis. We use the well known equations of displacements (see Goldenweizer (1961)), which after separation of variables with $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \sin(\omega t)$, obtain the form

$$\sum_{j=1}^{3} \left(L_{ij}(\mathbf{x}) + h_*^2 N_{ij}(\mathbf{x})\right) u_j + \lambda u_i = 0. \qquad i = 1, 2, 3 \tag{2.1}$$

Here $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{n}$ is the displacement vector, $\mathbf{e_1}$ and $\mathbf{e_2}$ are the unit vectors of the curvilinear coordinate system in the neutral surface, and $\mathbf{n} = \mathbf{e_1} \times \mathbf{e_2}$ is the unit normal to this surface. In some cases we also denote the displacement vector as $\mathbf{u} = u\mathbf{e_1} + v\mathbf{e_2} + w\mathbf{n}$. The main small parameter $h_* = h/R$ is equal to the relative shell thickness, where R is the typical linear shell dimension. The frequency parameter λ is equal to

$$\lambda = \frac{\rho R^2 \omega^2}{E}, \tag{2.2}$$

where ω is the unknown natural frequency, and ρ is the density of the shell material.

$L_{ij}(\mathbf{x})$ and $N_{ij}(\mathbf{x})$ denote linear differential operators with (in the general case) variable coefficients (see Goldenweizer et al. (1979), Goldenweizer (1961)). The system (2.1) at $h_* > 0$ is elliptic with partial derivatives of 8th order. If $h_* = 0$, the system (2.1) degenerates to the so-called membrane (or momentless) system of 4th order. The type of the membrane system can be elliptic, hyperbolic, or parabolic.

The system (2.1) is self-adjoint. At each edge it needs 4 boundary condition. We study the classical self-adjoint boundary conditions. At the edge $x =$ const the possible variants of the boundary conditions are given in Table 1.

Table 1. The classical boundary conditions.

$u = 0$	$v = 0$	$w = 0$	$\gamma_1 = 0$	1	Geometric restrictions
$T_1 = 0$	$S = 0$	$Q_1^* = 0$	$M_1 = 0$	0	Free conditions

Here γ_1 is the angle of rotation around the tangent to the edge, T_1 and S are the stress-resultant in the tangential plane, $Q_1^* = Q_1 + \partial H/\partial y$ is the generalized shear stress-resultant, and M_1 is the stress couple. In the first line of Table 1 the 4 generalized displacements are given, and in the second line the corresponding generalized forces. Taking every possible combination of geometric restrictions and free conditions, we study 16 variants of boundary conditions. Conditions with an elastic support (of the type $T_1 + cu = 0$) are not considered here. It is convenient to denote each boundary condition by a 4-digital number consisting of 1 and 0 according to the chosen condition given in the order of Table 1. For example, we denote a clamped edge $u = v = w = \gamma_1 = 0$ as 1111, a free edge $T_1 = S = Q_1^* = M_1 = 0$ as 0000, and a simply supported edge $T_1 = v = w = M_1 = 0$ as 0110.

We use the asymptotic approach to analyze the system (2.1). The important role in this analysis plays the index of variation p introduced by Goldenweizer (1961) by the relation

$$\max\left\{\left|\frac{\partial F}{\partial x}\right|, \left|\frac{\partial F}{\partial y}\right|\right\} \sim h_*^{-p} F, \tag{2.3}$$

where F is any function describing the stress-strain state of the shell.

The length L of the deformation pattern is connected with the index of variation by the relation $L \sim R h_*^p$. The system (2.1) is acceptable only if $p < 1$ because in this case $L \gg h$.

The index of variation helps Goldenweizer to introduce a classification of the types of the thin shell free vibrations (see Goldenweizer et al. (1979)). Let

$$\lambda \sim h_*^{-2r}. \tag{2.4}$$

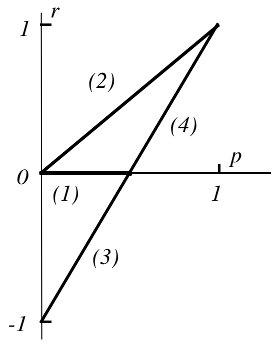

Figure 1: The dependence $r(p)$

Then there are the following 4 main types of free vibrations:
(1) the quasi-transversal vibrations with small variability
with $0 \le p < 1/2,\ r = p,\ w \gg \{u, v\}$,
(2) the quasi-tangential vibrations
with $0 \le p < 1,\ r = 0,\ w \ll \{u, v\}$,
(3) the vibrations of Rayleigh type
with $0 \le p < 1/2,\ r = -1 + 2p,\ w \gg \{u, v\}$,
(4) the quasi-transversal vibrations with large variability
with $1/2 \le p < 1,\ r = -1 + 2p,\ w \gg \{u, v\}$.

For the types (1)–(4) the dependence between the natural frequency of order r and the index of variation p is shown in Fig. 1.

