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Galactic Contraction and the Collinearity Principle

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In a spiral galaxy there is not only a Kepler force acting on an individual star but also a transverse pull opposing the motion. The relatively small transverse pull is due to the atmospheric drag exerted by interstellar gas (hydrogen, cosmic dust). It is also shown that the arms of a spiral galaxy consist of Ward spirals, that there is an orbital energy loss for each individual star in a contracting galaxy, and that the size of the Ward spiral observed can be used to predict the speed of the galaxy's contracting. For inside the galaxy's central sphere it is shown that the path of a star describes a logarithmic spiral and that there is an associated orbital energy loss.

1 Equations of Motion

The collinearity principle requires that the stars orbiting in a galaxy arrange themselves into a flat disk (Rimrott, 1998). Within the galactic disk (Figure 1) polar coordinates can be used for Newton's second law.

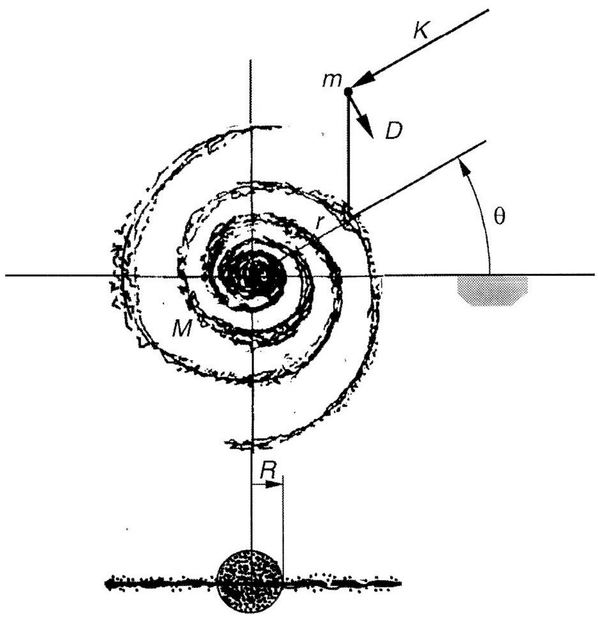

Figure 1. Model of a Spiral Galaxy

For an individual star m

$$-K \approx -\frac{GMm}{r^2} = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{1}$$

$$-D = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{2}$$

G = gravitional constant

M = mass of galaxy (without m)

D = (constant) drag force
According to Ward (2000) the term $\ddot{r}$ in equation (1) is often so small that it can profitably be neglected. This is apparently also the case for spiral galaxies. Consequently from equation (1)

$$mr\dot{\theta}^2 = \frac{GMm}{r^2}$$

or
$$\dot{\theta} = \sqrt{\frac{GM}{r^3}} \quad (3)$$

Differentiated with respect to time

$$\ddot{\theta} = -\frac{3}{2r}\sqrt{\frac{GM}{r^3}}\dot{r} \quad (4)$$

Equations (3) and (4) entered into equation (2) give

$$D = -\frac{1}{2}m\sqrt{\frac{GM}{r^3}}\dot{r} \quad (5)$$

which relates the drag to the radius change rate.

Solved for the radius change rate equation (5) becomes

$$\dot{r} = \frac{-2D}{m\sqrt{GM}}r^{3/2} \quad (6)$$

If $D > 0$ then $\dot{r} < 0$, characterizing a contracting galaxy.

2 Shape of the Galactic Arms

We begin with equations (5) which may also be written as

$$\frac{dr}{r^{3/2}} = -\frac{2D}{m\sqrt{GM}}dt \quad (6)$$

where we have chosen to use

$$D = -\frac{1}{2}m\dot{\theta}_0\dot{r}_0 = \text{constant} \quad (7)$$

for an initial interval. The subscript 0 refers to the star considered. For any subsequent interval the drag may have a different new value. On the other hand equation (7) may be valid, at least approximately, for an extended portion of, and possibly the whole spiral arm. We assume the latter to be the case.

Integrated between $r = r_0$ for $t = 0$ and $r = r$ for $t = t$ we arrive at a parameter equation

$$r = \frac{r_0}{\left(1+\frac{D}{m}\sqrt{\frac{r_0}{GM}}t\right)^2} \quad (8)$$

which was first obtained by C.A. Ward in connection with drag studies (see Rimrott and Salustri, 2001). Making use of equations (3) and (7) we may also write

$$r = \frac{r_0}{\left(1 - \frac{\dot{r}_0}{2r_0}t\right)^2} \tag{9}$$

instead of equation (8).

By using

$$\theta = \dot{\theta}_0 t \tag{10}$$

we obtain the Ward spiral in polar coordinates

$$r = \frac{r_0}{(1+c\theta)^2} \tag{11}$$

with

$$c = \frac{D}{K_0} \tag{12}$$

and

$$K_0 = \frac{GMm}{r_0^2} \tag{13}$$

From these results we conclude that the arms of a spiral galaxy are formed by Ward spirals. In Figure 2a Ward spiral with $c = 0.08$ is depicted.

