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On the Stability of Oscillatory Pipe Flows

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The linear stability of pure oscillatory pipe flow is investigated by solving the linearized disturbance equations as an initial value problem. The importance of the initial conditions on transient dynamics of the flow is analyzed. It is shown that transient growth can play an important role in the development of flow instability. The accuracy of the quasi-steady assumption is assessed. It is shown that the growth rates obtained with this assumption deviate considerably from the results obtained with a direct numerical solution of the linearized initial value problem.

1 Introduction

Linear stability of unsteady flows is a relatively new topic in hydrodynamic stability theory. Oscillatory flows represent an important subset of unsteady flows and often occur in engineering applications as well as in the field of physiological fluid mechanics. Grosch and Salwen (1968) studied linear stability of oscillatory flow superimposed on a steady plane Poiseuille flow (such a flow is known as a pulsatile flow in the hydrodynamic stability literature). They found that modulation of the pressure gradient has an important effect on the stability characteristics of the flow. In particular, for large modulation amplitude the flow is destabilized at lower mean Reynolds number. The results of Grosch and Salwen (1968) are re-examined by von Kerczek (1982). He found that the oscillating plane Poiseuille flow is more stable than the steady plane Poiseuille flow for a wide range of frequencies. However, the results of von Kerczek (1982) differ substantially from those of Grosch and Salwen (1968) for certain values of the parameters of the problem. A similar problem was recently solved by Straatman et al. (1998) for the range of parameter values which are of interest in physiological fluid mechanics. Modulated plane Poiseuille flow for high modulation frequencies is analyzed asymptotically by Hall (1975). It was shown that modulation destabilized the flow. Note that the Floquet theory is used in their stability analyses. It should be pointed out that the Floquet exponents (which are used to determine whether the given time-periodic flow is linearly unstable) are measures of the average long-term growth or decay of perturbations. The magnitudes of the Floquet exponents cannot be used to determine the transient behavior of a perturbation during one cycle of the imposed oscillation.

An exact unsteady solution of the Navier–Stokes equations for the case of a rigid wall oscillating transversely in a viscous fluid was found by Stokes and is known as the Stokes layer. The linear stability of the Stokes layer was investigated by von Kerczek and Davis (1974), Hall (1978), Cowley (1987), Blennerhassett and Bassom (2002) and Hall (2003). Von Kerczek and Davis (1974) introduced a second boundary away from the oscillating wall and found no unstable modes for Reynolds numbers up to 400. Hall (1978) presented an improved model without an upper boundary but he also could not find unstable modes for Reynolds numbers up to 160. Cowley (1987) used the method of multiple scales to demonstrate that, for sufficiently large Reynolds numbers, disturbances can experience a significant growth over a part of the oscillating cycle. The analysis by Blennerhassett and Bassom (2002) showed that the Stokes layer becomes unstable at Reynolds numbers about 708. Since this result is inconsistent with the previous studies, in a recent paper Hall (2003) tried to resolve this inconsistency. He found that there are no unstable Floquet modes at high Reynolds numbers.

The instability of a pure oscillatory flow in a pipe (which is the most interesting flow geometry from a practical point of view) has been analyzed by Sergeev (1966) and Hino et al. (1976). Three types of flow regimes were analyzed in their experiments: laminar flow, weakly turbulent flow and conditionally turbulent flow. It was found that, in the conditionally turbulent flow, turbulence is generated in the decelerating phase but the flow returns to laminar in the accelerating phase. Thus, the experiments by Hino et al. (1976) confirmed that some oscillatory pipe flows can be unstable only over part of the oscillating cycle.

A major difficulty in comparing experimental data for oscillatory pipe flows with theoretical results is the absence of a neutral stability curve. An oscillatory pipe Poiseuille flow (like a steady pipe flow) is found to be linearly stable in accordance with the linear stability theory (no unstable Floquet modes were found). The transient growth mechanism was used in the last decade in order to explain the transition in steady pipe flows (see, for example, O'Sullivan and Breuer (1994), Schmid and Henningson (1994), Schmid and Henningson (2001)). It is based on the fact that the superposition of stable modes can initially experience a large growth before the perturbation starts to decay asymptotically (for large time). The energy amplification factor can be large enough to trigger instability even if the flow is exponentially stable in accordance with the Floquet theory.