Instead of system (2.1) for the point $p = 1/2,\ r = 0$ in Fig. 1, the more simple system of Donnell type may be used. This system is acceptable to describe approximately all types of vibrations except the type (2) of the quasi-tangential vibrations because in this system the tangential inertia forces are neglected. We write the Donnell system in the form

$$\begin{aligned} &\mu^2 \Delta\Delta w - \lambda w + \Delta_\kappa \Phi = 0, \\ &\mu^2 \Delta\Delta \Phi + \Delta_\kappa w = 0, \end{aligned} \qquad \mu^4 = \frac{h_*^2}{12(1-\nu^2)}, \tag{2.5}$$

where $\mu > 0$ is a small parameter, Φ is the stress function, and the linear differential operators Δ and Δ_κ are

$$\Delta w = \frac{1}{A_1 A_2} \left(\frac{\partial}{\partial x} \left(\frac{A_2}{A_1} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{A_1}{A_2} \frac{\partial w}{\partial y} \right) \right),$$
$$\Delta_\kappa w = \frac{1}{A_1 A_2} \left(\frac{\partial}{\partial x} \left(\frac{\kappa_2 A_2}{A_1} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\kappa_1 A_1}{A_2} \frac{\partial w}{\partial y} \right) \right),$$

where A_1, A_2 are the Lamé coefficients of the neutral surface, and κ_1, κ_2 are its main curvatures.

Now by using asymptotic expansions we study various cases of the vibration modes localization.

3 Axisymmetric Vibrations of Shells of Revolution

The problem of the axisymmetric vibrations of a shell of revolution is one-dimensional. Let the shell be bounded by two parallels $s = s_1$ and $s = s_2$ where s is the generatrix. The problem can be reduced to the singularly perturbed ordinary differential equation of 6th order

$$-\mu^4 \sum_{k=1}^{6} a_k(s) \frac{d^k w}{ds^k} + \sum_{k=1}^{2} b_k(s) \frac{d^k w}{ds^k} = 0, \quad b_2(s) = \lambda - \kappa_2^2(s) \tag{3.1}$$

where $\kappa_2(s)$ is the generatrix curvature.

Equation (3.1) is very simple for a numerical solution, but it is complex due to the presence of turning points s_* for which $b_2(s_*) = 0$ (see Goldenweizer et al. (1979), Tovstik (1967)). Let us introduce the frequency interval Λ

$$\Lambda = [\Lambda^-, \Lambda^+], \qquad \Lambda^- = \min_s\{\kappa_2^2(s)\}, \quad \Lambda^+ = \max_s k_2^2(s)\} \tag{3.2}$$

in which the turning point is contained. It is to be remarked that for cylindrical and for spherical shells the turning points are absent. In this section we do not study these shells.

For $\lambda \notin \Lambda$ the turning point is absent and the general solution of equation (3.1) consists of the linear combination of four bending solutions

$$w_n(s, \mu) = \sum_{k=0}^{\infty} \mu^k A_{kn}(s) \exp\left(\frac{1}{\mu}\int q_n(s)\, ds\right), \quad q_n^4 = b_2, \quad n = 1, 2, 3, 4, \tag{3.3}$$

and of two membrane solutions

$$w_n(s) = \sum_{k=0}^{\infty} \mu^{4k} w_{kn}(s), \qquad b_2 \frac{d^2 w_{0n}}{ds^2} + b_1 \frac{dw_{0n}}{ds} + b_0 w_{0n} = 0, \quad n = 5, 6. \tag{3.4}$$

In the case with the turning point s_* for which $b_2(s_*) = 0,\ b_2'(s_*) \neq 0$, the expansions (3.3) and (3.4) are inapplicable near $s = s_*$ because $A_{kn} \to \infty$ at $s \to s_*$, and equation (3.4) has a singular point at $s = s_*$. Near the turning point five linearly independent solutions of equation (3.1) have the asymptotic expansions

$$w^{(n)}(s, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon A_k(s) v_k^{(n)}(\eta) + \varepsilon \delta_{n5} w^*(s, \mu), \qquad n = 1, 2, 3, 4, 5, \tag{3.5}$$

$$\varepsilon = \mu^{4/5}, \quad \eta(s) = \frac{1}{\varepsilon}\left(\frac{5}{4}\int_{s_*}^{s} b_2^{1/4}(s)\, ds\right)^{4/5}.$$

Expansions (3.5) are expressed by the following standard functions $v_k^{(n)}$

$$\frac{d^5 v_k^{(0)}}{d\eta^5} - \eta \frac{dv_k^{(0)}}{d\eta} - v_k^{(0)} = 0, \qquad v_{k+1}^{(n)}(\eta) = \int v_k^{(n)}(\eta)\, d\eta, \quad k = 0,\, 1, \ldots \tag{3.6}$$

which are the entire functions of the complex argument η. The 6th solution of equation (3.1) has the same form (3.4) where the functions w_{k6} are regular at $s = s_*$.

In the case of the turning point expansions, (3.3) and (3.4) are valid for $|s - s_*| \gg \varepsilon$ and expansion (3.5) is valid for $|s - s_*| \ll 1$. To construct the solution in the whole interval $[s_1,\, s_2]$ it is necessary to combine solution (3.5) with solutions (3.3) and (3.4). Here we use that at $\eta \to \pm\infty$ the solutions (3.5) may be expressed through the more simple solutions (3.3) and (3.4) (see Bauer et al. (1993)). Unfortunately, uniform asymptotic expansions of equation (3.1) are unknown (evidently such expansions do not exist).

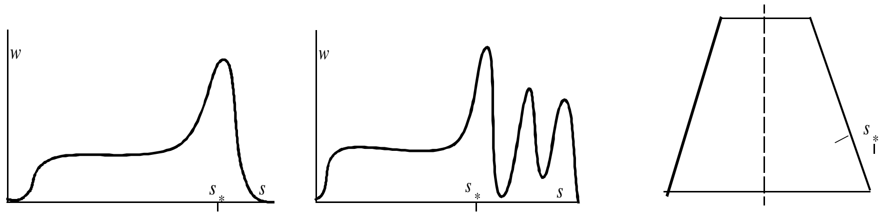

Figure 2: The vibration modes for a conic shell of revolution with the turning point

In Fig. 2 the vibration modes for a conic shell of revolution are presented. As it is easy to see the turning point separates the shell surface into two parts. In the first part the variability of the normal deflection $w(s)$ is large (the index of variation $p = 1/2$), and in the second the function $w(s)$ changes slowly with s ($p = 0$). The maximal deflection is near the turning point and its value is of the order ε^{-1}.