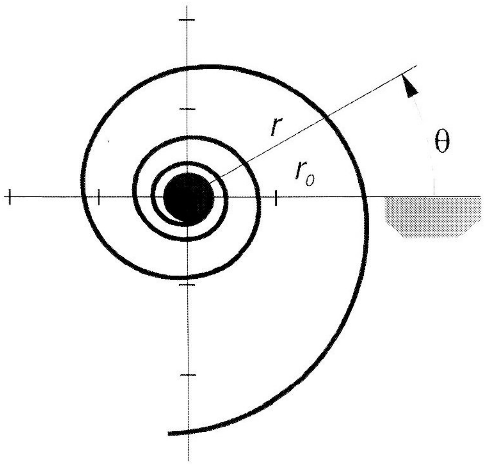

Figure 2. A Ward Spiral with $c = 0.08$

3 Orbital Energy

The orbital energy of a single star m on a circular orbit can be shown (Rimrott, 1989) to be

$$E = -\frac{GMm}{2r} \tag{14}$$

If subject to radius change rate $\dot{r}$ the associated energy change rate is

$$\dot{E} = \frac{GMm}{2r^2}\dot{r} = -\sqrt{\frac{GM}{r}}D \tag{15}$$

In particular for a contracting galaxy $\dot{r} < 0$ and consequently $\dot{E} < 0$, in other words there is an energy loss for the star m in a contracting galaxy, as required by the collinearity principle.

4 Speed of Contracting

For the physical appearance of a spiral arm one can infer as to how fast a spiral galaxy will contract on its way to become an elliptical galaxy. With the coefficient c known from observation, the initial contracting speed $\dot{r}_0$ can be calculated. From equations (5), (11) and (12)

$$D = cK_0 = -\frac{1}{2}m\sqrt{\frac{GM}{r_0^3}}\dot{r}_0$$

or

$$\dot{r}_0 = -2c\sqrt{\frac{GM}{r_0}} \tag{16}$$

5 Numerical Values for Sun and Milky Way

For our own Milky Way and our own Sun (Greiner, 1989) which is located in one of the spiral arms, we have

$$m = 1.989\left(10^{30}\right)\text{kg} \qquad \text{for the Sun}$$
$$r_0 = 5\left(10^{20}\right)\text{m}$$
$$\dot{\theta}_0 = 6\left(10^{-16}\right)\text{s}^{-1}$$
$$v_0 = r_0\dot{\theta}_0 = 5(6)10^4 = 3\left(10^5\right)\text{m/s}$$
$$M = 5.967\left(10^{41}\right)\text{kg} \qquad \text{for the Milky Way}$$
$$G = 6.67\left(10^{11}\right)\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

If the Sun moves closer to the galactic center at a rate of, say, $\dot{r}_0 = -4.5(10^4)\,\text{m/s}$ then we have for the energy rate

$$\dot{E} = \frac{6.67(10^{-11})\,5.967(10^{41})\,1.989(10^{30})}{1(5^2)10^{40}}\left(-4.5(10^4)\right) = -7.125\left(10^{24}\right)\text{W}$$

for the drag in terms of the radius change rate

$$D = -\frac{1}{2}m\dot{\theta}_0\dot{r}_0 = \frac{1}{2}1.989(10^{30})6\left(10^{-16}\right)4.5\left(10^4\right) = 2.685\left(10^{19}\right)\text{N}$$

for the Kepler force magnitude

$$K_0 = \frac{GMm}{r_0^2} = \frac{6.67(10^{-11})5.967(10^{41})1.989\left(10^{30}\right)}{5^2\left(10^{40}\right)} = 3.166(10^{20})\,\mathrm{N}$$

and for the ratio

$$c = \frac{D}{K_0} = 0.0848$$

A ratio of $c = 0.08$ has been used for the Ward spirals of Figures 1 and 2.

A contracting rate of $\dot{r}_0 = -45\,\mathrm{km/s} = -4.5(10^4)\,\mathrm{m/s}$ may seem to be rather large. But consider this: For a galaxy of, say, $r_0 = 5(10^{20})$ m radius, shrinking at a rate of 4.5 (10^4) m/s to half its size would mean that it takes

$$t = \frac{1}{2}\frac{5(10^2)}{4.5(10^4)} = 5.556(10^{15})\ \mathrm{s} = 1.762\left(10^8\right)\mathrm{a}$$

i.e. 176.2 million years!