The perturbation dynamics of unsteady non-periodic flows is analyzed in Zhao (2003) and Zhao et al. (2004) by solving the corresponding initial value problem. The dependence of the energy growth of perturbation on the initial conditions is analyzed and the importance of transient growth mechanisms for stability analyses of non-periodic unsteady flows is investigated. In the present paper, linear stability of pure oscillatory flow in a circular pipe is investigated. The linearized stability equations are solved as an initial value problem. The initial value problem approach allows one to analyze the behavior of perturbations for short time (transient growth) and large time (asymptotic growth). In addition, the role of the initial conditions on perturbation dynamics is discussed.

2 Stability Model and Numerical Solution

Consider an axisymmetric unsteady flow in a circular pipe of radius R of the following structure,

$$\widetilde{W} = \widetilde{W}(\tilde{r}, \tilde{t}); \;\; \widetilde{P} = \widetilde{P}(\tilde{z}, \tilde{t}) \tag{1}$$

where $\tilde{t}$, $\tilde{z}$ and $\tilde{r}$ are the dimensional time, axial and radial coordinates, respectively, and $\widetilde{W}$ and $\widetilde{P}$ are dimensional axial velocity and pressure. The functions $\widetilde{W}$ and $\widetilde{P}$ satisfy the following equation of motion (Schlichting, 1979),

$$\frac{\partial \widetilde{W}}{\partial \tilde{t}} = -\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial \tilde{z}} + \nu\left(\frac{\partial^2 \widetilde{W}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \widetilde{W}}{\partial \tilde{r}}\right) \tag{2}$$

where ρ is the density of fluid and ν is the kinematic viscosity. Consider the case of a pure oscillatory flow where the pressure gradient is given by

$$-\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial \tilde{z}} = K\cos\widetilde{\omega}\tilde{t}. \tag{3}$$

Here K is a constant and $\widetilde{\omega}$ is the dimensional frequency. Let the measure of the length, velocity and time be R, $U = K/\widetilde{\omega}$ and R/U, respectively. Then equation (2) can be written in the form

$$\frac{\partial W}{\partial t} = \omega\cos\omega t + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right) \tag{4}$$

where the un-tilded variables are dimensionless variables and the Reynolds number is defined by Re $= RU/\nu$.

The solution of equation (4) can be found in Hino et al. (1976), Yang and Yih (1977), and Schlichting (1979) and has the form

$$W(r,t) = \Re\left\{-ie^{i\omega t}\left[1 - \frac{J_0(r\sqrt{-i\omega\mathrm{Re}})}{J_0(\sqrt{-i\omega\mathrm{Re}})}\right]\right\} \tag{5}$$

where $\Re$ means real part, $i = \sqrt{-1}$ is the imaginary unit and $J_m(q)$ is the Bessel function of the first kind of order m.

Following Hino et al. (1976) we introduce the notations $\beta = \sqrt{2}e^{\frac{3}{4}\pi i}$ and $\lambda = \sqrt{\frac{\widetilde{\omega}}{2\nu}}R$ so that $\beta\lambda = \sqrt{-i\omega\mathrm{Re}}$. The Reynolds numbers R_e reported by Hino et al. (1976) are based on the cross-sectional mean velocity amplitude $(\widehat{W})$ and pipe diameter. Since the amplitude of the cross-sectional mean velocity is

$$\widehat{W} = U\left[1 - \frac{2}{\beta\lambda}\frac{J_1(\beta\lambda)}{J_0(\beta\lambda)}\right] \tag{6}$$

the Reynolds number can be written in the form

$$\mathrm{Re} = \frac{UR}{\nu} = \frac{\widehat{W}(2R)}{\nu}\frac{U}{2\widehat{W}} = \frac{1}{2}R_e\left[1 - \frac{2}{\beta\lambda}\frac{J_1(\beta\lambda)}{J_0(\beta\lambda)}\right]^{-1}. \tag{7}$$