4 Non-symmetric Vibrations of Shells of Revolution

In this case after separation of variables $w(s, \varphi) = w(s)\cos(m\varphi)$, $m = 1, 2, \ldots$, the problem again becomes one-dimensional. Here m is the number of waves in the circumferential direction, and φ is the angular coordinate.

If the number of waves m is fixed and small enough, then this case in the asymptotic point of view does not differ from the axisymmetric case. Therefore we suppose that the number m is large ($m \gg 1$) and we put

$$m = \mu^{-1}\rho. \tag{4.1}$$

Then the system (3.1) can be rewritten in the standard form of a first order system

$$\mu\frac{d\mathbf{y}}{ds} = \mathbf{A}(s,\mu)\mathbf{y}, \qquad \mathbf{A}(s,\mu) = \sum_{k=0}^{\infty}\mathbf{A}_k(s)\mu^k, \quad \mathbf{y} = \{y_1, \ldots, y_8\}. \tag{4.2}$$

The system (4.2) has 8 WKB solutions (see Goldenweizer et al. (1979), Bauer et al. (1993))

$$\mathbf{y}^{(n)}(x,\mu) = \sum_{k=0}^{\infty}\mu^k\mathbf{y}_k^{(n)}(x)\exp\left(\frac{1}{\mu}\int i\,p_n(s)\,ds\right), \quad n = 1, \ldots, 8, \quad i = \sqrt{-1}. \tag{4.3}$$

The functions $p_n(s)$ satisfy the algebraic equation of 8th order

$$\det\left(\mathbf{A}_0(s) - i\,p\mathbf{E}\right) = 0, \tag{4.4}$$

where $\mathbf{E}$ is the unit matrix of 8th order. Equation (4.4) can be solved with respect to the frequency parameter λ

$$\lambda = (p^2+q^2)^2 + \frac{(k_2p^2+k_1q^2)^2}{(p^2+q^2)^2} \equiv f(p,q,s), \qquad q = \frac{\rho}{B(s)}, \tag{4.5}$$

where $B(s)$ is the distance between the point on the neutral surface and the axis of rotation. It is supposed here and below that all linear dimensions of the neutral surface (here $k_1^{-1}, k_2^{-1}, s, B(s)$) are related to R. Then all variables in (4.5) are dimensionless. If the root $p_n(s)$ of equation (4.5) is multiple at the (turning) point $s = s_*$, then the corresponding solution (4.3) is not valid near this point because $\mathbf{y}_k^{(n)}(x) \to \infty$ when $s \to s_*$.

Let us introduce the value

$$\lambda_0 = \min_{p,q,s} f(p,q,s) = f(p_0, q_0, s_0) \tag{4.6}$$

for all real p, q and for $s \in [s_1, s_2]$.

Real roots $p_n(s)$ of equation (4.5) correspond to the oscillating solutions (4.3) of the system (4.2). If $\lambda < \lambda_0$ then there are no real roots, and all solutions (4.3) increase or decrease exponentially. Therefore in this case the vibration modes may be localized near one of the shell edges, and the existence of such modes depends on the boundary conditions at this edge (see below in this Section).

In the case $\lambda > \lambda_0$ equation (4.5) has one pair or two pairs of real roots and therefore equation (4.2) has oscillating solutions which generate the vibration modes for arbitrary boundary conditions. If $\lambda > \lambda_0$ and λ is close to λ_0 then the oscillations are concentrated near the point $s = s_0$ which is called the weakest point. Near the weakest point the first vibration modes are localized. Here we study two cases of the weakest point position.

The weakest point is far from the shell edges. Let equation (4.5) have no more than one pair of real roots (the case of two pairs is studied in Goldenweizer et al. (1979)). Let the frequency parameter λ be close enough to λ_0 so that the real roots of equation (4.5) are at $s_*^{(1)} \le s \le s_*^{(2)}$, where the turning points $s_*^{(k))}$ satisfy the inequality

$$s_1 < s_*^{(1)} < s_0 < s_*^{(2)} < s_2. \tag{4.7}$$

Then the localized modes occupy the interval between the turning points and exponentially decrease with increasing distance from this interval. The corresponding eigen-values $\lambda^{(n)}$ can be found from the equation (see Goldenweizer et al. (1979))

$$\frac{1}{\mu}\int_{s_*^{(1)}}^{s_*^{(2)}} p_0(s)\,ds = \pi\left(\frac{1}{2}+n\right) + O(\mu), \qquad n = 0, 1, 2, \ldots, \tag{4.8}$$

where $\pm p_0(s)$ are real roots of equation (4.5) at $s \in [s_*^{(1)}, s_*^{(2)}]$.

The set of the localized modes is two-parametric and depends on the number m of waves in circumferential direction and the number n in equation (4.8), which is connected with the number of waves in the longitudinal direction. Let the integer number m be fixed and close to $m_0 = \mu^{-1} q_0 B(s_0)$. We introduce the function

$$f^{(m)}(p, s) = f(p, q(s), s), \qquad q(s) = \frac{\rho}{B(s)}, \quad \rho = \mu m, \tag{4.9}$$

and find

$$\lambda_0^{(m)} = \min_{p,s} f^{(m)}(p, s) = \min_{s}\left\{\frac{\rho^4}{B^4(s)} + \kappa_2^2(s)\right\} = \frac{\rho^4}{B^4(s_0^{(m)})} + \kappa_2^2(s_0^{(m)}). \tag{4.10}$$

Here it is remarked that the weakest parallel may (slightly) depend on m.