6 Inside the Central Sphere

Physical evidence has it that spiral galaxies still have a nucleus that can be modeled by a small central sphere (Figure 1) of uniform star distribution. The gravitational field for a point mass *m* inside a solid sphere *M* of uniform mass (=star) distribution and of outer radius *R* (Rimrott, 1989) is

$$U = -\frac{GMm}{2R}\left\{3 - \frac{r^2}{R^2}\right\} \tag{17}$$

Leading to a Kepler force inside the sphere of

$$K = -\frac{\partial U}{\partial r} = -\frac{GMm}{R^3}r \tag{18}$$

Newton's second law requires then, that in radial direction

$$-\frac{GMm}{R^3}r = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{19}$$

With $\ddot{r} = 0$ as an approximation we have

$$\dot{\theta} = \sqrt{\frac{GM}{R^3}} = \text{constant} \tag{20}$$

and

$$\ddot{\theta} = 0 \tag{21}$$

The velocity of the star *m* is

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = r\dot{\theta}\sqrt{1 + \frac{\dot{r}^2}{r^2\dot{\theta}^2}} \tag{22}$$

In view of $\dot{r}^2 \ll r^2\dot{\theta}^2$ we simplify and write

$$v = r\dot{\theta} = r\sqrt{\frac{GM}{R^3}} \tag{23}$$

We also let the drag force decrease accordingly towards the galactic center and write for it

$$\frac{r}{R}D \tag{24}$$

and get for Newton's second law, in transverse direction,

$$-\frac{r}{R}D = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{25}$$

With $\ddot{\theta} = 0$ from equation (21), we have

$$\dot{r} = -\frac{D}{2mR\dot{\theta}}r \tag{26}$$

which integrated between the limits $r = R$ for $t = 0$ and $r = r$ for $t = t$ gives

$$r = R\exp\left(-\frac{D}{2mR\dot{\theta}}t\right) \tag{27}$$

or with

$$t = \frac{\theta}{\dot{\theta}} \tag{28}$$

we have

$$r = R\exp\left(-\frac{D}{2K_R}\theta\right) \tag{29}$$

a logarithmic spiral, with a Kepler force of magnitude

$$K_R = mR\dot{\theta}^2 = \frac{GMm}{R^2} \tag{30}$$

The kinetic energy of a star m inside the central sphere is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}mr^2\frac{GM}{R^3} = \frac{1}{2}\frac{GMm}{R^3}r^2 \tag{31}$$

Together with the potential energy from equation (17) we have for the star's orbital energy

$$E = U + T = -\frac{GMm}{R}\left\{\frac{3}{2} - \frac{r^2}{R^2}\right\} \tag{32}$$

and for its time derivative

$$\dot{E} = \frac{2GMm}{R^3} r\dot{r} \tag{33}$$

We conclude that

$$\dot{E} < 0 \qquad \text{if} \qquad \dot{r} < 0 \tag{34}$$

i.e. contraction is associated with an orbital energy loss. We also have

$$\dot{E} = \frac{2GMm}{R^2}\dot{r} \qquad \text{at} \qquad r = R \tag{35}$$

and

$$\dot{E} = 0 \qquad \text{at} \qquad r = 0 \tag{36}$$

The orbital energy change rate may also be expressed in terms of the drag, equation (26), as

$$\dot{E} = -\frac{\dot{\theta}}{R} r^2 D = -\frac{\sqrt{GM}}{R\sqrt{R^3}} r^2 D \tag{37}$$

at $r = R$ the orbital energy change rate becomes

$$\dot{E} = -\sqrt{\frac{GM}{R}} D \tag{38}$$

7 The Interface

At the interface $r = R$ of galactic outer disk and inner sphere, we find that all quantities, with one exception, are continuous.

For the drag force we compare equations (7) and (24). For the orbital energy we compare equations (14) and (32). For the orbital energy change rate we compare equations (15) and (35). For the magnitude of the Kepler force we compare equations (1) and (30). For the angular velocity we compare equations (3) and (20).

The single exception is the radial velocity component which changes from

$$\dot{r} = -\frac{2D}{m\sqrt{GM}} R^{3/2} \tag{39}$$

as given by equation (6), when approached from outside the central sphere, to

$$\dot{r} = -\frac{D}{2m\sqrt{GM}} R^{3/2} \tag{40}$$

from equation (26), when approached from inside the central sphere. Moving inwards its magnitude decreases by a factor of four and provides an explanation for the sudden increase of the star population from disk to sphere at the interface.

8 Conclusion

By the simple expedient of taking into account the influence of a small atmospheric drag, exerted on each individual star of a shrinking spiral galaxy, the spiral nature of the galaxy, consisting of Ward spirals outside the central sphere and logarithmic spirals inside the central sphere, reveals itself. Another consequence is that, in accordance with the collinearity principle, each individual star gives up some of its orbital energy in the process. The shape of the Ward spirals is also a measure of how quickly a spiral galaxy shrinks moving towards becoming an elliptical galaxy, the subsequent stage in the evolution process of galaxies.

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