Linear stability of flow (5) is analyzed below. We assume that $u(r,\theta,z,t)$, $v(r,\theta,z,t)$, $w(r,\theta,z,t)$ and $p(r,\theta,z,t)$ are the perturbations of the velocity components and pressure, respectively (here θ is the azimuthal coordinate). Adding these perturbations to (5), substituting the sum into the Navier–Stokes equations, linearizing the equations in the neighborhood of the base flow (5) and assuming that

$$[u(r,\theta,z,t);v(r,\theta,z,t);w(r,\theta,z,t);p(r,\theta,z,t)]^T=[\hat{u}(r,t);\hat{v}(r,t);\hat{w}(r,t);\hat{p}(r,t)]^T\,e^{in\theta+i\alpha z}, \tag{8}$$

where α and n are axial and azimuthal wavenumbers, respectively, we obtain the following system

$$\frac{1}{r}\frac{\partial(r\hat{u})}{\partial r}+\frac{in}{r}\hat{v}+i\alpha\hat{w} = 0, \tag{9}$$

$$\frac{\partial\hat{u}}{\partial t}+i\alpha W\hat{u} = -\frac{\partial\hat{p}}{\partial r}+\frac{1}{\mathrm{Re}}\left(N\hat{u}-\frac{\hat{u}}{r^2}-\frac{2in}{r^2}\hat{v}\right), \tag{10}$$

$$\frac{\partial\hat{v}}{\partial t}+i\alpha W\hat{v} = -\frac{in}{r}\hat{p}+\frac{1}{\mathrm{Re}}\left(N\hat{v}-\frac{\hat{v}}{r^2}+\frac{2in}{r^2}\hat{u}\right), \tag{11}$$

$$\frac{\partial\hat{w}}{\partial t}+i\alpha W\hat{w}+\hat{u}\frac{\partial W}{\partial r} = -i\alpha\hat{p}+\frac{1}{\mathrm{Re}}N\hat{w}, \tag{12}$$

where $N=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}-\dfrac{n^2}{r^2}-\alpha^2$.

The boundary conditions at $r=1$ are

$$\hat{u}(1)=0,\quad \hat{v}(1)=0,\quad \hat{w}(1)=0,\quad \left.\frac{\partial\hat{u}}{\partial r}\right|_{r=1}=0. \tag{13}$$

The boundary conditions at $r=0$ depend on the azimuthal wavenumber and have the following form (details are given by Moin & Kim (1980) and Lopez et al. (2002)):

(a) For $n=0$,
$$\hat{u}(0)=0,\quad \hat{v}(0)=0,\quad \hat{w}(0)\equiv\text{finite},\quad \hat{p}(0)\equiv\text{finite}. \tag{14}$$

(b) For $n=1$,
$$\hat{u}(0)+i\hat{v}(0)=0,\quad \hat{w}(0)=0,\quad \hat{p}(0)=0,\quad \left.\left[\frac{\partial\hat{u}}{\partial r}+i\frac{d\hat{v}}{dr}\right]\right|_{r=0}=0. \tag{15}$$

(c) For $n=2,3,\ldots$,
$$\hat{u}(0)=0,\quad \hat{v}(0)=0,\quad \hat{w}(0)=0,\quad \hat{p}(0)=0. \tag{16}$$

A semi-implicit scheme suggested by Moin and Kim (1980) is used to solve the problem. Chebyshev polynomials are used to discretize the system in the radial direction. Details of the numerical scheme are given in Zhao et al. (2004). The initial conditions for the functions $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{p}$ are chosen in the following way. First, we generate a random number from a standardized normal distribution at each radial node. Second, the linearized equations without viscous terms are solved in order to obtain the functions which satisfy the incompressibility condition (9) and these functions are used as initial conditions.