For the first some eigen-values $\lambda^{(m,n)}$ and the corresponding vibration modes more simple explicit relations (instead of equation (4.8)) can be found (see Bauer et al. (1993)). The frequency parameter is equal to

$$\lambda^{(m,n)} = \lambda_0^{(m)} + \mu\left(\frac{1}{2}+n\right)\sqrt{f_{pp}^{(m)} f_{ss}^{(m)}} + O(\mu^2), \qquad n = 0, 1, \ldots, \tag{4.11}$$

where the partial derivatives $f_{pp}^{(m)}$ and $f_{ss}^{(m)}$ are calculated at $s = s_0^{(m)}$, $p = 0$. The vibration mode has the asymptotic representation

$$w^{m,n}(s, \varphi) = e^{-\xi^2/2}\left(H_n(\xi) + O(\mu^{1/2})\right)\cos m(\varphi - \varphi_0), \tag{4.12}$$

where

$$\xi = \sqrt{\frac{c}{\mu}}\left(s - s_0^{(m)}\right), \qquad c = \sqrt{\frac{f_{pp}^{(m)}}{f_{ss}^{(m)}}}$$

and $H_n(\xi)$ are the Hermite polynomials, $H_0(\xi) = 1$, $H_1(\xi) = \xi$. We remark that the first eigen-values do not depend on the boundary conditions.

In Fig. 3 the first vibration mode of an elongated ellipsoid of revolution is shown. The weakest parallel coincides with its equator.

The weakest point coincides with the edge $s = 0$. If $\lambda > \lambda_0$ and λ is close enough to λ_0 then near the edge $s = 0$ there is a turning point $s = s_*$. One of the solutions of equation (4.2) oscillates at $s < s_*$ and exponentially decrease at $s > s_*$. The asymptotic expansion of this solution contains the standard Airy function and its derivative (see Wasow (1965))

$$w_0(s, \mu) = a(s, \mu)\,\mathrm{Ai}(\eta) + \mu^{1/3} b(s, \mu)\mathrm{Ai}'(\eta), \qquad \eta = \left(\frac{3}{2\mu}\int_{s_*}^{s} p_0(s)\,ds\right)^{2/3} \tag{4.13}$$

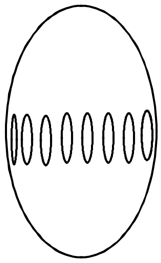

Figure 3: The Vibration Mode of an Ellipsoid of Revolution

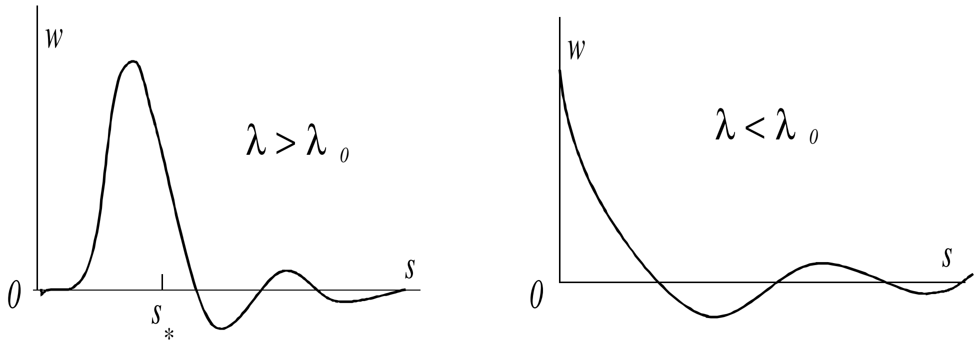

Figure 4: The vibration modes localized near the edge

where $p_0(s)$ is the real root of equation (4.4) at $s < s_*$, and the functions $a(s,\mu)$ and $b(s,\mu$ are the asymptotic series in powers of μ with the coefficients regular in s.

To satisfy the given boundary conditions at the edge $s = 0$, we add to function (4.13) three solutions (4.3) exponentially decreasing with increasing distance from the edge $s = 0$. As a result we get the vibration mode in the form

$$w(s,\varphi) = \left(w_0(s,\mu) + \sum_{n=2}^{4} C_n(s,\mu) e^{ip_n s/\mu} \right) \cos m(\varphi - \varphi_0), \quad \operatorname{Im} p_n > 0 \tag{4.14}$$

The first vibration mode for the clamped edge $s = 0$ is shown in the left side of Fig. 4.

Such modes exist for arbitrary boundary conditions at $s = 0$ and the first some of them do not depend on the boundary conditions at the opposite edge. The asymptotic expansion of the first eigen-values is the following (see Goldenweizer et al. (1979))

$$\lambda^{(m,n)} = \lambda_0^{(m)} + \mu^{2/3}\lambda_1^{(m,n)} + \mu\lambda_2^{(m,n)} + O(\mu^{4/3}) \tag{4.15}$$

where $\lambda_0^{(m)}$ is the same as in (4.10), $\lambda_1^{(m,n)}$ depends on the roots of equation $Ai(\eta) = 0$ and does not depend on the boundary conditions at $s = 0$, and only $\lambda_2^{(m,n)}$ depends on the boundary conditions.

The cases of a weakly supported edge. Here we suppose that $\lambda < \lambda_0$. In this case all solutions of equation (4.2) exponentially decrease or increase. We try to satisfy 4 given boundary conditions at the edge $s = 0$ by 4 solutions which decrease away from this edge. We seek the vibration mode in the form

$$w(s,\varphi) = \left(\sum_{n=1}^{4} C_n(s,\mu) e^{ip_n s/\mu} \right) \cos m(\varphi - \varphi_0), \quad \operatorname{Im} p_n > 0 \tag{4.16}$$

The problem is reduced to the equation

$$\Delta_4(\lambda) = 0, \qquad \lambda < \lambda_0 \tag{4.17}$$

where Δ_4 is the determinant of the 4th order with elements depending on λ. All 16 classical variants of boundary conditions (see Table 1) are examined. The numerical calculations show that there exist only 6 variants for which

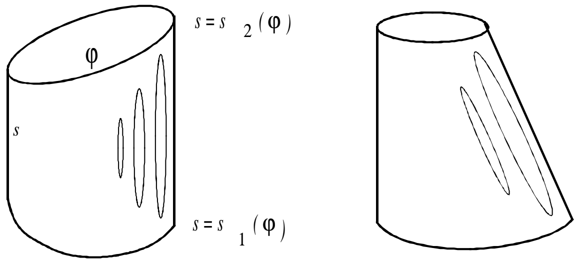

Figure 5: The localized vibration modes of cylindrical and conic shells

equation (4.17) has roots. It occurs that no more than one root exists. These 6 variants are given in Table 2. We call such boundary conditions weak boundary conditions. Among these conditions there are the free edge, 4 conditions with one geometrical restriction, and one condition with two restrictions ($v = \gamma_1 = 0$).

Table 2. The weak boundary conditions.

N	Boundary conditions	Geometric restrictions	$\frac{\lambda}{\lambda_0}$
1	0000	–	0.469
2	0001	$\gamma_1 = 0$	0.537
3	0100	$v = 0$	0.842
4	0101	$v = \gamma_1 = 0$	0.880
5	0010	$w = 0$	0.987
6	1000	$u = 0$	0.995

The numerical example in Table 2 is given for the following values of parameters $q = 1$, $k_1 = 1.5$, $k_2 = 1$. From this example it is easy to see which geometrical restriction is more essential to prevent the existence of localized vibration modes with $\lambda < \lambda_0$.

5 The Localized Vibration Modes for Non-circular Cylindrical and Conic Shells with Slanted Edges

Vibration modes of circular cylindrical and conic shells with straight edges occupy the entire shell surface. But if shells are non-circular and/or its edges are slanted then a localization of the vibration modes is possible. The modes are localized near the weakest generatrix $\varphi = \varphi_0$.

In Fig. 5 the localized vibration modes are shown. The variability of these modes in the circular direction is much larger than its variability in the longitudinal direction

$$\left|\frac{\partial w}{\partial \varphi}\right| \gg \left|\frac{\partial w}{\partial s}\right|. \tag{5.1}$$

The corresponding stress–strain state is called a semi-momentless state because the stress couple M_1 may be neglected compared with the couple M_2. Relation (5.1) allows us to simplify the Donnell equations (2.5) to perform the asymptotic separation of variables. In this problem the asymptotic expansions are the same as in the corresponding buckling problem (see Tovstik and Smirnov (2001)). For simplicity we here study only the cylindrical shells.

The simplified equations of the Donnel type have the form

$$\mu_*^4 \Delta\Delta w + \kappa_2(\varphi)\frac{\partial^2 \Phi}{\partial s^2} - \lambda_* w = 0, \qquad \mu_*^4 \Delta\Delta \Phi - \kappa_2(\varphi)\frac{\partial^2 w}{\partial s^2} = 0 \tag{5.2}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \varphi^2} \approx \frac{\partial^2 w}{\partial \varphi^2}, \qquad \lambda = \mu^2 \lambda_*, \qquad \mu_*^2 = \mu.$$

Here λ_* and μ_* are the new frequency and the small parameter, respectively.

The asymptotic solution has the form

$$w(s,\varphi,\mu_*) = \sum_{n=0}^{\infty} \mu_*^{n/2} w_n(s,\xi) \exp\{i(\mu_*^{-1/2} q\xi + (1/2)a\xi^2)\}, \qquad \xi = \frac{\varphi - \varphi_*}{\sqrt{\mu_*}},$$
$$\lambda_* = \lambda_0 + \mu_*\lambda_1 + \mu_*^2\lambda_2 + \dots, \qquad q > 0, \qquad \operatorname{Im} a > 0. \tag{5.3}$$

Here $w_n(s,\xi)$ are the polynomials in ξ, in particular in the zeroth approximation $w_0(s,\xi) = H_m(\xi)W_0(s)$, $m = 0,1,\dots$, where $H_m(\xi)$ are the Hermite polynomials. Due to $\operatorname{Im} a > 0$ the solution of equation (5.2) exponentially decreases with decreasing distance from the weakest generatrix $\varphi = \varphi_0$. The asymptotic solutions (5.3) and (4.12) are similar to the expansions constructed by Maslov (1977).