The results of the computations are compared with experimental data found in Hino et al. (1976). We selected three runs for comparison, namely, runs 4, 9 and 14 from Hino et al. (1976), which represent laminar or distorted laminar flow, weakly turbulent and conditionally turbulent flow, respectively. The following table gives the flow parameters for these three cases.

Test No.	Pipe Diameter (m)	Period (s)	$\hat{W}$ (m/s)	ν (m^2/s)	R_e $(-)$	λ $(-)$	ω $\times 10^3$	Flow Regime
4	0.03	6.0	0.355	1.5×10^{-5}	710	2.76	27.2	laminar
9	0.03	3.0	2.13	1.5×10^{-5}	4260	3.90	10.6	weakly turbulent
14	0.0145	3.0	6.03	1.5×10^{-5}	5830	1.91	1.08	conditionally turbulent

Table 1: Parameters for test cases.

3 Transient and Asymptotic Growth

As pointed out earlier, the initial value problem approach allows one to investigate the full temporal dynamics of perturbations including short-term transients and long-term asymptotics. The energy growth of a perturbation can be characterized by the following integral which represents the kinetic energy per unit density of a three-dimensional perturbation contained in a single wave length:

$$E(t) \equiv \frac{1}{2\mathcal{V}} \int_0^{2\pi/\alpha} \int_0^{2\pi} \int_0^1 (u^2 + v^2 + w^2) r dr d\theta dz \tag{17}$$

where $\mathcal{V}$ is the volume for a complete wave length.