The function $W_0(s)$ satisfies the ordinary boundary value problem of the beam type

$$\kappa_2^2(\varphi)\frac{d^4 W_0}{ds^4} + (q^8 - \Lambda q^4)W_0 = 0. \tag{5.4}$$

Equation (5.4) contains the values φ, q, and Λ as parameters. At each edge $s = s_1(\varphi)$ and $s = s_2(\varphi)$ it is possible to satisfy only 2 (main) boundary conditions. The other 2 conditions may be satisfied by the edge effect solutions which exponentially decrease away from the edges. The edge effect solutions in the case, when the edge does not coincide with the curvature lines of the surface, are found by Goldenweizer (1961). The problem how to choose 2 main boundary conditions for the given 4 conditions is discussed in detail for all 16 variants of the classical boundary conditions in the book by Tovstik and Smirnov (2001). For the beam 4 variants of the main conditions are possible: $W_0 = dW_0/ds = 0$ (the clamped edge), $W_0 = d^2W_0/ds^2 = 0$ (the simply supported edge), $dW_0/ds = d^3W_0/ds^3 = 0$ (the weakly supported edge), and $d^2W_0/ds^2 = d^3W_0/ds^3 = 0$ (the free edge). The solution of equation (5.4) with the main conditions coincides with the function, which describes the transversal vibrations of a beam

$$\frac{d^4 W_0}{ds^4} - \frac{\alpha^4}{l^4} W_0 = 0, \tag{5.5}$$

where l is the beam length. For example, for both edges clamped we have $\alpha = 4.73$, for simply supported edges $\alpha = \pi$. The 16 variants of the full boundary conditions are separated into 4 groups according to the main boundary conditions. The list of the full boundary conditions, which correspond to these 4 groups of the main conditions in the case when $ds_1/d\varphi = 0$ is given in Table 3.

Table 3. The groups of boundary conditions.

N	The groups of main conditions	The full boundary conditions
1	The clamped group	1111, 1110, 1011, 1101, 1100, 1010
2	The simply supported group	0111, 0110, 0011, 0101, 0100, 0010
3	The weakly supported group	1001, 1000
4	The free edge group	0001, 0000

For the given boundary conditions we find α and then we obtain

$$\Lambda = q^4 + \frac{\alpha^4 \kappa_2^2(\varphi)}{q^4 l^4(\varphi)} = f(q,\varphi) \qquad l(\varphi) = s_2(\varphi) - s_1(\varphi). \tag{5.6}$$

After minimization of the function $f(q,\varphi)$ we get values λ_0, q, and the weakest generatrix φ_0 near which the vibration mode is localized.

$$\lambda_0 = \min_{q,\varphi} f(q,\varphi) = \min_{\varphi} \frac{2\alpha^2 k_2(\varphi)}{l^2(\varphi)} = \frac{2\alpha^2 k_2(\varphi_0)}{l^2(\varphi_0)}, \qquad q^4 = \frac{\alpha^2 k_2(\varphi_0)}{l^2(\varphi_0)}. \tag{5.7}$$

The value λ_1 depends only on the derivatives of the function $f(q,\varphi)$ at $\varphi = \varphi_0$

$$\lambda_1 = \lambda_1^{(m)} = \left(m + \frac{1}{2}\right)\sqrt{f_{\varphi\varphi} f_{ss} - f_{\varphi s}^2} \tag{5.8}$$

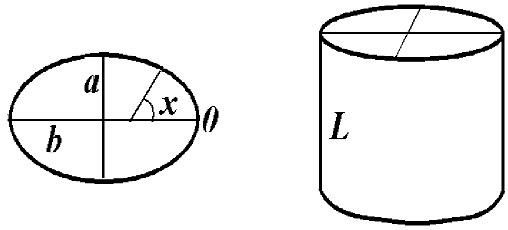

Figure 6: The elliptic cylinder

and the next term λ_2 in the asymptotic expansion (5.3) depends on all the boundary conditions (see Tovstik and Smirnov (2001)).

The eigen-functions (5.3) are complex. Therefore their real and imaginary parts are also eigen-functions of the problem (5.2). But it is incorrect to think that the corresponding eigen-values λ_* are double. Due to the asymptotic character of the series (5.3) in reality two very close eigen-values $\lambda_*^{(1)}$ and $\lambda_*^{(2)}$ with the coinciding asymptotic expansions (5.3) exist

$$\left|\lambda_*^{(1)} - \lambda_*^{(2)}\right| = O\left(e^{-c/\mu_*}\right), \qquad c > 0. \tag{5.9}$$

To these eigen-values the eigen-functions with the following asymptotic expansions

$$w^{(j)} = C_1^{(j)} \mathrm{Re}(w(s,\varphi,\mu_*)) + C_2^{(j)} \mathrm{Im}(w(s,\varphi,\mu_*)), \qquad j = 1,2, \tag{5.10}$$

correspond with the definite constants $C_1^{(j)}$ and $C_2^{(j)}$.

In the paper by Naumova (2001) the free vibrations of a non-straight circular conic shell (as in the right side of Fig. 6) are investigated. The asymptotic solution is compared with the numerical solution obtained by the Finite Element Method.

These solutions are in good agreement, but the numerical solution gives only the first of two asymptotically double eigen-functions.

The eigen-values and eigen-functions of an elliptic cylindrical shell. As an example, which illustrates the asymptotically double eigen-values we study the free low-frequency vibrations of a thin elastic shell with simply supported edges in the form of an elliptical cylinder (see Fig. 6).

Instead of the coordinate φ in circular direction, we use the angle x shown in Fig. 6 as an independent variable. After the separation of variables $w(x,y) = w(x)\sin(\pi s/L)$ (which is possible for simply supported edges) we rewrite the system (5.2) in a dimensionless form (see Krotov and Tovstik (1997))

$$\varepsilon^4 k^4(x)\frac{d^4 w}{dx^4} - \lambda w + k(x)\Phi = 0, \qquad \varepsilon^4 k^4(x)\frac{d^4 \Phi}{dx^4} - k(x) w = 0, \tag{5.11}$$

where

$$\varepsilon^8 = \frac{h^2 L^4}{12(1-\nu^2)a^6}, \quad \lambda = \frac{\rho\omega^2 L^4}{\pi^2 \varepsilon^4 E R^2}, \quad k(x) = e^{-2}\left(\sin^2 x + e^2\cos^2 x\right)^{3/2}, \quad e = \frac{b}{a} > 1.$$

Here ε is the small parameter, and $k(x)$ is the curvature of the ellipse.

We seek the eigen-values of the frequency parameter λ for which there exist the 2π-periodic solutions $w(x)$, $\Phi(x)$ of the system (5.11).

The problem has 2 weakest generatrixes $x = \pm\pi/2$ therefore we expect to have 4 eigen-values, which are very close each other and the asymptotically fourfold eigen-value which may be found by relations similar to (5.3),

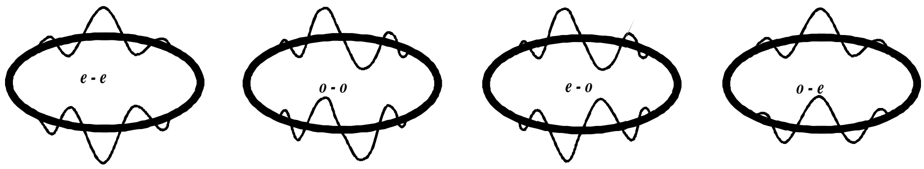

Figure 7: The even and odd eigen-functions (scheme)

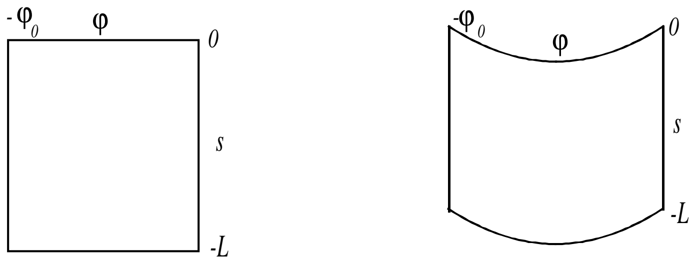

Figure 8: Rectangular plate Cylindrical panel

(5.7), and (5.8) is

$$\lambda^{(m)} = \frac{2}{e^2} + \varepsilon\sqrt{\frac{192(e^2-1)}{e^7}}\left(\frac{1}{2}+m\right) + O(\varepsilon^2), \qquad m = 0, 1, \ldots \tag{5.12}$$

We take the following numerical parameters $m = 0$, $e = 1.4$, $\varepsilon = 0.1$ and numerically calculate the first eigen-values. Due to symmetry of the problem it is possible to seek even and odd eigen-functions with respect to the ellipse diameters (see Fig. 7) and also to separately find the corresponding eigen-values.

For this aim we solve the problem (5.11) in the interval $0 \le x \le \pi/2$ and take the boundary conditions

$$\begin{aligned} & w = \frac{d^2 w}{dx^2} = \Phi = \frac{d^2\Phi}{dx^2} = 0 \qquad (\text{odd}) \\ & \frac{dw}{dx} = \frac{d^3 w}{dx^3} = \frac{d\Phi}{dx} = \frac{d^3\Phi}{dx^3} = 0 \qquad (\text{even}) \end{aligned} \qquad \text{or} \tag{5.13}$$

at the ends of this interval.

Calculations show that 4 asymptotically fourfold eigen-values differ from each other by no more than 10^{-5}. It is interesting to remark that these 4 eigen-values may be gathered in two groups: {even–even, odd–odd} and {even–odd, odd–even} and the difference between eigen-values within the groups is much smaller, namely of the order of 10^{-10}.

6 The Localized Vibration Modes of Cylindrical Panels with a Weakly Supported Rectilinear Edge

We study the free low-frequency vibrations of a thin circular cylindrical panel with radius R and length L (see Fig. 8, right). For the low-frequency vibrations the inequality (5.1) is fulfilled and the asymptotic separation of variables is possible, after which the problem is reduced to a one-dimensional boundary value problem. Therefore it is possible to obtain the approximate asymptotic solutions for arbitrary boundary conditions. In this sense this problem for cylindrical panel vibrations is simpler than the problem for rectangular plate vibrations (see Fig. 8, left). The last problem has no analytical solution for arbitrary boundary conditions.

Here we suppose that one of the rectilinear edges $y = 0$ is weakly supported so that the vibration mode is localized near this edge. The boundary conditions at the curvilinear edges $x = 0$ and $x = -l$ may be arbitrary, but at first we begin from the simply supported curvilinear edges for which the exact separation of variables is possible.

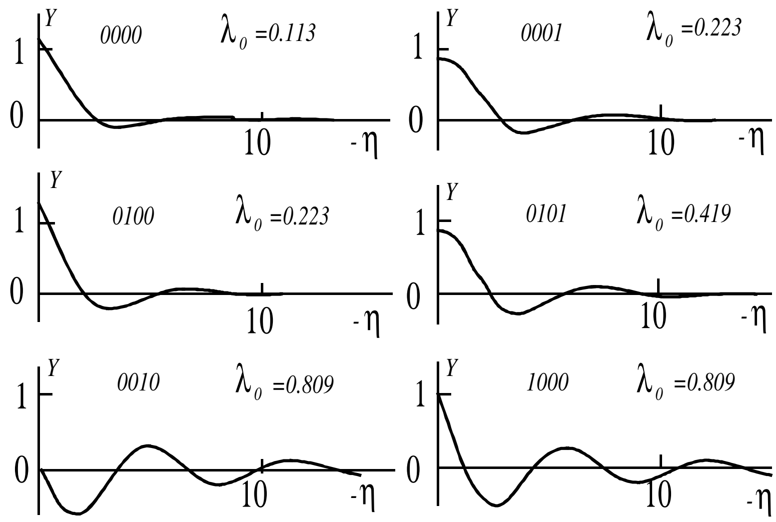

Figure 9: Forms of the localized eigen-functions

The equation which describes the low-frequency vibrations of a circular cylindrical shell may be written in the dimensionless form