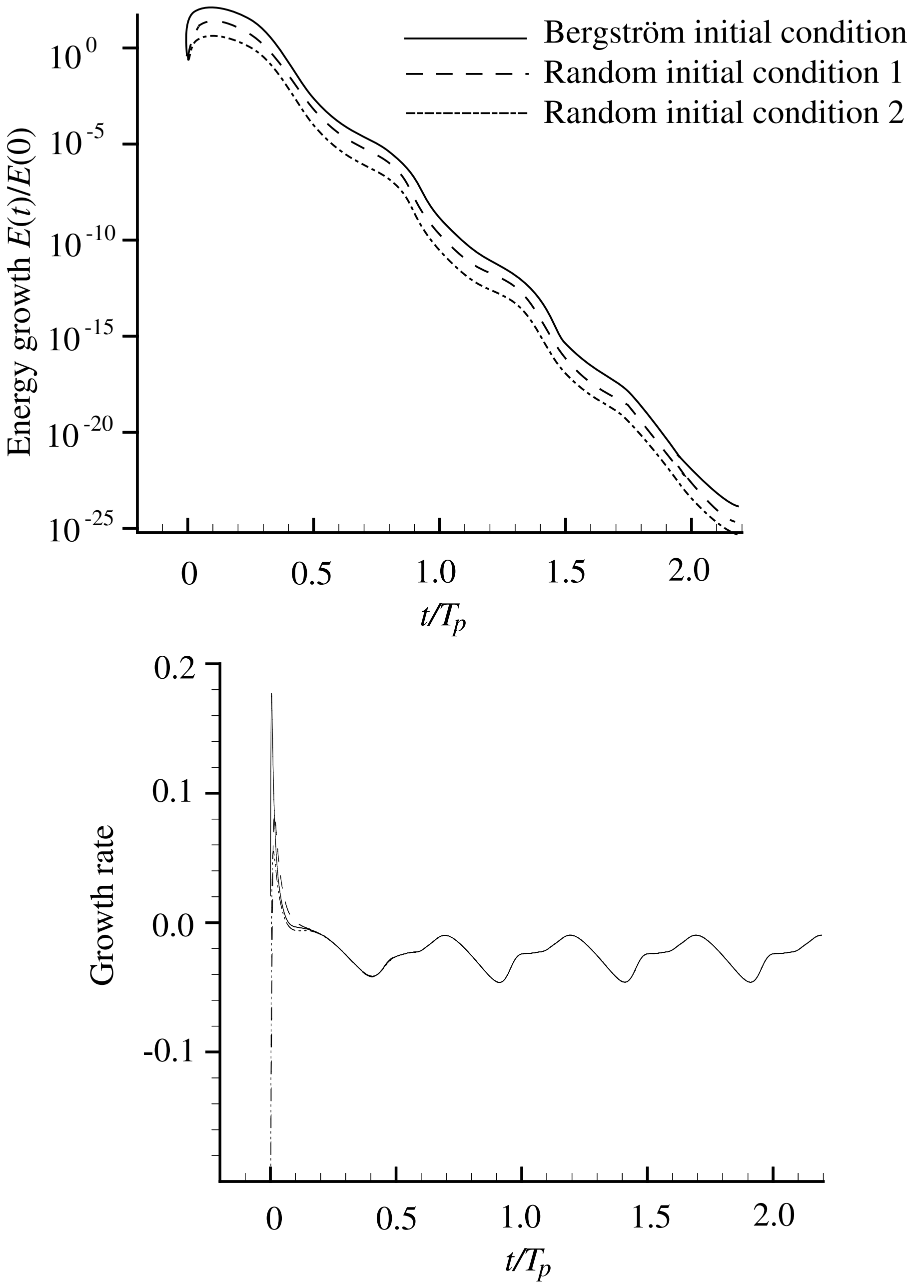


Figure 1: (Top) Growth behavior of perturbations for case 9, and (bottom) growth rate for different initial conditions.

The energy growth curves for three different initial conditions (namely, the initial condition suggested by Bergström (1993) and two randomly chosen initial conditions) and the corresponding growth rates are shown in Fig. 1 for the

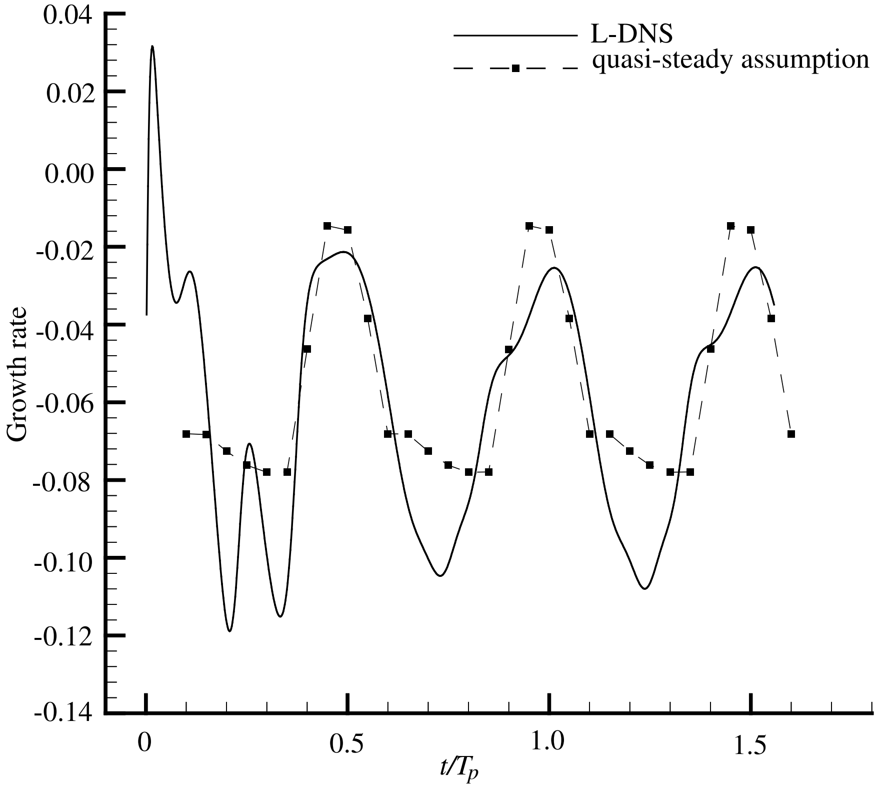
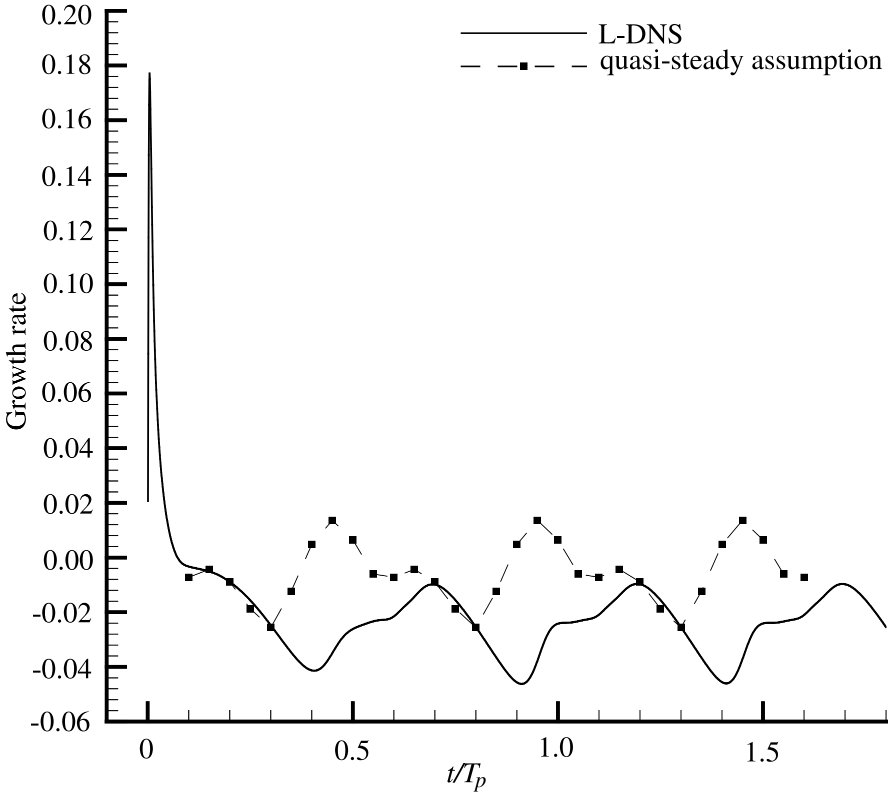