$$\mu_*^8\Delta^4 w - \lambda\left(\Delta^2 w - \frac{\partial^2 w}{\partial\varphi^2}\right) + \frac{\partial^4 w}{\partial x^4} = 0, \qquad \Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \varphi^2}, \tag{6.1}$$

where $-l = -L/R \leq x \leq 0$; φ is the angle in circular direction, $-\varphi_0 \leq \varphi \leq 0$, $\varphi_0 = y_0/R$, and

$$\mu_*^8 = \frac{h^2}{12(1-\nu^2)R^2}, \qquad \lambda = \frac{\rho\omega^2 R^2}{E}.$$

After the scaling of the variable $\varphi = \mu_*\sqrt{l/\pi}\eta$ and the separation of variables $w(x,\varphi) = Y(\eta)\sin(\pi x/l)$ we get as the zeroth approximation for the unknown function $Y(\eta)$ the ordinary differential equation

$$\frac{d^8 Y}{d\eta^8} - 2\lambda_0\frac{d^4 Y}{d\eta^4} + Y = 0, \qquad \lambda = \frac{2\mu^2\pi^2}{l^2}\lambda_0, \tag{6.2}$$

where $\lambda_0 = 1$ corresponds to the minimal frequency of a circular cylindrical shell closed in the circular direction.

We seek solutions of equation (6.2), which satisfy the given boundary conditions at the edge $\eta = 0$ and the decreasing condition

$$Y(\eta) \to 0 \quad \text{at} \quad \eta \to -\infty. \tag{6.3}$$

In the paper of Ershova and Tovstik (1998) the 6 variants of the weakly supported edge are found for which such a solution of equation (6.2) exists. The eigen-values λ_0 and the corresponding eigen-functions are presented in Fig.9.

We remark that in this problem the variants of the weakly supported rectilinear edge coincide exactly with the variants of the weakly supported curvilinear edge of a shell of revolution. For the free edge 0000 there exists the second eigen-value $\lambda_0^{(2)} = 0.973$ for which the corresponding eigen-function decreases more slowly than in Fig.9.

More exact than (6.2) the asymptotic relation for the frequency parameter λ for arbitrary boundary conditions at the curvilinear edges has the form (see Ershova and Tovstik (1998))

$$\lambda = \frac{2\mu^2\alpha}{l^2}\left(\lambda_0 + \mu_*^2\lambda_2 + O(\mu_*^3)\right), \tag{6.4}$$

Here the parameter λ_0 depends on the boundary conditions at the weakly supported rectilinear edge $\varphi = 0$ and it is the same as in Fig. 9. The parameter α appears when we asymptotically separate the variables as in equation

(4.5). This parameter depends only on the groups of the main boundary conditions at the curvilinear edges $x = 0$ and $x = -l$. The list of its possible values is the following ($\alpha_{ij} = \alpha_{ji}$)

$$\begin{aligned} &\alpha_{11} = 4.71, \quad \alpha_{12} = 3.93, \quad \alpha_{22} = 3.14 = \pi, \quad \alpha_{13} = 2.37, \\ &\alpha_{14} = 1.88, \quad \alpha_{23} = 1.57, \quad \alpha_{24} = \alpha_{33} = \alpha_{34} = \alpha_{44} = 0 \end{aligned} \tag{6.5}$$

where the indices indicate the numbers of the groups for $x = 0$ and $x = -l$ (see Table 3). For the boundary conditions with $\alpha_{ij} = 0$ the relation (6.4) (and also relation (5.3)) is inapplicable. The parameter λ_2 in relation (6.4) depends on all boundary conditions at the rectilinear and at the curvilinear edges.

7 Conclusions

By the asymptotic approach for thin elastic shells, two types of the free vibration modes localized near the weakest lines are investigated. In the first of them the localization is possible if the neutral surface is heterogeneous (for example the surface curvatures are not constant). To find the asymptotic expansions of the modes in this case it is necessary to use the complex asymptotic constructions including turning point and lines. The second type of localization is connected with the weakly supported edges. The 6 variants of weak boundary conditions are found. It is surprising that for two essentially different problems (with the curvilinear edge of a shell of revolution and with the rectilinear edge of a cylindrical panel) these variants are identical.

The used asymptotic approach is based on a single small parameter (the relative shell thickness h_*), and the other parameters are supposed to be of the order of unity. In the other case the obtained asymptotic relations become incorrect. For example the relation (6.4) is not valid for very long panels $L/R \gg 1$, or for very short panels $L/R \ll 1$, or for very narrow panels $\varphi_0 \ll 1$. In the last case the opposite rectilinear edge begins to influence the frequency.

It is interesting to construct the vibration modes localized near the (weakest) point. For the buckling problems such localization is studied in the book by Tovstik and Smirnov (2001). For a shell of an ellipsoidal form with three different axes, the weakest point coincides with one of the poles. It is also interesting to study cases when $\alpha_{ij} = 0$ for the cylindrical panel.

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