Figure 2: Growth rates obtained from L-DNS and the quasi-steady assumption for case 4 ($n = 1, \alpha = 0.5$).

case $n = 1, \alpha = 0.5$. The dependence of the perturbation energy on the initial condition for short time is clearly seen from the graphs. For large times all three graphs in top Fig. 1 are almost parallel to each other, therefore the corresponding growth rates coincide (see bottom Fig. 1). The graph in bottom Fig. 1 shows that the flow is asymptotically stable for large times (since the growth rates are negative). We performed calculations for other values of α, namely, $\alpha = 0$, 0.5, 1 and 2 and found that in all cases the growth rates are negative for large time. Thus, the results of our calculations indicate that the flow is asymptotically stable (however, these calculations by no means represent a proof of stability). It is interesting to note, however, that the growth rates are positive during a short time interval. In addition, the magnitudes of the growth rates are quite large (the value are larger than 0.1 for some time interval). This is the transient growth which cannot be obtained by means of the Floquet theory. Perturbations may experience sufficient growth during a short-time interval and this may lead to flow instability.

4 Quasi-steady Assumption for Oscillatory Flows

In this section we analyze the accuracy of the quasi-steady assumption for stability analysis of oscillatory pipe flows. This assumption, which is widely used in hydrodynamic stability theory, assumes that the base flow varies slowly compared with the growth of a perturbation. In other words, the base flow is treated as a steady state using an instantaneous “frozen” profile. The classical approach in this case is to use the Floquet theory where the stability of the flow depends on the sign of the real part of the Floquet exponents (von Kerczek 1982). Instead of calculating the Floquet exponents by solving an eigenvalue problem, to calculate the asymptotic growth rate for a particular velocity profile, we fix the velocity profile and use the solution of an initial value problem for sufficiently large time. The results obtained in this way are based on the quasi-steady assumption. The results without quasi-steady assumption are referred to as L-DNS (linear direct numerical simulation). As Fig. 1 indicates, the choice of the initial condition is unimportant if one is interested in long-time asymptotics since the growth rate will be the same for all perturbations at sufficiently large times. The computational results are presented in Fig. 2 for case 4 and in Fig. 3 for cases 9 (top) and 14 (bottom) corresponding to the three different flow regimes reported by Hino et al. (1976) (see Table 1).



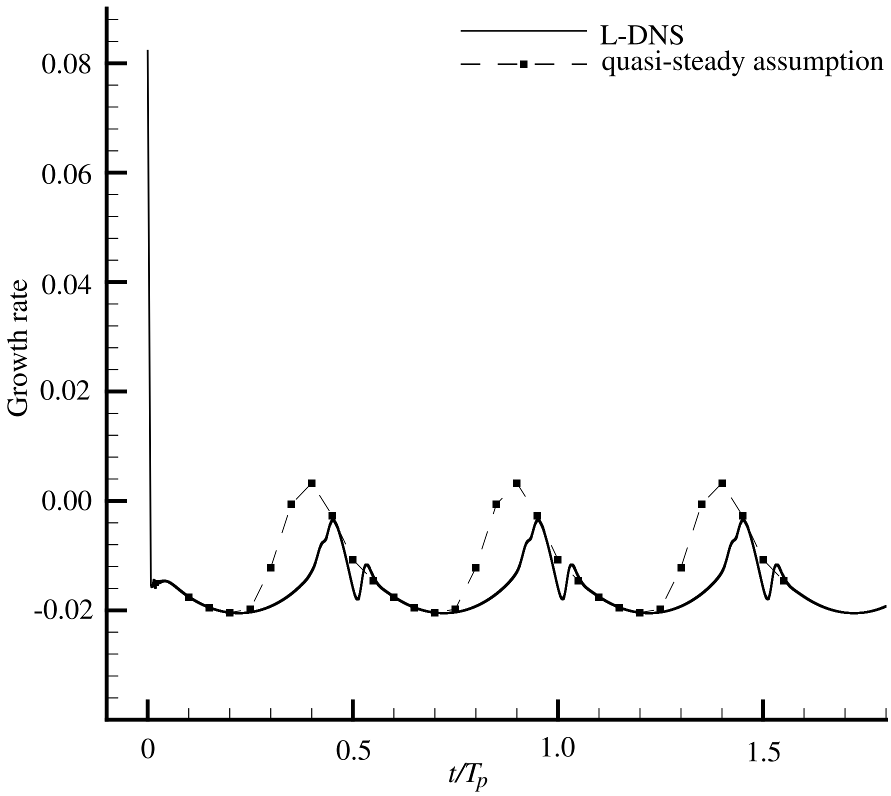


Figure 3: Growth rates obtained from L-DNS and the method of normal modes wiith $n = 1$ and $\alpha = 0.5$ for case 9 (top) and case 14 (bottom).

All the figures indicate the presence of a region of transient growth (where the growth rate is positive for a short time) followed by a region of asymptotic decay (the growth rates are all negative for sufficiently large time). The magnitudes of the growth rates are considerably higher for case 9 and case 14 in comparison with case 4. This may explain why the instability is not observed in case 4 — the growth rate is not large enough and the perturbation does not have enough time to grow. Note that the transient growth cannot be captured by Floquet theory which can be used only to analyze long time asymptotic stability. All three graphs also show that even in the long-time limit the growth rates obtained by means of the quasi-steady assumption deviate considerably from the L-DNS results. In particular, for cases 9 and 14 the quasi-steady approach shows that for large times the flow become unstable for a part of the oscillation cycle while the L-DNS approach shows no instability for large times. Therefore we suggest the use of the L-DNS approach for stability studies of oscillatory pipe flows.

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